NOTES ON THE REPRESENTATION THEORY OF $SL_2(\mathbb{R})$

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ABSTRACT. Introductory notes with a view toward recent work on automorphic cohomology, covering: (1) finite-dimensional irreducible representations; (2) parabolic induction and principal series representations; (3) Eisenstein series; (4) modular forms; (5) cuspidal automorphic forms; and (6) automorphic cohomology. The two appendices treat supplementary topics: (I) $L^2(SL_2(\mathbb{R}))$ and discrete series representations; and (II) Poincaré series.

INTRODUCTION

What follows is an expanded writeup of my talks at the NSF/CBMS workshop on "Hodge Theory, Complex Geometry, and Representation Theory" (Fort Worth, TX, June 18-22, 2012). Two major themes of this meeting were:

(a) the use of representation theory to study the complex geometry and automorphic cohomology of arithmetic quotients $\Gamma\backslash D = \Gamma\backslash G(\mathbb{R})/H$ of generalized period domains (arising from Hodge theory); and

(b) the use of arithmetic geometry of such quotients $\Gamma\backslash D$ – particularly non-algebraic ones – to attack the Langlands program for automorphic representations.

From either perspective, a central role is played by the decomposition of the space ${}^{\circ}\mathcal{A}(G,\Gamma)$ of cuspidal automorphic forms into irreducible submodules, and the computation of certain Lie algebra cohomology groups of these submodules. This connection was described at length in the lectures of P. Griffiths at the workshop, and is exploited in [Ca, GGK, Ke] for Sp_4, $SU(2,1)$, and groups of higher rank.

Though the portion of automorphic representation theory involved is relatively small, and (so far) limited to the archimedean setting, it is challenging

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for a researcher starting from the Hodge-theoretic side to build the required intuition, particularly for the concrete computations underlying the abstract classification results. These notes are intended to help with that process, by telling the whole story in the simplest nontrivial case (i.e., SL_2), starting with finite-dimensional representations (§1) and modular forms (§4) and showing how infinite-dimensional (§2) and automorphic (§5) representations grow out of them. (After §2, the discussion is limited to the discrete and principal spherical series.) We explain how Maass forms (§5) and Eisenstein series (§3) enter, and continue with a discussion (§6) of $\mathfrak{n}$-cohomology (which, while simple, may be particularly useful). This is the material which was treated in my CBMS lectures. The two appendices (§§7-8) explain an alternative approach to discrete series representations via $L^2(G)$, and how this story ultimately gets related back to that in §5.

We have chosen to concentrate on developing the material in what we hope is an intuitive and approachable manner, rather than on technical precision and completeness (for which the reader may may consult the many excellent references). Nontrivial but straightforward exercises are included in every section, to engage the reader in some of the computations. These notes are based in part on a set which was prepared by the author as a warm up to the joint writing of [GGK] with Griffiths and Green. That set, which is in some ways more extensive and contains solutions to many of the exercises, is available upon request.

Remark. As suggested by the title, and in the interest of keeping the abstraction in these notes to a minimum, we have suppressed any discussion of automorphic forms on the adele group $SL_2(\mathbb{A})$. The reader should be aware that what are here called “automorphic representations” are only the archimedean components thereof.

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1. Irreducible representations of sl_2

Definition 1.1. A *Lie algebra* (over a field $\mathbb{F}$) is a vector space $\mathfrak{g}$ (over $\mathbb{F}$) with an antisymmetric bilinear form $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the Jacobi identity

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0.$$

Example 1.2. (a) For any vector space V, take $\mathfrak{g} := End(V)$ and $[x, y] := xy - yx$.

(b) Inside $End(\mathbb{R}^2)$, we have

$$sl_{2,\mathbb{R}} := \langle Y, N_+, N_- \rangle := \left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\rangle$$

with $[Y, N_\pm] = \pm 2N_\pm$ and $[N_+, N_-] = Y$.

Definition 1.3. A *representation* (V, ρ) of $\mathfrak{g}$ is an $\mathbb{F}$-linear map $\mathfrak{g} \xrightarrow{\rho} End(V)$ such that

$$\rho([x, y]) = [\rho(x), \rho(y)].$$

(i.e., a morphism of $\mathbb{F}$-Lie algebras).

Exercise 1.4. Check that the Jacobi identity implies that

$$\mathrm{ad} : \mathfrak{g} \to End(\mathfrak{g})$$

$$x \mapsto [x,\ \cdot\ ]$$

is a Lie algebra representation.

Example 1.5. Given a finite-dimensional matrix Lie group G (over $\mathbb{R}$ or $\mathbb{C}$) with identity id_G, consider the tangent space $\mathfrak{g} = Lie(G) := T_{\mathrm{id}_G}G$. Writing $\Psi_g \in Aut(G)$ for conjugation by g and $\mathrm{Ad}(g) := d\Psi_g \in Aut(\mathfrak{g})$, one defines $\mathrm{ad} : \mathfrak{g} \to End(\mathfrak{g})$ as the differential of $\mathrm{Ad} : G \to Aut(\mathfrak{g})$. From $e^{\epsilon x}e^{\epsilon y}e^{-\epsilon x}e^{-\epsilon y} = \mathrm{id}_G + \epsilon^2(xy - yx)$+higher-order terms, it follows that $[x, y] := (\mathrm{ad}x)(y) = xy - yx$, from which the Jacobi identity is immediate – making $\mathfrak{g}$ a Lie algebra. If G is connected and simply-connected, and V is finite-dimensional, then there is a one-to-one correspondence

$$\boxed{\begin{array}{c}\text{representations}\\ \rho:\mathfrak{g}\to End(V)\end{array}} \underset{(d\pi)_{\mathrm{id}_G}}{\overset{\exp(x)\mapsto\exp(\rho(x))}{\rightleftarrows}} \boxed{\begin{array}{c}\text{representations}\\ (C^{\infty}\text{ homomorphisms})\\ \pi:G\to GL(V)\end{array}} \tag{1.1}$$

Now the left-to-right part of (1.1) does not apply directly to $SL_2(\mathbb{R})$, which (unlike, say, $SL_2(\mathbb{C})$) is not simply connected. However, there is an easy "fix": any complex representation of $sl_{2,\mathbb{R}}$ may be bootstrapped up to a representation of $sl_{2,\mathbb{C}} = sl_{2,\mathbb{R}} \oplus i \cdot sl_{2,\mathbb{R}}$. Applying (1.1) gives a representation of $SL_2(\mathbb{C})$ which we may restrict to $SL_2(\mathbb{R})$, recovering a 1-to-1 correspondence between finite-dimensional representations of $sl_{2,\mathbb{R}}$ and $SL_2(\mathbb{R})$. Call this (1.1)*.

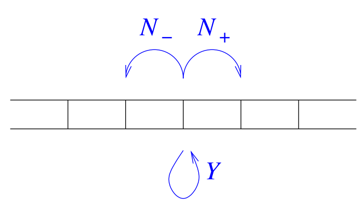
Let $\rho : \mathfrak{g} = sl_{2,\mathbb{R}} \to End(V)$ be a Lie algebra representation. Diagonalizing $\rho(Y)$ produces a decomposition

$$V = \bigoplus_{j\in J} V_j := \bigoplus_{j\in J} E_j(\rho(Y))$$

into eigenspaces called "weight spaces"; let $n := \max(J)$ be the highest weight occurring.[1] For $v \in V_j$, we have

$$Y(N_+v) = N_+(\underbrace{Yv}_{jv}) + \underbrace{[Y, N_+]}_{2N_+}v = (j+2)N_+v$$

and similarly $Y(N_-v) = (j-2)N_-v$, which yields a picture



Therefore, for $v_0 \in V_n$, we must have $N_+v_0 = 0$, whereupon

$$N_+N_-v_0 = [N_+, N_-]v_0 + 0 = Yv_0 = nv_0,$$

$$N_+N_-^2v_0 = \underbrace{[N_+, N_-]}_{Y}N_-v_0 + N_-\underbrace{N_+N_-v_0}_{nv} = \underbrace{(n + (n-2))}_{2(n-1)}N_-v_0,$$

[1] *a priori*, the weights (i.e. eigenvalues) are ordered by real part; below it will become clear that they are integers, at least in the finite-dimensional case.

