Brief Research Statement

Algebraic cycles, Hodge theory, and Arithmetic

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Introduction

The antique origins of Algebraic Geometry lie in the study of solution sets of polynomial equations, in which complex, symplectic, and arithmetic geometry are bound tightly together. Many of the most spectacular recent developments in the subject have occurred through the consideration of these aspects in tandem: for example, the duality between symplectic and complex geometry that is mirror symmetry, and the Beilinson conjectures [1] on the transcendental invariants of generalized algebraic cycles defined over number fields. Beilinson [2] (and independently Bloch) also made conjectures on the structure of algebraic cycle groups over $\mathbb{C}$; here the arithmetic content only becomes apparent when one looks at recent efforts to construct the filtration they predict. Likewise, the arithmetic side to the Hodge Conjecture (cf. [3]) is revealed by its bifurcation into absoluteness of Hodge cycles (proved for abelian varieties by Deligne [4]) and validity of the (slightly weakened) conjecture for varieties over $\bar{\mathbb{Q}}$, emphasized in recent work of Voisin [5].

The invariants appearing in these longstanding and celebrated problems are formalized Hodge-theoretically — that is, in terms of integrals of algebraic differentials on the variety called *periods*. Roughly speaking, the conjectures can be thought of as predicting the existence of certain algebraic cycles explaining what algebraic structure these otherwise transcendental periods *do* have. But cycles are more ubiquitous (and useful!) than this formulation might suggest, manifesting themselves through generalized normal functions in diverse contexts, from number theory (modular forms [6] and higher Green's functions [7]; Apéry numbers [8]) to mathematical physics (asymptotics of local instanton numbers [9]; prediction of open Gromov-Witten invariants [10]; Feynman integrals [11, 12]; topological string theory [13]) and differential equations. What is more, the discovery of a "motivating cycle" can be the key to proving properties of such functions and generating more examples. This is a first major theme of my research.

The periods described above are typically packaged in a linear-algebraic object called a Hodge structure (or mixed Hodge structure), and this is used to do far more than predict the existence of cycles. For instance, they can be used to study degeneration of algebraic varieties in one or more variables via limiting mixed Hodge structures. A more representation-theoretic approach to the algebraic structure of periods is obtained by considering the (linear-algebraic) symmetry groups of Hodge structures. These Mumford-Tate groups and their classification [14] lead (particularly in higher weight) to strong constraints

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on degeneration types [15] and to the prediction of exotic families of algebraic (e.g. Calabi-Yau) varieties [16]. More importantly, the study of related homogeneous classifying spaces for Hodge structures turns out to have significant applications to the infinite-dimensional representation theory of real Lie groups.

In what follows, I will elaborate on the three major themes of cycles, degenerations, and classifying spaces from the standpoint of Hodge theory: first as they have appeared in my research to date; and then briefly as to how I plan to develop them in the future.

Summary of Research

Background on invariants of algebraic cycles. Algebraic cycles on a variety are simply formal sums of irreducible subvarieties of some fixed codimension; they are considered modulo rational equivalence (divisors of rational functions) in the cycle groups (known as Chow groups). Their study began historically with divisors (codimension-1 cycles) on algebraic curves, and the famous Bezout, Riemann-Roch and Abel-Jacobi theorems. The last of these says that the degree-0 divisors (modulo divisors of functions) are parametrized by an abelian variety, the Jacobian. When divisor and curve vary together in an algebraic family, one gets certain holomorphic sections of the corresponding family of Jacobians, termed *normal functions* by Poincaré [17]. These were subsequently used by Lefschetz to prove his $(1, 1)$ theorem [18] on fundamental classes of divisors on smooth projective varieties, which the Hodge Conjecture attempts to extend to cycles of codimension >1.

In the late 60's, Griffiths [19] introduced Abel-Jacobi (AJ) invariants with values in "intermediate Jacobians" for such cycles, together with an accompanying generalization of normal functions. The intuition built up for higher-codimension cycles by analogy with divisors was quickly demolished as one learned more about these AJ maps. Their target was no longer algebraic; they failed to be surjective, meaning that Lefschetz's proof would not generalize (to prove Hodge); and they failed spectacularly (by a theorem of Mumford [20]) to be injective, meaning that *they could not detect a huge swath of cycle classes.*

New Hodge-theoretic invariants. To remedy this situation and detect the nontrivial cycles in $\ker(AJ)$, consider their field of definition: cycles defined *a priori* over $\mathbb{C}$ have, in fact, a "minimal" field of definition finitely generated over $\bar{\mathbb{Q}}$. Exchanging this field extension for additional geometry by taking the "$\bar{\mathbb{Q}}$-spread", one can then compute certain graded pieces of AJ invariants of the spread-out cycle. Useful in this connection is Arapura's result that the Leray filtration is motivic [21].

Drawing on (and bridging a significant technical gap between) work of Griffiths-Green [22] and Lewis [23], in [24], [25], and [26] I gave explicit geometric and Hodge-theoretic descriptions of *higher* Abel-Jacobi maps defined in this way on the kernel of AJ, or more precisely, on the graded pieces of

Lewis's candidate Bloch-Beilinson filtration (BBF). These maps are related nontrivially to differential characters and transcendence theory in [24], iterated integrals in [26], and a *new* candidate BBF using kernels of *higher normal functions* in [27]. Moreover, as a simple consequence of my result on exterior products of cycles [25], one gets a class of 0-dimensional cycles on products of smooth curves of positive genera not previously known to be rationally inequivalent to zero, with nonzero image under a higher AJ map (improving a result of [28]).

Regulators on higher Chow complexes. The generalized cycle groups alluded to above in connection with [1] are called the *algebraic K-groups* or *higher Chow groups* of a variety X. Their elements may be represented by "relative" algebraic cycles on $X \times (\mathbb{G}_a, \{0,1\})^n$, or by homology groups of the *higher Chow complex* [29]. These groups are equipped with cycle-class maps into (real resp. integral) Deligne cohomology called (real resp. integral) regulators; the integral regulators, which carry more information, are also known as Abel-Jacobi (AJ) maps. The real regulators are the main ingredient in Beilinson's conjectures, and a general formula for these had been derived by Goncharov [30]. His formula was not the end of the story, however, since the AJ map had potential applications to polylogarithms, height pairings, and detecting torsion cycles; and (because it varies holomorphically) also to differential equations and physics.

In [31] and [32] I computed the *rational AJ* map directly for the relative algebraic cycles, and used this (together with techniques from commutative algebra and Hodge theory) to prove its *vanishing* on all higher cycles arising from Milnor K-theory on *sufficiently general* complete intersection varieties. (The vanishing result is related to Nori's work on Hodge-theoretic connectivity [33]; related results for families of abelian varieties of Hodge type follow from recent work with Keast [34].) This rational AJ computation was then developed into a morphism of complexes called the *KLM map* with Lewis and Müller-Stach in [35], based on a difficult moving lemma [27]. Our work (and my thesis) was cited by Levine in the K-theory handbook [36]. One basic application [38] is to construct explicit higher Chow cycles with Hodge realization *exactly* the extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(n)$ with period $Li_n(\zeta)$ (for any n and any root of unity ζ); more will be described below.

Subsequently the KLM map has been extended in various ways. Using geometric measure theory, Lewis and I reworked it into a map of double complexes computing AJ on motivic cohomology of many singular algebraic varieties [27]; this was used crucially in my work with Doran [9] and Griffiths-Green [39] described below. A version for simplicial complexes was given in [40] and employed to give (with my undergraduate student P. Lopatto) a simple geometric proof of the Kummer-Spence functional equation for the trilogarithm, and to construct a rational regulator directly on the complex computing homology of the general linear group. And at last, forthcoming work of my student Muxi Li will yield an integral version of the KLM map, putting O. Petras's computations of integral generators for K_3^{ind} of number fields [37], as well as the many expected consequences of [38] for torsion in integral motivic cohomology, on solid ground.

Just as for algebraic cycles, one needs higher regulators to capture the information missed by AJ, and these were introduced and used to detect new higher Chow cycles in [27], improving in particular a result of Collino and Fakhruddin [41] about indecomposable higher cycles on Jacobians of curves.