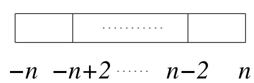
$$\vdots$$

lead to the general formula

$$N_+ N_-^k v_0 = k(n-k+1) N_-^{k-1} v_0. \tag{1.2}$$

Now suppose (V, ρ) is irreducible. Then the images of our highest weight vector v_0 by powers of N_+ and N_- must span V. By virtue of (1.2), it is clear that $\{N_-^k v_0\}_{k \geq 0}$ spans V, and thus that $\dim V_j = 1$ $(\forall j \in J)$.

Further assuming $\dim V < \infty$, we must have $N_-^m v = 0$ but $N_-^{m-1} v \neq 0$ for some $m \in \mathbb{N}$. From (1.2) it follows that $m(n-m+1) = 0$, so $n \geq 0$ and $m = n+1$. This implies $-n$ is the lowest weight and $J = \{-n, -n+2, \ldots, n\}$:

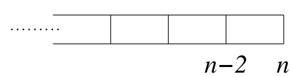


This is the picture for all finite-dimensional irreducible representations of $sl_{2,\mathbb{R}}$ (and $su(2)$). Under (1.1)*, the corresponding irreducible representation of $G = SL_2(\mathbb{R})$ is the n^{th} symmetric power

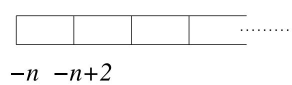
$$W_n := Sym^n(\text{St}) \tag{1.3}$$

of the two-dimensional standard representation.

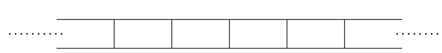
If we allow $\dim V = \infty$, then we may have $n < 0$, and $N_-^k v \neq 0$ for all $k \geq 0$:



In case $-n \in \mathbb{N}$, this picture and its reflection



are related to the discrete series representations below. The bi-infinite ladder



which has no highest or lowest weight, will correspond to principal series representations (in case the weights are even or odd integral). However, for infinite-dimensional representations, (1.1)* does not apply. Our next task is to construct infinite-dimensional sl_2-representations with essentially these weight pictures,[2] in such a way that it is clear that they come from (irreducible) representations of G.

Good general references on finite-dimensional representation theory include [Kn1] and [FH].

2. Parabolic induction

For the remainder of these notes, G will denote $SL_2(\mathbb{R})$. By $C^\infty_{\mathbb{C}}(G)$, we shall mean the smooth complex-valued functions on G. Unless otherwise mentioned, we consider G to act on $\phi \in C^\infty_{\mathbb{C}}(G)$ by right translation:

$$(g_0.\phi)\,(g) := \phi(gg_0). \tag{2.1}$$

The goal of this section is to explicitly construct infinite-dimensional representations of G inside $C^\infty_{\mathbb{C}}(G)$.

By applying Gram-Schmidt to its columns, any given $g \in SL_2(\mathbb{R})$ may be written uniquely

$$g = \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} =: p_\tau k_\theta =: g_{\tau,\theta}$$

(with $\tau := x + iy$) as the product of an upper triangular matrix with positive diagonal entries times an orthogonal matrix. This leads to the *Iwasawa decomposition*

$$G = \underbrace{P}_{\substack{\text{parabolic} \\ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}}} \cdot \underbrace{K}_{\text{compact}} = \underbrace{N}_{\text{unipotent}} \cdot \underbrace{A \cdot M}_{\text{Levi}} \cdot K,$$

where $M = \{\pm\mathbb{I}\} \cong \mathbb{Z}/2\mathbb{Z} = P \cap K$, by writing

$$\pm p_\tau = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}.$$

A function $\phi \in C^\infty_{\mathbb{C}}(G)$ is called *right K-finite* if the right K-translates $\{k.\phi \mid k \in K\}$ span a finite-dimensional vector space.

[2]In fact, Y will effectively be replaced by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and the weights will be in $i(2\mathbb{Z})$ or $i(2\mathbb{Z}+1)$, but otherwise the weight diagrams will be these.

Let

$$\sigma : AM \cong \mathbb{R}^* \to \mathbb{C}^*$$

be a (1-dimensional) complex representation of the Levi,

$$\tilde{\sigma} : P \to \mathbb{C}^*$$

$$\pm p_\tau \mapsto \chi^\epsilon(\pm 1) \cdot y^{\lambda/2}$$

denote its pullback (trivial on N, with $\lambda \in \mathbb{C}$), and

$$\begin{aligned} \Delta_P := \mathrm{Ad}(\cdot)|_{Lie(N)} : \quad & P \to \mathbb{C}^* \\ & \pm p_\tau \mapsto y \end{aligned}$$

the *modular character.* (Here χ is the nontrivial character on M, and $\epsilon = 0$ or 1.)

Definition 2.1. The *principal series representation* induced from σ is

$$I_{\pm,\lambda} := I_P(\sigma) := Ind_P^G(\underbrace{\Delta_P^{1/2}\tilde{\sigma}}_{=\hat{\sigma}})$$

$$:= \left\{ \phi \in C^\infty_{\mathbb{C}}(G) \,\middle|\, \begin{array}{c} \phi(pg) = \hat{\sigma}(p)\phi(g), \\ \phi \text{ right } K\text{-finite} \end{array} \right\} \tag{2.2}$$

$$\xrightarrow[\text{restrict}]{\cong} \left\{ \mathsf{f} \in C^\infty_{\mathbb{C}}(K) \,\middle|\, \begin{array}{c} \mathsf{f}(\pm k) = \chi^\epsilon(\pm 1)\mathsf{f}(k), \\ \mathsf{f} \text{ right } K\text{-finite} \end{array} \right\}. \tag{2.3}$$

The "$\pm$" in the subscript is determined by $(-1)^\epsilon$.

Remark 2.2. (i) In fact $I_{\pm,\lambda}$ as just defined is only the underlying *Harish-Chandra* ($(\mathfrak{g}, K)$-)*module* of a genuine representation of G. The latter is obtained by removing the K-finite condition (which says that ϕ must be a finite sum of K-eigenvectors). This larger space is closed under the action (2.1) of G by right translation. The action of $\mathfrak{g}$ resp. K (under which $I_{\pm,\lambda}$ is actually closed) is the differential resp. restriction of this action.

(ii) In the definition, we "twist" by the square root of the modular character in order that σ unitary ($\lambda \in i\mathbb{R}$) imply $I_P(\sigma)$ unitary, and so that λ becomes the so-called "Harish-Chandra parameter".

The reader who has not previously encountered Definition 2.1 may wonder both why we have begun with induced representations and why their definition takes the form (2.2). We can partially address the latter by noting that for finite groups, with (V, σ) a representation of $H \leq G$, the underlying vector space of $Ind_H^G(\sigma) := V \underset{H}{\otimes} \mathbb{C}[G]$ consists of V-valued functions on $H\backslash G$. For topological groups, this naturally generalizes to sections of a vector bundle (of rank $\dim V$), which is what (2.2) really is. More importantly, by *Harish-Chandra's subquotient theorem*, every irreducible representation of $SL_2(\mathbb{R})$ which is *admissible* (i.e. having all $\dim V_j < \infty$) is a subquotient of some principal series representation.

Turning to the computation of $I_P(\sigma)$, a basis for (2.3) is given by

$$\mathsf{f}_n(\theta) := e^{in\theta} \text{ with } \begin{cases} n \text{ even, if } \epsilon = 0 \\ n \text{ odd, if } \epsilon = 1 \end{cases} .$$

Noting that

$$p_{x_0+iy_0,} \cdot g_{x+iy,\theta} = g_{(xy_0+x_0)+iyy_0,\theta},$$

a basis for (2.2) is

$$\phi_n(g_{x+iy,\theta}) := y^{\frac{\lambda+1}{2}} \mathsf{f}_n(\theta).$$

The correspondence between the two bases, or the two spaces more generally (viz., $\phi \longleftrightarrow \mathsf{f}$), is given by multiplication by $y^{\frac{\lambda+1}{2}}$ going one way, and evaluating at $\tau = i$ or dividing by $y^{\frac{\lambda+1}{2}}$ going back. Notice that functions in (2.2) are independent of x.) Using Remark 2.2(i) and $g_{\tau,\theta} k_\phi = g_{\tau,\theta+\phi}$, K acts via

$$k_\phi.\phi_n = e^{in\theta}\phi_n. \tag{2.4}$$

We shall briefly discuss how to obtain a formula for the action of G (extending (2.4)) on the spaces (2.2)-(2.3) without the K-finite condition. Define $\tilde{\tau}$ and $\tilde{\theta}$ by

$$g_{\tau,\theta} \cdot g_{\mu,\phi} =: g_{\tilde{\tau},\tilde{\theta}},$$

so that

$$(g_{\mu,\phi}.\phi)\,(g_{\tau,\theta}) = \phi(g_{\tilde{\tau},\tilde{\theta}})$$

and put

$$g_{\mu,\phi}.\mathsf{f} := \frac{g_{\mu,\phi}.\phi}{y^{\frac{\lambda+1}{2}}}.$$

(Here μ and ϕ are fixed.) In order to make this any more explicit, we need to find $\tilde{\tau}$ and $\tilde{\theta}$. The idea for this is to consider the two projections

$$\begin{array}{ccccc} G & \twoheadrightarrow & G/K & \cong & \mathfrak{H} \\ g_{\tilde{\tau},\tilde{\theta}} & \mapsto & g_{\tilde{\tau},\tilde{\theta}}\left[\begin{pmatrix} i \\ 1 \end{pmatrix}\right] & = & \left[\begin{pmatrix} \tilde{\tau} \\ 1 \end{pmatrix}\right] \end{array}$$

and

$$\begin{array}{ccccccc} G & \twoheadrightarrow & P\backslash G & \cong & \mathbb{P}^1(\mathbb{R}) & \cong & S^1 \\ g_{\tilde{\tau},\tilde{\theta}} & \mapsto & [(0,1)]\,.g_{\tilde{\tau},\tilde{\theta}} & = & \left[\left(-\sin\tilde{\theta},\cos\tilde{\theta}\right)\right] & \mapsto & 2\tilde{\theta} \end{array}.$$

Exercise 2.3. (i) Compute that $\left(g_{\mu,\phi}.\mathsf{f}\right)(\theta) = \frac{\Im(\mu)^{\frac{\lambda+1}{2}}}{|\cos\theta - \mu\sin\theta|^{\lambda+1}}\mathsf{f}(\tilde{\theta})$ by partially carrying out this program.

(ii) Show $\langle \mathsf{f}_1, \mathsf{f}_2\rangle := \int_{S^1}\mathsf{f}_1\overline{\mathsf{f}_2}d\theta$ is G-invariant if $\lambda \in i\mathbb{R}$. Replacing the K-finiteness by the obvious L^2 condition in (2.3) therefore gives a unitary representation of G.

(iii) Prove that $dg := \frac{dx\wedge dy\wedge d\theta}{2\pi y^2}$ is (both left and right) G-invariant.

What we are really after is the action of $\mathfrak{g}$ by infinitesimal right translation: given $X \in \mathfrak{g}$,

$$(\mathcal{L}_X\phi)(g) := \frac{d}{dt}\phi(ge^{tX})\Big|_{t=0}.$$

Together with the action of K, this will give a $(\mathfrak{g},K)$-module structure on (2.2) hence (2.3). Since the eigenvalues of $\mathfrak{k} = \mathbb{R}\left\langle\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}\right\rangle$ are imaginary, it will be convenient to extend $\mathbb{C}$-linearly to the action of

$$\mathfrak{g}_{\mathbb{C}} = \mathbb{C}\langle W, E_+, E_-\rangle := \mathbb{C}\left\langle \begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1\end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1\end{pmatrix}\right\rangle$$

by *defining* Lie derivatives

$$\mathcal{L}_W := -i\mathcal{L}_{\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}}, \quad \mathcal{L}_{E_\pm} := \frac{1}{2}\mathcal{L}_{\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}} \pm \frac{i}{2}\mathcal{L}_{\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}}.$$

Note that $[W, E_\pm] = \pm 2E_\pm$ and $[E_+, E_-] = W$.

To describe our approach to computing $\mathcal{L}_X\phi$ (for $X \in sl_{2,\mathbb{R}}$), begin by writing

$$g_{\tau,\theta}e^{tX} =: g_{\tilde{\tau},\tilde{\theta}} = p_{\tilde{\tau}}\cdot k_{\tilde{\theta}} =$$

$$p_\tau \cdot \left(1 + t\frac{\tilde{y}'(0)}{2y}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) + t\frac{\tilde{x}'(0)}{y}\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) + \cdots\right) \cdot k_\theta \cdot \left(1 + t\tilde{\theta}'(0)\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) + \cdots\right)$$

where $\tilde{\tau} = \tilde{x} + i\tilde{y}$ and $\tilde{\theta}$ are considered as functions of t. Applying $\frac{d}{dt}$ and evaluating at $t = 0$ gives $g_{\tau,\theta}X =$

$$\frac{\tilde{y}'(0)}{2y}p_\tau \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} k_\theta + \frac{\tilde{x}'(0)}{y}p_\tau \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} k_\theta + \tilde{\theta}'(0)p_\tau k_\theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Taking the bottom two matrix entries alone gives

$$\left(-\frac{\sin\theta}{\sqrt{y}}, \frac{\cos\theta}{\sqrt{y}}\right) \cdot X = \tilde{y}'(0)\left(\frac{\sin\theta}{2y^{3/2}}, -\frac{\cos\theta}{2y^{3/2}}\right) + \tilde{\theta}'(0)\left(-\frac{\cos\theta}{\sqrt{y}}, -\frac{\sin\theta}{\sqrt{y}}\right),$$

which enables us to solve for $\tilde{y}'(0)$ and $\tilde{\theta}'(0)$ once we fix X. The result is then plugged into

$$(\mathcal{L}_X\phi)(g_{\tau,\theta}) = \frac{\partial\phi}{\partial x} \cdot \tilde{x}'(0) + \frac{\partial\phi}{\partial y} \cdot \tilde{y}'(0) + \frac{\partial\phi}{\partial\theta} \cdot \tilde{\theta}'(0),$$

where for the functions we consider $\frac{\partial f}{\partial x} = 0$.

Applying the $\mathbb{C}$-linear extension mentioned previously, we obtain for $\phi \in$(2.2)

$$\mathcal{L}_W\phi = -i\frac{\partial\phi}{\partial\theta}\ , \quad \mathcal{L}_{E_\pm}\phi = e^{\pm 2i\theta}\left\{y\frac{\partial\phi}{\partial y} \mp \frac{i}{2}\frac{\partial\phi}{\partial\theta}\right\}, \tag{2.5}$$

which induces operations on (2.3); in particular, we get

$$\mathcal{L}_W\mathsf{f} = n\mathsf{f}_n\ , \quad \mathcal{L}_{E_\pm}\mathsf{f}_n = \frac{\lambda \pm n + 1}{2}\mathsf{f}_{n\pm 2}. \tag{2.6}$$

(Details are left to the reader.) The $\mathcal{L}_{E_\pm}$ are commonly referred to as "raising and lowering operators".

Remark 2.4. From the $(\mathfrak{g}, K)$-module structure described by (2.6), we can immediately recover the inducing parameter λ. More generally, given an arbitrary irreducible representation of G inside $C^\infty(G)$, it is less immediately obvious how to determine which $I_{\pm,\lambda}$ it "belongs" to. We need an operator which commutes with all $\mathcal{L}_X$, i.e. in $Z(\mathfrak{g})$ – the center of the universal enveloping algebra – which may be regarded as comprising the left-invariant differential operators on G.