Families of algebraic varieties. A C^∞ form on a smooth complex projective variety (or more generally a Kähler manifold) decomposes into a sum of (p, q) components according to the number of dz's and $d\bar{z}$'s. The corresponding *Hodge decomposition* of the complex cohomology groups, records complex structure moduli of the variety, generalizing Riemann's period matrices of algebraic curves. Going further, the variation of Hodge structure (VHS) associated to a family of varieties (with smooth generic fiber), gives a powerful tool for studying their local and global behavior. For instance, under what circumstances can one have a non-isotrivial family of varieties of a certain type with *no singular fibers*? For elliptic curves Kodaira [42] showed that this is impossible; whereas Atiyah [43] gave a positive example in the case of $K3$ surfaces.

Griffiths, Green and I considered this problem for Calabi-Yau (CY) threefolds in [44], motivated by the role of CY families in string theory and the important work [45]. Using the Grothendieck-Riemann-Roch theorem and curvature computations for canonically extended Hodge bundles, we proved that (over a curve) there are no such families if the Euler characteristic of the CY's in question is at least -24, and derived constraints on the numbers and types of singular fibers. Furthermore, fundamental work of Schmid [46] and Steenbrink [47] had showed that one could associate a limit mixed Hodge structure (LMHS) to the singular fibers in such a family. For CY 3-folds with $h^{2,1} = 1$, we classified the possible $\mathbb{Z}$-LMHS in [48] (leading to a Torelli theorem in [44]) and computed these in some examples using data from physics papers (e.g. [49]). Subsequent work with da Silva and Pearlstein [50] made use of results of Iritani [51] to determine LMHS for the hypergeometric CY families studied by Doran and Morgan [45].

Associated to a family of algebraic varieties is a (non-algebraic) family of intermediate Jacobians, a holomorphic horizontal section of which gives a normal function. (In particular, a family of homologically trivial algebraic cycles on the family of varieties gives rise to such a section.) By a recent result of Griffiths and Green [52], the behavior of certain normal functions in the neighborhood of degeneracies of a family has a tight relation to the Hodge conjecture. This has inspired a blizzard of important works [53]-[60] on singularities, limits, and zero-loci of *admissible normal functions* (ANF), surveyed in [61] with G. Pearlstein. A key result has been my construction with Griffiths and Green of a "Néron model" for graphing ANFs over a curve [39] (generalizing fundamental work of Clemens [62]), and our corresponding interpretation (for families of algebraic cycles) of limits of AJ maps [39]. Our Néron model was subsequently generalized in various ways [63], [64], [65] to a base of arbitrary dimension; while the limit AJ result was extended to families of higher Chow cycles (which give rise to *higher* normal functions) and non-semistable degenerations in [13].

Algebraic K-theory, modular forms, and physics. The genus-zero Gromov-Witten (GW) numbers $N_d(\in \mathbb{Q})$ are enumerative invariants of the symplectic structure on (for example) a complex variety, which arose as path integrals computing topological string amplitudes in physics. Local GW invariants of open CY 3-folds were introduced in [66] to understand the contribution made to the N_d by an embedded toric Fano surface; it was found experimentally [67] that their exponential growth rate (as the degree d runs to ∞) was related to special values of L-functions. Meanwhile, in his study of (logarithmic) Mahler measure for families of Laurent polynomials defining elliptic curves, Rodriguez-Villegas [68] (amplified by [69],[70]) had noticed that for certain "tempered" families, precomposing a lift of the Mahler measure with an automorphic function produced an integral of an Eisenstein series.

In the lengthy paper [9] with C. Doran, I constructed algebraic K_n-classes called "toric symbols" in the total space of certain 1-parameter families of CY hypersurfaces in toric Fano n-folds; under this heading fall CY 3-folds, $K3$ surfaces, and elliptic curves. The families arise from Laurent polynomials satisfying explicit combinatorial and arithmetic conditions, and existence of the cycle class implies the rationality of periods of extended Haar measure in [71]. In the case where the family was classically modular, we explicitly related the cycles to Beilinson's Eisenstein symbol; computing the KLM map on both kinds of cycles (which improves on work of [72] in the latter case), this led to a general and computationally effective explanation of the examples of [68]-[70].

Furthermore, reinterpreting a conjecture from the local mirror symmetry literature [66],[73] in the elliptic curve case, Doran and I were able to explain (and compute) the observed GW asymptotics by applying the AJ map (from [27]) to the restriction of the toric symbol to a singular fiber. (For this we received a citation from S.-T. Yau in his Scholarpedia article on Calabi-Yau manifolds [74].) Similar techniques were applied recently in [13] to check a prediction of a conjecture relating spectral theory and enumerative geometry [75].

The local mirror symmetry conjecture was later proved with Bloch and Vanhove in [12] by combining Iritani's work [51] and a delicate degeneration argument proposed in [76], as part of work applying the KLM formula to evaluate Feynman integrals: as a sum of elliptic trilogarithms in [11] (for the equal-mass three-banana graph); and in terms of the local Gromov-Witten prepotential for the del Pezzo surface of degree 6 in [12] (for the unequal-mass sunset graph). These papers have so far received around 60 citations, largely in the physics literature! I should also mention here that in his thesis [77], my student Yu Yang showed how to use similar techniques to tackle the Feynman integral for the wheel with three spokes and related graphs.

Another case where both toric and modular constructions work well is that of the indecomposable K_1 of a $K3$ surface. The most natural source for families of such classes should be cycles supported on semistable singular fibers of elliptically fibered $K3$'s. In [79], [80] we carry out regulator computations for two different examples, opening up intriguing connections to Tauberian theory and to higher Green's functions and the thesis of A. Mellit [7]. Forthcoming work of my student Tokio Sasaki will give a constructive proof of the Hodge D-Conjecture for K_1 cycles on certain families of surfaces, which also takes advantage of toric geometry.

Thus far, the story has focused primarily on cycles and varieties, with Hodge theory playing its traditional supporting role of providing invariants. In what follows, we change gears and emphasize the complex geometry, arithmetic, and representation theory associated to the classifying spaces for Hodge structures themselves.

Background on period maps. A *period domain* D is a classifying space for polarized Hodge structures with given Hodge numbers, and is always a homogeneous space for the action of (the real points of) a reductive algebraic group G. For a family of algebraic varieties, the period matrices (corresponding to cohomology in a given degree) yield a *period map* from the coefficient parameter space S to a quotient of $D = G(\mathbb{R})/H$ by an arithmetic group $\Gamma \leq G(\mathbb{Q})$. The local lifts of its image (to D) satisfy the infinitesimal period relation (IPR), due to Griffiths, which can be thought of as a differential ideal in $\Omega^*(D)$. Those domains (or subdomains) on which this IPR is (or pulls back to) zero, are called "classical," and include the Siegel upper-half-spaces classifying level-1 Hodge structures.

Now, each D has an algebraic structure arising from the Hodge (period integral) parameters, which is quite different from what one needs to algebraize its quotient and the period map φ. In the classical cases, this is remedied by the construction of automorphic forms for Γ, which essentially give algebro-geometric coefficients (i.e., functions embedding $\Gamma\backslash D$ in projective space) as functions of period integrals. Non-classical period domains, in contrast, are frequently non-Hermitian-symmetric, and may have no nontrivial holomorphic automorphic forms with respect to a given Γ, meaning that only the image $\varphi(S)$ (and not all of $\Gamma\backslash D$) is algebraic. Consequently, many arithmetic aspects of VHS and period maps (related to conjectures of Hodge, Bloch-Beilinson, André-Oort, Grothendieck) remain mysterious. See [78] for an expository account.

Mumford-Tate groups and domains. Mumford-Tate groups are the natural ($\mathbb{Q}$-algebraic) symmetry groups of HS, in the sense of stabilizing the Hodge substructures of all tensor powers of a HS (or MHS) H and its dual. They were introduced by Mumford in [81] to give a Hodge-theoretic characterization of certain families of abelian varieties studied by Kuga and Shimura, and in that context turned out to have spectacular applications to the Hodge conjecture and Shimura varieties. In higher weight they have been less explored, and in a recent monograph [14] with Griffiths and Green we have attempted to address this. (In particular, this allows for many more types of group G than in the period domain case, bringing representation theory to the fore.) A first key result of our work states that a $\mathbb{Q}$-simple algebraic group G is a MT group of a polarized HS if and only if $G(\mathbb{R})$ has a compact maximal torus T; these are precisely the cases where G has nontrivial discrete series representations.