Exercise 2.5. Define the *Casimir operator*

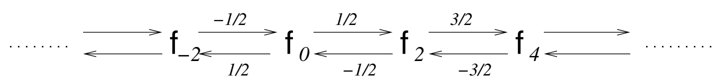
$$\Omega := -\frac{1}{4}\left(\mathcal{L}_W\right)^2 + \frac{1}{2}\mathcal{L}_W - \mathcal{L}_{E_+}\mathcal{L}_{E_-}.$$

(a) Check, using $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$, that $[\Omega, \mathcal{L}_{E_\pm}] = 0 = [\Omega, \mathcal{L}_W]$.

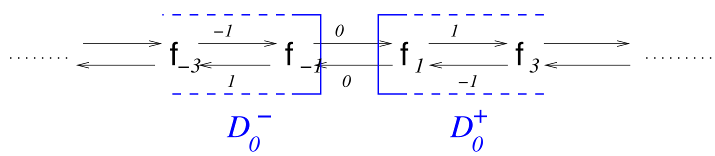
(b) Show that for $\phi = y^{\frac{\lambda+1}{2}}\mathsf{f}$, we have $\Omega\phi = \frac{1}{4}(1-\lambda^2)\phi$. Hence, applied to any vector in our irreducible representation, Ω recovers λ up to sign.

Here are two concrete illustrations of induced representations and their subquotients.

Example 2.6. ($\lambda = 0$) From (2.6), we may read off the action of $E_\pm$ on $I_{+,0} = I_P(1)$

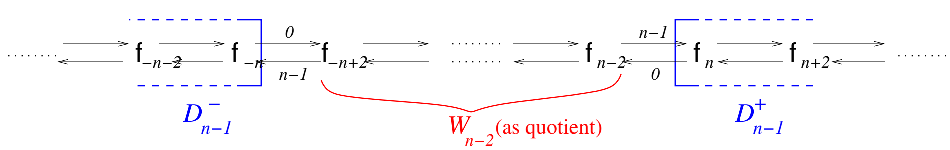


and $I_{-,0} = I_P(\chi)$



Clearly $I_{+,0}$ is irreducible, with Casimir eigenvalue $\frac{1}{4}$; such representations arise in connection with *Maass forms* as we shall see later. On the other hand, the picture shows that $I_{-,0}$ splits into a direct sum $D_0^- \oplus D_0^+$; the summands underlie the two *limits of discrete series* for $SL_2(\mathbb{R})$.

Example 2.7. ($\lambda = n-1 \in \mathbb{Z}_+$, $\pm =$ parity of n) Applying (2.6) to $I_{\pm,n-1}$ gives



in which the irreducible *submodules* D_{n-1}^+ and D_{n-1}^- underlie discrete series and the *quotient* $I_{\pm,n-1}\big/ D_{n-1}^+ \oplus D_{n-1}^-$ is the finite-dimensional representation W_{n-2} in (1.3). For $\lambda \in \mathbb{Z}_-$, the reverse situation occurs.

In all other cases, $I_{\pm,\lambda}$ is irreducible, and this produces all irreducible admissible $(\mathfrak{g}, K)$-modules. We have also showed (by construction) that they each have a "globalization" to an irredicble representation of G. This globalization is not unique,[3] but a unitary one is unique when it exists. The following representations are unitarizable:

- trivial (1-dimensional quotient of $I_{+,1}$) – obvious;
- spherical unitary principal series $P_+(\nu) := I_{+,i\nu}$ ($\nu \in \mathbb{R}$) – clear from Exercise 2.3(ii);
- non-spherical unitary principal series $P_-(\nu) := I_{-,i\nu}$ ($\nu \in \mathbb{R}^*$) – ditto;
- limits of discrete series D_0^-, D_0^+ – ditto (since they lie in $I_{-,0}$);
- discrete series D_k^+, D_k^- ($k \in \mathbb{Z}_+$) – will arise inside a larger unitary representation in §5;
- complementary series $I_{+,\lambda}$ ($0 < |\lambda| < 1$) – we will not treat these.

Finally, we should mention that there is some redundancy in our classification of induced representations (hence of irreducible representations). Let $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, which we can think of as representing the nontrivial element of the Weyl group of the pair (G, K). Define a map $J : C^\infty(N\backslash G) \to C^\infty(N\backslash G)$ by

$$(J\phi)(g_{x+iy,\theta}) := \int_{x_0 \in \mathbb{R}} \phi(w g_{x_0+iy,\theta}) dx_0.$$

Exercise 2.8. Check that

(i) J intertwines[4] the action of G (on the right), and

(ii) $J(I_{\pm,\lambda}) \subset I_{\pm,-\lambda}$ (ignoring the K-finite restriction).

(iii) By computing $J\phi_n$, show that unless $\lambda \in \mathbb{Z}$ with parity opposite to the "$\pm$", $J : I_{\pm,\lambda} \to I_{\pm,-\lambda}$ is nonzero hence an isomorphism.

Good general references for admissible representations include [Kn2], [Wa], and [Vo].

3. Eisenstein series

Let $\Gamma := SL_2(\mathbb{Z})$. In this section we shall sketch one way in which the discrete series arise as (archimedean components of) automorphic representations,

[3] For instance, we can remove the K-finite condition or (if $\lambda \in i\mathbb{R}$) replace it by an L^2-condition.

[4] i.e. commutes with: we are showing this is a morphism of representations

i.e. as submodules of the space of automorphic forms on G

$$\mathcal{A}(G,\Gamma) := \left\{ \Phi \in C^\infty(\Gamma\backslash G) \,\middle|\, \begin{array}{c} |\Phi| \text{ of polynomial growth in } y, \\ \Phi \text{ right } K\text{-finite, } \Omega\text{-finite} \end{array} \right\} \tag{3.1}$$

which is a $(\mathfrak{g}, K)$-module under (infinitesimal) right translation. This will not yet prove that the $D_k^\pm$ are unitarizable, which will require the complementary perspective taken in §5 where (allowing more general Γ) we impose rapid decrease (rather than polynomial growth) at the cusps.

Consider, for ℓ even, the functions

$$\phi_{\lambda,\ell}(g_{\tau,\theta}) := y^{\frac{\lambda+1}{2}} e^{i\ell\theta} \in C^\infty(\pm N\backslash G) \subset C^\infty(\pm\Gamma_N\backslash G),$$

where $\Gamma_N = \begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix}$. To get into $C^\infty(\Gamma\backslash G)$, we will need to average over (equivalently)

$$\begin{array}{ccccc} \pm\Gamma_N\backslash\Gamma & = & [(0,1)].\Gamma & = & \mathbb{P}^1(\mathbb{Q}) \\ \pm\Gamma_N \overbrace{\begin{pmatrix} a & b \\ p & q \end{pmatrix}}^{\gamma} & \mapsto & [(p,q)] & \mapsto & \frac{q}{p} = \kappa \\ \text{cosets} & & \text{relatively prime pairs} & & \text{rational boundary points} \end{array}.$$

Writing $\gamma g_{\tau,\theta} =: g_{\tilde\tau,\tilde\theta}$, we have

$$\tilde\tau = \gamma\langle\tau\rangle := \frac{a\tau+b}{p\tau+q}\,, \quad \tilde y = \mathrm{Im}(\gamma\langle\tau\rangle) = \frac{y}{|p\tau+q|^2}.$$

Exercise 3.1. Show that $\tilde\theta = \theta - \arctan\left(\frac{y}{x+\kappa}\right)$ and thus that $e^{2i\tilde\theta} = e^{2i\theta}\frac{p\bar\tau+q}{p\tau+q}$.