Contained within a general period domain are natural subvarieties defined over algebraic extensions $K/\mathbb{Q}$, the Mumford-Tate (M-T) domains, which are the orbits of points $p \in D$ under the action of the M-T group of the HS V_p. In [14] we give a general algorithm for classifying these subdomains, which blends representation theory of Lie algebras, Hodge theory, and the Galois theory of CM fields. The Galois-theoretic connection arises from the fact that all MT domains contain polarized Hodge structures with CM (complex multiplication),[2] which therefore become natural base points of MT domains. (Furthermore, we prove a lemma computing dimensions, rational points, and Lie algebras of MT groups of arbitrary CM Hodge structures.) Applying the algorithm to HS of weight 3 with Hodge numbers $(1, 1, 1, 1)$ yields eight distinct classes of MT domains, one of which is defined by a Hodge tensor of degree four and exhibits fascinating arithmetic behavior.

More broadly, reformulating the theory of period maps arising from VHS in terms of MT domains as in [14] clarifies many issues, from the nature of global monodromy to the meaning of the conjectures mentioned above. It gives insight into recent constructions of geometric variations with exceptional monodromy [83, 84, 85], and was used to give a direct geometric proof of G_2 monodromy for a family of surfaces from [85] by my student Genival da Silva Jr. In conjunction with a result of Kostant [87], it leads to a (very short) classification of Hermitian VHS which can admit nontorsion normal functions (and higher normal functions) under a finite pullback [34]. Keast and I use this to obtain vanishing results for AJ-images of cycles on families of (e.g. Weil and quaternionic) abelian varieties, in analogy to the theorem of Green and Voisin for projective hypersurfaces [88]; we also prove a new countable-generation result for Griffiths groups of Weil abelian 6-folds.

My work with Griffiths and Green has also had some influence on those working in moduli theory, differential geometry, and transcendence theory (e.g. [16],[89],[90],[91]). In particular, Robles's paper [89] showed that the horizontal subvarieties in a MT domain $D = G(\mathbb{R})/H$ (H compact) are infinitesimally "in the linear span" of the horizontal Schubert varieties X in its compact dual $\check{D} := G(\mathbb{C})/P$ (P parabolic). So it was natural to ask: when are they (compact duals of) MT subdomains? Such an X would of course be smooth, and also Hermitian symmetric; but in general, smooth Schubert varieties need not be homogeneous. In [92], Robles and I make use of a recent result of Hong and

[2] On the other hand, my student Ryan Keast observed in [82] that many CM Hodge structures are never contained in positive-dimensional horizontal MT subdomains or rational nilpotent orbits.

Mok [93] to show that any *smooth horizontal* Schubert variety in $\check{D}$ is in fact a homogeneously embedded Hermitian symmetric domain.

Automorphic cohomology and boundary components. When holomorphic automorphic forms are absent for a given M-T domain D, one can consider instead its *automorphic cohomology* [94] — that is, the higher Dolbeault (coherent) cohomology groups of $\Gamma\backslash D$, which can typically be computed in terms of discrete series and limits thereof. These groups play a central role in some of Carayol's work ([95], [96], [97]), which explored (for $G = \mathrm{SU}(2,1)$) nonclassical domains as a possible source of arithmetic structure on automorphic representations (of $G(\mathbb{R})$) with archimedean component a totally degenerate limit of discrete series (TDLDS). The protagonists in this story are the subspaces $H^q_\circ$ of "cuspidal" classes in automorphic cohomology groups $H^q(\Gamma\backslash D, \mathcal{O}(L_\mu))$, where D has the special form $G(\mathbb{R})/T$ (T compact Cartan). A key role is played by *Penrose transforms*, which (in his case) map spaces of Picard modular forms to non-classical automorphic H^1 spaces.