We can now carry out the average, defining "Eisenstein series" on $\Gamma\backslash G$ by

$$\begin{aligned} E_{\lambda,\ell}(g_{\tau,\theta}) &:= \sum_{\pm\Gamma_N.\gamma\in\pm\Gamma_n\backslash\Gamma} \phi_{\lambda,\ell}(\gamma g_{\tau,\theta}) \\ &= \sum_{\gcd(p,q)=1} \tilde y^{\frac{\lambda+1}{2}} \mathsf{f}_\ell(\tilde\theta) \\ &= y^{\frac{\lambda+1}{2}} e^{i\ell\theta} \sum_{\gcd(p,q)=1} \frac{1}{|p\tau+q|^{\lambda+1}} \left(\frac{p\bar\tau+q}{p\tau+q}\right)^{\frac{\ell}{2}}. \end{aligned}$$

As a function of λ, this is evidently absolutely convergent (uniformly on compact sets) for $\Re(\lambda) > 1$. Assume λ is not an even integer. Then writing

$\tilde{\zeta}(\lambda+1) = \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{k^{\lambda+1}}$, $(kp,kq) = (m,n)$, and $\sum'_{m,n\in\mathbb{Z}^2} := \sum_{m,n\in\mathbb{Z}^2\setminus\{(0,0)\}}$, the last displayed expression

$$= y^{\frac{\ell}{2}} e^{i\ell\theta} \left\{ \frac{1}{\tilde{\zeta}(\lambda+1)} \sum_{m,n\in\mathbb{Z}^2}{}' \frac{y^{\frac{\lambda-\ell+1}{2}}}{(m\tau+n)^{\frac{\lambda+\ell+1}{2}}(m\bar{\tau}+n)^{\frac{\lambda-\ell+1}{2}}} \right\} \tag{3.2}$$
$$=: y^{\frac{\ell}{2}} e^{i\ell\theta} \mathcal{E}_{\lambda,\ell}(\tau).$$

Example 3.2. We recover holomorphic Eisenstein series of weight $2j$ on the upper half plane as a special case:

$$\mathcal{E}_{2j-1,2j}(\tau) = \frac{1}{2\zeta(2j)} \sum_{m,n}{}' \frac{1}{(m\tau+n)^{2j}}.$$

A basic result is that we may meromorphically continue (in λ, fixing $g_{\tau,\theta}$) to obtain Eisenstein series for "most" $\Re(\lambda) \leq 1$; poles occur at $\lambda = 1$ and (as (3.2) suggests) for λ any critical zero of the Riemann zeta function. Observing that the averaging map $\phi \mapsto E$ is right G-equivariant, we conclude the

Theorem 3.3. *$\{E_{\lambda,\ell}\}_{\ell\in 2\mathbb{Z}}$ gives a copy of $I_{+,\lambda} \subset \mathcal{A}(G,\Gamma)$ for "most" λ.*

Remark 3.4. Notice in particular the distinguished role played by the holomorphic Eisenstein series in the copy of D^+_{2j-1}: it is the lowest-weight vector. Before further discussing automorphic representations, we shall give a brief review of holomorphic modular forms, as they play this role quite generally for copies of holomorphic discrete series in $\mathcal{A}(G,\Gamma)$.

For a much more general perspective on Eisenstein series, see §5 of [Ga].

4. Modular forms

To start off a bit more generally, let $\Gamma \leq SL_2(\mathbb{Z})$ be any *arithmetic* subgroup, i.e. one that is commensurable with some congruence subgroup[5]

$$\Gamma(N) := \ker\{SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N\mathbb{Z})\}.$$

For $f \in \mathcal{O}(\mathfrak{H})$ and $\gamma \in \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, write

$$f|^k_\gamma(\tau) := \frac{f(\gamma\langle\tau\rangle)}{(c\tau+d)^k}.$$

[5]If $N = 1$, $\Gamma(N)$ is *defined* to be $SL_2(\mathbb{Z})$.

Exercise 4.1. Verify that $(f|_{\gamma}^{k})|_{\eta}^{k} = f|_{\gamma\eta}^{k}$.

Definition 4.2. f belongs to the space $M_k(\Gamma)$ [resp. $S_k(\Gamma)$] of *modular* [resp. *cusp*] *forms* of weight k and level Γ, iff *(i)* $f \equiv f|_{\gamma}^{k}$ $(\forall \gamma \in \Gamma)$ and *(ii)* $\lim_{\tau\to i\infty}(f|_{\gamma}^{k})(\tau)$ is finite [resp. 0] $(\forall \gamma \in SL_2(\mathbb{Z}))$.

The definition has the following geometric interpretation. Let

$$\begin{array}{ccccc} \mathcal{E} & \supset & E_\tau & = & \mathbb{C}/\mathbb{Z}\langle 1,\tau\rangle \\ \downarrow & & \downarrow & & \\ \mathfrak{H} & \ni & \tau & & \end{array}$$

be the universal elliptic curve, with Γ acting via

$$\gamma.(\tau,u) := \left(\gamma\langle\tau\rangle, \frac{u}{c\tau+d}\right).$$

Exercise 4.3. Check this action is well-defined.

Assuming Γ is neat,[6] we may obtain a universal family of elliptic curves with level structure Γ by

$$\mathcal{E}_\Gamma := \Gamma\backslash\mathcal{E} \xrightarrow{p} \Gamma\backslash\mathfrak{H} =: Y_\Gamma.$$

Its k^{th} fiber self-product admits a compactification

$$\begin{array}{ccc} \mathcal{E}_\Gamma^k & \to & Y_\Gamma \\ \cap & & \cap \\ \overline{\mathcal{E}_\Gamma^k} & \to & \overline{Y_\Gamma} \end{array}$$

due to Shokurov [Sh]. Noting that on $\mathcal{E}$ we have

$$\gamma^* du = \frac{du}{c\tau+d}\ , \quad \gamma^* d\tau = \frac{(c\tau+d)a-(a\tau+b)c}{(c\tau+d)^2}d\tau = \frac{d\tau}{(c\tau+d)^2}, \tag{4.1}$$

and with some more delicate work "at the boundary", one deduces the

Proposition 4.4. *For $f \in \mathcal{O}(\mathfrak{H})$, the form $f(\tau)d\tau\wedge du_1\wedge\cdots\wedge du_k$ descends to $\Omega^{k+1}(\overline{\mathcal{E}_\Gamma^k})\langle\log \overline{\mathcal{E}_\Gamma^k}\backslash\mathcal{E}_\Gamma^k\rangle$ [resp. $\Omega^{k+1}(\overline{\mathcal{E}_\Gamma^{k+1}})$] if and only if $f \in M_{k+2}(\Gamma)$ [resp. $S_{k+2}(\Gamma)$].*

[6] i.e. the eigenvalues of its elements generate a torsion-free subgroup of $\mathbb{C}^*$. In particular, for $\Gamma(N)$ to be neat we must have $N \geq 4$, so when $N = 1,2,3$ more care is required in constructing the family and compactification.

We will not use this interpretation until §6, so for simplicity we shall until then take $\Gamma = SL_2(\mathbb{Z})$ (though of course the results will all generalize). In Definition 4.2, *(i)* becomes

$$\left\{ \begin{array}{l} f(\tau+1) = f(\tau) \\ f(-\frac{1}{\tau}) = \tau^k f(\tau) \end{array} \right. , \tag{4.2}$$

the first line of which implies

$$f(\tau) = F(\underbrace{e^{2\pi i\tau}}_{=:q}) = \sum_{n\in\mathbb{Z}} a_n q^n;$$

(ii) then simply says

$$a_n = 0 \text{ for } n < 0 \text{ [resp. } n \leq 0]. \tag{4.3}$$

Note that $\Gamma \ni -\mathrm{id} \implies$ there are no modular forms of odd weight.

Example 4.5. Let $k \geq 4$ be an even integer; then

$$E_k(\tau) := \frac{1}{2\zeta(k)} \sum\nolimits'_{m,n\in\mathbb{Z}^2} \frac{1}{(m\tau+n)^k} \in M_k(\Gamma). \tag{4.4}$$

A Mittag-Leffler computation shows that

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n\geq 1} \sigma_{k-1}(n) q^n, \tag{4.5}$$

where $\sigma_{k-1}(n) := \sum_{\left\{ \substack{m>0 \\ m|n} \right.} m^{k-1}$. From the special cases $E_4 = 1 + 240q + 2160q^2 + \cdots \in M_4(\Gamma)$ and $E_6 = 1 - 504q - 16632q^2 - \cdots \in M_6(\Gamma)$, one constructs the modular discriminant

$$\Delta(\tau) := \frac{E_4^3 - E_6^2}{1728} = q - 24q^2 + \cdots \in S_{12}(\Gamma)$$

of Weierstrass.

Exercise 4.6. Check (4.4), either by directly verifying (4.2)-(4.3), or by using the fact that it equals $\mathcal{E}_{k-1,k}$ from (3.2).

Example 4.7. Define the Dedekind eta function

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{\ell\geq 1} (1 - q^\ell) \in \mathcal{O}(\mathfrak{H}).$$

Clearly η^{24} is invariant under $\tau \mapsto \tau + 1$, while residue theory gives that $\eta(-\frac{1}{\tau})^{24} = \tau^{12}\eta(\tau)^{24}$; it follows that $\eta^{24} = q + \cdots \in S_{12}(\Gamma)$.

Now on $\mathbb{P}^1 \cong \overline{\Gamma\backslash\mathfrak{H}}$, the local coordinate w at 0 [resp. 1, ∞] looks like $(\tau - i)^2$ [resp. $(\tau - \zeta_3)^3$, q]; so $d\tau$ becomes essentially $dw/\sqrt{w}$ [resp. $dw/w^{\frac{2}{3}}$, dw/w]. Therefore $f \in M_{2k}(\Gamma)$ is equivalent to $fd\tau^{\otimes k}$ descending to a section of $\mathcal{O}_{\mathbb{P}^1}(-2k + \left\lfloor \frac{k}{2} \right\rfloor [0] + \left\lfloor \frac{2k}{3} \right\rfloor [1] + k[\infty])$ (for $S_{2k}(\Gamma)$, replace the coefficient of $[\infty]$ by $k-1$). This gives the

Proposition 4.8. *For $k \geq 2$,* $\dim M_{2k}(\Gamma) = \dim S_{2k}(\Gamma) + 1 = \left\lfloor \frac{k}{2} \right\rfloor + \left\lfloor \frac{2k}{3} \right\rfloor - k + 1.$

Example 4.9. *(a)* $\dim S_{12}(\Gamma) = 1 \implies \eta^{24} = \Delta$

(b) $\dim M_8 = 1 \implies E_4^2 = E_8 \implies$

$$\sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m) = \frac{\sigma_7(n) - \sigma_3(n)}{120}$$

for every n.

One can also show that the ring $\oplus_{k\geq 0} M_k(\Gamma) = \mathbb{C}[E_4, E_6]$. A good introductory reference on modular forms is [Za].

5. Cuspidal automorphic forms

Let G, Γ, dg, etc. be as above, continuing to assume for simplicity that $\Gamma = SL_2(\mathbb{Z})$ so that there is only one cusp. Denote by $^\circ L^2(\Gamma\backslash G)$ the L^2-closure (with respect to Haar measure dg) of

$$\left\{ \Phi \in L^2(\Gamma\backslash G) \,\middle|\, \int_0^1 \Phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) dx = 0 \ (\forall g \in G) \right\}.$$

This is a unitary representation under the action of G by right translation

$$(\pi(g_0).\Phi)(g) := \Phi(gg_0).$$

The irreducible subrepresentations are therefore unitary, and we are interested in which ones appear (and with what multiplicity). By a result in representation theory, they occur discretely, so that there are countably many, and $^\circ L^2(\Gamma\backslash G)$ is their Hilbert direct sum.

It will be convenient to study a slightly smaller space:

Definition 5.1. The *cuspidal automorphic forms*

$$^\circ\mathcal{A}(G,\Gamma) := \mathcal{A}(G,\Gamma) \cap {}^\circ L^2(\Gamma\backslash G)$$

are the smooth, Ω- and K-finite vectors in $^\circ L^2(\Gamma\backslash G)$; infinitesimal right translation endows them with a $(\mathfrak{g}, K)$-module structure.

In fact, $^\circ\mathcal{A}(G,\Gamma)$ is the algebraic direct sum of the underlying Harish-Chandra modules of the irreducible summands of $^\circ L^2(\Gamma\backslash G)$. Our first step in analyzing the decomposition is to convert automorphic forms to smooth functions on $\mathfrak{H}$. Now $^\circ\mathcal{A}(G,\Gamma)$ is the direct sum of its K-types

$$^\circ\mathcal{A}_m(G,\Gamma) := \left\{ \Phi \in {}^\circ\mathcal{A}(G,\Gamma) \,\middle|\, \pi(k_\theta)\Phi = e^{im\theta}\Phi \right\},$$

and we define (for each $m \in \mathbb{Z}$)

$$^\circ C^\infty_m(\mathfrak{H},\Gamma) := \left\{ f \in C^\infty(\mathfrak{H}) \,\middle|\, \begin{array}{c} \bullet\, f|^m_\gamma = f \text{ for every } \gamma \in \Gamma \\ \bullet\, \int_{\Gamma\backslash\mathfrak{H}} |f|^2 y^{m-2} dx \wedge dy < \infty \\ \bullet\, \int_0^1 f(x+iy)dx = 0 \;\forall y \in \mathbb{R}_+ \\ \bullet\, f \text{ is } \omega_m\text{-finite} \end{array} \right\},$$

where

$$\omega_m := -y^2 \underbrace{\left(\partial_x^2 + \partial_y^2\right)}_{\Delta} + 2imy\partial_{\bar\tau} + \frac{m}{2}\left(1 - \frac{m}{2}\right).$$

Proposition 5.2. *For each m, the map $f^{[m]} : {}^\circ\mathcal{A}_m(G,\Gamma) \to {}^\circ C^\infty_m(\mathfrak{H},\Gamma)$ defined by*

$$f^{[m]}_\Phi(\tau) := y^{-\frac{m}{2}}\Phi(p_\tau), \tag{5.1}$$

is an isomorphism.

Proof. (Sketch) The main idea is to write down an inverse map

$$\Phi^{[m]} : {}^\circ C^\infty_m(\mathfrak{H},\Gamma) \to {}^\circ\mathcal{A}_m(G,\Gamma),$$

namely

$$\Phi^{[m]}_f(g_{\tau,\theta}) := e^{im\theta} y^{\frac{m}{2}} f(\tau). \tag{5.2}$$

Given $f \in C^\infty(\mathfrak{H})$, indeed one has $f^{[m]}_{\Phi^{[m]}_f}(\tau) = y^{-\frac{m}{2}}\Phi^{[m]}_f(p_\tau) = f(\tau)$, while for Φ satisfying $\Phi(g_{\tau,\theta+\theta_0}) = e^{im\theta}\Phi(g_{\tau,\theta})$, we obtain $\Phi^{[m]}_{f^{[m]}_\Phi}(g_{\tau,\theta}) = e^{im\theta} y^{\frac{m}{2}} f^{[m]}_\Phi(\tau) =$

$e^{im\theta}\Phi(p_\tau) = \Phi(g_{\tau,\theta})$; however, there are still all the Γ-automorphy, L^2, cuspidal, and finiteness conditions to match up. We shall check the first of these: note that the right-hand side of (5.2)

$$= e^{im\theta}\frac{f(p_\tau\langle i\rangle)}{(0i+\frac{1}{\sqrt{y}})^m} = e^{im\theta}(f|^m_{p_\tau})(i) = \frac{(f|^m_{p_\tau})(k_\theta\langle i\rangle)}{(-i\sin\theta+\cos\theta)^m} = (f|^m_{p_\tau}|^m_{k_\theta})(i)$$

so that

$$\Phi_f^{[m]}(g_{\tau,\theta}) = (f|^m_{g_{\tau,\theta}})(i), \tag{5.3}$$

and then

$$(f_\Phi^{[m]}|^m_\gamma)(\tau) = \frac{(f_\Phi^{[m]}|^m_\gamma|^m_{p_\tau})(i)}{y^{\frac{m}{2}}} \overset{(5.3)}{=} y^{-\frac{m}{2}}\Phi^{[m]}_{f_\Phi^{[m]}}(\gamma p_\tau) = y^{-\frac{m}{2}}\Phi(\gamma p_\tau).$$

Assuming Φ is left Γ-invariant, the last expression

$$= y^{-\frac{m}{2}}\Phi(p_\tau) = f_\Phi^{[m]}(\tau)$$

as desired. $\square$

Exercise 5.3. Check that the remaining conditions identify; in particular, show (referring to Exercise 2.5) that

$$\omega_m = f^{[m]}\circ\Omega\circ\Phi^{[m]}.$$

Remark 5.4. For m odd, ${}^\circ\mathcal{A}_m(G,\Gamma) = 0$ since $-\mathbb{I}\in\Gamma \implies \Phi(g) = \Phi(-g) = \Phi(ge^{i\pi}) = \pi(k_\pi)\Phi(g) = -\Phi(g)$.