My joint monographs [98, 99] with Griffiths and Green contain new results on TDLDS, coherent cohomology, and $\bar{\mathbb{Q}}$-algebraicity. We frame Carayol's work in a more invariant and less computational setting, extending his Penrose transforms to all groups of Hermitian type (and degrees of automorphic cohomology) and his "isomorphy result" [96] to $G = Sp_4$. We also prove an algebraicity result for the "values" of automorphic cohomology classes at CM points in a correspondence space, generalizing a classical result of Shimura. In [100], I established an Sp_4 analogue of [95] for cup products $H^1 \otimes H^1 \to H^2$, which allows one to describe "all of the TDLDS-related automorphic cohomology" for $G = Sp_4$ in terms of spaces of Siegel modular forms.

My recent work with Pearlstein [15] is motivated by Carayol in a different way. In the classical cases, one has a toroidal compactification of $\Gamma\backslash D$ [101] by certain boundary components, which may be viewed as parametrizing limiting mixed Hodge structures [102]. The very difficult "extension" of this to the non-classical setting (a dream of Griffiths in [103]) has only recently been worked out by Kato and Usui [104] for period domains, resulting in a log-analytic partial compactification of the quotient $\Gamma\backslash D$. In [15], we study *boundary components* $\bar{B}(\sigma)$ (associated to nilpotent cones $\sigma := \mathbb{R}_{>0}\langle N_1, \ldots, N_r\rangle \subset Lie(G)$) in the more general MT setting, computing the MT group of the generic LMHS they parametrize, and using this to present them as double-coset spaces. (The specific implications for limiting periods of G_2-VHS are worked out in [50], and applied to the geometric example of Dettweiler-Reiter.) Now, Carayol's amazing realization in [95] was that a *non-algebraic* $\Gamma\backslash D$ could have *algebraic* boundary components, and that one could try to algebraize its automorphic cohomology via "Fourier coefficients" in the boundary cohomology! Moreover, one can hope to generalize his example whenever $\Gamma\backslash D$ has a CM abelian variety boundary component, and in [15] we are able to determine precisely when this happens.

This work was pushed further with Pearlstein in [105], whose subject is the topological boundary of D in its compact dual $\check{D}$. The points of $\check{D}$ carry a mixed-Hodge-theoretic interpretation, which we use to study the decomposition of ∂D into $G(\mathbb{R})$-orbits called *naive boundary strata*. A natural "Hodge-theoretic accessibility" question is: which strata can a period map limit into? By studying the "naive limit maps" $\mathscr{F}^{\sigma}_{lim} : B(\sigma) \to \partial D$, we obtain a complete answer in root-theoretic terms. Subsequently Robles and I carried out an analysis in the case where $\check{D}$ is an *adjoint variety* (with minimal homogeneous enbedding in $\mathbb{P}\mathfrak{g}$), which include the "interesting" nonclassical MT domains for weight-two HS. All the strata of ∂D are "accessible", but the $\bar{B}(\sigma)$ associated to the codimension-one strata have a particularly beautiful description: they are the intermediate Jacobian bundles attached to the maximal Hermitian VHSs of CY type from [16]. We also examine the differential properties of period maps into adjoint domains, especially the characteristic variety of the associated VHS, and find a striking relationship between the latter and the homogeneous Legendrian varieties of [106].

In a recent preprint with both Pearlstein and Robles [107], we introduce a relation on real conjugacy classes of SL(2)-orbits in a MT domain D which is compatible with natural partial orders on the sets of nilpotent orbits in the corresponding Lie algebra and boundary orbits in the compact dual. A generalization of the SL(2)-orbit theorem of [108] to such domains leads to an algorithm for computing this relation, which is worked out in several examples and special cases including period domains, Hermitian symmetric domains, and complete flag domains, and used to define a "secondary poset" of equivalence classes of multivariable nilpotent orbits on D. The overall effect is to demystify the constraints on several-variable degenerations of HS, and to provide a coarse classification of nilpotent cones (which until now had seemed a totally "wild" problem).

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