The next step is to use the isomorphisms of Proposition 5.2 to explicitly realize the $(\mathfrak{g},K)$-module structure on ${}^\circ\mathcal{A}(G,\Gamma)$, on the left-hand side of

$${}^\circ C^\infty(\mathfrak{H},\Gamma) := \bigoplus_{m\in\mathbb{Z}} {}^\circ C^\infty_m(\mathfrak{H},\Gamma) \cong \bigoplus_{m\in\mathbb{Z}} {}^\circ\mathcal{A}_m(G,\Gamma).$$

The formulas (2.5) for the action of the Lie derivatives were for functions on G which are constant in x. Redoing the computation without this assumption (to get the missing ∂_x term) gives

$$\mathcal{L}_W = -i\partial_\theta,\ \mathcal{L}_{E_+} = e^{2i\theta}\left\{2iy\partial_\tau - \frac{i}{2}\partial_\theta\right\},\ \mathcal{L}_{E_-} = e^{-2i\theta}\left\{-2iy\partial_{\bar\tau} + \frac{i}{2}\partial_\theta\right\} \tag{5.4}$$

for the action of $\mathfrak{g}$ on ${}^{\circ}\mathcal{A}(G,\Gamma)$; it is clear that $\mathcal{L}_{E_\pm}$ sends ${}^{\circ}\mathcal{A}_m(G,\Gamma)$ to ${}^{\circ}\mathcal{A}_{m\pm 2}(G,\Gamma)$. To transfer these to ${}^{\circ}C^{\infty}(\mathfrak{H},\Gamma)$, one defines

$$L_W^{[m]} := f^{[m]} \circ \mathcal{L}_W \circ \Phi^{[m]} : {}^{\circ}C^{\infty}_m(\mathfrak{H},\Gamma) \to {}^{\circ}C^{\infty}_m(\mathfrak{H},\Gamma),$$
$$L_{E_\pm}^{[m]} := f^{[m\pm 2]} \circ \mathcal{L}_{E_\pm} \circ \Phi^{[m]} : {}^{\circ}C^{\infty}_m(\mathfrak{H},\Gamma) \to {}^{\circ}C^{\infty}_{m\pm 2}(\mathfrak{H},\Gamma)$$

and computes that

$$L_W^{[m]} = m,\ L_{E_-}^{[m]} = -2iy^2\partial_{\bar\tau},\ L_{E_+}^{[m]} = 2i\partial_\tau + \frac{m}{y}; \tag{5.5}$$

these then "paste together" to give operators L_W, $L_{E_\pm}$ on all of ${}^{\circ}C^{\infty}(\mathfrak{H},\Gamma)$.

Exercise 5.5. Verify the formulas (5.4) and (5.5).

From the form of L_{E_-} we arrive at the first main result.

Theorem 5.6. *The number of independent copies of* D^+_{m-1} $(m \geq 2)$ *in* ${}^{\circ}\mathcal{A}(G,\Gamma)$ *is zero for m odd and* $\left\lfloor \frac{m}{4} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor - \frac{m}{2}$ *for m even.*

Proof. We claim that

$$Hom_{(\mathfrak{g},K)}(D^+_{m-1}, {}^{\circ}\mathcal{A}(G,\Gamma)) \cong S_m(\Gamma), \tag{5.6}$$

whereupon the result follows from taking dimensions on both sides. Indeed, the left-hand side of (5.6) identifies with the space $\ker \mathcal{L}_{E_-} \cap {}^{\circ}\mathcal{A}_m(G,\Gamma) \cong \ker L_{E_-} \cap {}^{\circ}C^{\infty}_m(\mathfrak{H},\Gamma)$ of lowest-weight vectors in weight m. Since $\ker L_{E_-} = \ker \partial_{\bar\tau}$ is just the holomorphic functions, this identifies with

$$\left\{ f \in \mathcal{O}(\mathfrak{H}) \;\middle|\; \begin{array}{c} \bullet\ f|^m_\gamma = f \text{ for all } \gamma \in \Gamma \\ \bullet\ \int_0^1 (a_0 + a_1 q + \cdots)dx = 0 \end{array} \right\},$$

i.e. the cusp forms of weight m. $\square$

Remark 5.7. (i) The number of copies of D^-_{m-1} in ${}^{\circ}\mathcal{A}(G,\Gamma)$ is the same, since complex conjugation $\Phi \mapsto \bar\Phi$ inverts weights and $\overline{\mathcal{L}_{E_\pm}} = \mathcal{L}_{E_\mp}$. On $\mathfrak{H}$, this is

$$\begin{aligned} {}^{\circ}C^{\infty}_m(\mathfrak{H},\Gamma) &\to {}^{\circ}C^{\infty}_{-m}(\mathfrak{H},\Gamma) \\ f &\longmapsto y^m \overline{f}. \end{aligned}$$

(ii) An obvious corollary of the theorem is that the discrete series are unitary, at least for those occurring with positive multiplicity (though this issue can be eliminated by using more general Γ). See §7 for another approach.

(iii) The identification (5.6) means that to each cusp form f is attached a cuspidal automorphic representation π_f. In the literature, the representation is itself often referred to (in a standard abuse of notation) as a cusp form.

It remains to determine which "bi-infinite ladder" type irreducible representations live inside ${}^\circ\mathcal{A}(G,\Gamma)$. By Remark 5.4, these must pass through ${}^\circ\mathcal{A}_0(G,\Gamma)$; and we know that they must belong to the list of unitarizable representations from §2. Now

$$\begin{aligned}{}^\circ C_0^\infty(\mathfrak{H},\Gamma) &= \left\{ f\in C^\infty(\Gamma\backslash\mathfrak{H}) \;\middle|\; \begin{array}{c} \bullet\ \int_{\Gamma\backslash\mathfrak{H}} |f|^2\frac{dx\wedge dy}{y^2}<0 \\ \bullet\ \int_0^1 f(x+iy)dx=0 \\ \bullet\ f \text{ is } \omega_0\text{-finite}\end{array}\right\} \\ &= \bigoplus_{\xi\in\Xi} {}^\circ C_0^\infty(\mathfrak{H},\Gamma)\cap\ker\{\omega_0-\xi\} \\ &=: \bigoplus_{\xi\in\Xi}\mathcal{M}_\xi,\end{aligned}$$

where $\Xi(\subset\mathbb{C}$ *a priori*) is some countable index set and $\omega_0=-y^2\Delta$ is the hyperbolic Laplacian.

Definition 5.8. The elements of $\mathcal{M}_\xi$ are the *Maass cusp forms of eigenvalue* ξ.

A short computation using integration by parts

$$\langle -y^2\Delta f,f\rangle=\int_{\Gamma\backslash\mathfrak{H}}(-y^2\Delta f)\bar{f}\frac{dx\wedge dy}{y^2}=\int_{\Gamma\backslash\mathfrak{H}}\left\{\left|\frac{\partial f}{\partial x}\right|^2+\left|\frac{\partial f}{\partial y}\right|^2\right\}dx\wedge dy>0$$

shows that ω_0 is positive definite, so that $\Xi\subset\mathbb{R}_+$.

If $I_{+,\lambda}(\cong I_{+,-\lambda})$ is an irreducible representation occurring in ${}^\circ\mathcal{A}(G,\Gamma)$, then by Exercise 2.5 Ω operates as multiplication by $\frac{1}{4}(1-\lambda^2)$ on it. It follows that

$$Hom_{(\mathfrak{g},K)}(I_{+,\lambda},{}^\circ\mathcal{A}(G,\Gamma))\cong\mathcal{M}_{\frac{1}{4}(1-\lambda^2)}, \tag{5.7}$$

and $\omega_0>0$ then forces $\frac{1}{4}(1-\lambda^2)>0$. We have therefore either $0<|\lambda|<1$ (and $I_{+,\lambda}$ in the complementary series) or $\lambda\in i\mathbb{R}$ (and $I_{+,\lambda}$ in the principal spherical series). In fact, a result of Selberg says that for $\Gamma=SL_2(\mathbb{Z})$ (as we are assuming) Maass forms must have eigenvalue $\xi \geq \frac{1}{4}$, eliminating the first option.[7] Writing $\lambda = i\nu$ and taking dimensions in (5.7) gives the

Theorem 5.9. *The number of independent copies of* $P_+(\nu)$ *occurring in* ${}^\circ\mathcal{A}(G,\Gamma)$ *is given by* $\dim \mathcal{M}_{\frac{1+\nu^2}{4}}$.

In conclusion, ${}^\circ\mathcal{A}(G,\Gamma)$ is the algebraic (countable) direct sum of $(\mathfrak{g}, K)$-modules of type D^+, D^-, and P_+ with multiplicities described by Theorems 5.6 and 5.9.

Remark 5.10. The Eisenstein series of §3 are a complement to this in the sense that (a) they lie outside ${}^\circ\mathcal{A}(G,\Gamma)$ (even in some cases $\mathcal{A}(G,\Gamma) \cap L^2$) and (b) λ varies continuously.

Some excellent introductory notes are [Ga] for automorphic forms and [Sc] for discrete series. (In particular, Gan's notes treat automorphic forms on $SL_2(\mathbb{A})$.)

6. Cohomology

We now shall permit $\Gamma \leq SL_2(\mathbb{Z})$ to be any arithmetic subgroup, so that ${}^\circ\mathcal{A}_m(G,\Gamma)$ can be nonzero for m odd, and limits of discrete series $D_0^\pm$ and non-spherical unitary principal series $I_{-,i\nu}$ can occur. Write

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{n} \oplus \mathfrak{n}^+ = \mathbb{C}\langle W \rangle \oplus \mathbb{C}\langle E_- \rangle \oplus \mathbb{C}\langle E_+ \rangle.$$

In general, the *Lie algebra cohomology* $H^*(\mathfrak{n}, V_\pi)$ of a representation (V_π, π) of $G (= SL_2(\mathbb{R}))$ with respect to $\mathfrak{n}$ is cohomology of a complex

$$0 \to V_\pi \xrightarrow{d} \mathfrak{n}^\vee \otimes V_\pi \xrightarrow{d} \bigwedge^2 \mathfrak{n}^\vee \otimes V_\pi \xrightarrow{d} \cdots . \tag{6.1}$$

K acts by Ad on $\mathfrak{n}$ (and of course also on V_π) hence on (6.1) and $H^*(\mathfrak{n}, V_\pi)$.

In this case, (6.1) is simply

$$\begin{array}{ccccccc} 0 & \longrightarrow & V_\pi & \xrightarrow{d} & Hom(\mathfrak{n}, V_\pi) & \longrightarrow & 0 \\ & & & & {\scriptstyle \mathrm{ev}_{E_-}} \big\downarrow {\scriptstyle \cong} & & \\ & & & & V_\pi & & \end{array} \tag{6.2}$$

[7] The generalization of this to other choices of Γ, which is not known, is a version of the Generalized Ramanujan Conjecture (cf. [Sa] for an even more general discussion).

where for $v \in V_\pi$

$$(dv)(E_-) = E_-.v,$$

and if V_π is a subrepresentation of ${}^\circ\mathcal{A}(G,\Gamma)$ then $E_-.v$ means $\mathcal{L}_{E_-}v$. In any case, (6.2) gives

$$\begin{cases} H^0(\mathfrak{n}, V_\pi) \cong \ker(E_-) = \text{space of lowest weight vectors} \\ H^1(\mathfrak{n}, V_\pi) \cong \operatorname{coker}(E_+) = \text{space of highest weight vectors} \end{cases}.$$

For V_π one of the irreducible representations we have classified, a quick look at (2.6) shows that this can only be nonzero for discrete series and the two limits of discrete series. More precisely, we have

Proposition 6.1. *For V_π an admissible irreducible representation of G,*

$$H^0(\mathfrak{n}, V_\pi)_k = \begin{cases} \mathbb{C} & \textit{if } k \geq 1 \textit{ and } V = D^+_{k-1} \\ 0 & \textit{otherwise} \end{cases} \quad \textit{and}$$

$$H^1(\mathfrak{n}, V_\pi)_k = \begin{cases} \mathbb{C} & \textit{if } k \leq 1 \textit{ and } V = D^-_{1-k} \\ 0 & \textit{otherwise} \end{cases},$$

where the subscript indexes K-eigenspaces.

Remark 6.2. *(a)* Note that in the H^1 case, $\mathfrak{n}^\vee$ contributes $+2$ to the K-type.

(b) The fact that only two representations have any cohomology in weight k reflects a general result of Casselman and Osborne [CO].

Denoting the unitary dual of $SL_2(\mathbb{R})$ by $\hat{G}$, we know that

$$(6.3) \qquad {}^\circ\mathcal{A}(G,\Gamma) = \bigoplus_{\pi \in \hat{G}} V_\pi^{\oplus m_\pi(\Gamma)},$$

where only countably many of the $m_\pi(\Gamma)$ are nonzero. From the foregoing, it is clear that $H^*(\mathfrak{n}, {}^\circ\mathcal{A}(G,\Gamma))$ singles out DS and LDS representations inside the cuspidal automorphic forms. To see the implications of this observation for cohomology in the geometric setting, we assume (for simplicity) that Γ is neat and reintroduce the elliptic modular surface $\mathcal{E}_\Gamma \xrightarrow{p} Y_\Gamma$ of §4.

Writing $\omega := p_*\Omega^1_{\mathcal{E}_\Gamma/Y_\Gamma}$ for its Hodge bundle, we see from (4.1) that ω has formal K-type -1 and that the canonical bundle $K_{Y_\Gamma} \cong \omega^{\otimes 2}$. Now ω admits a canonical extension $\bar{\omega} \to \overline{Y}_\Gamma$ (compatible with taking tensor powers), with a Hermitian metric of logarithmic growth along $D := \overline{Y}_\Gamma \backslash Y_\Gamma$. Using Proposition

4.4 we find that

$$\Gamma(\overline{Y_\Gamma}, \bar{\omega}^{\otimes k}) = M_k(\Gamma) \text{ and } \Gamma(\overline{Y_\Gamma}, \bar{\omega}^{\otimes k}(-D)) = S_k(\Gamma);$$

in particular, we have

$$K_{\overline{Y_\Gamma}} = \bar{\omega}^{\otimes 2}(-D). \tag{6.4}$$

We are now ready to introduce a "toy model" related to the spaces studied in [GGK, sec. 4].

Definition 6.3. The *cuspidal automorphic cohomology* of Y_Γ is

$$S^*(Y_\Gamma, \omega^{\otimes k}) := \ker \left\{ H^*(\overline{Y_\Gamma}, \bar{\omega}^{\otimes k}) \to H^*(D, \bar{\omega}^{\otimes k}|_D) \right\}.$$

This turns out to be equal to the L^2 Dolbeault cohomology

$$H^*_{(2)}(Y_\Gamma, \omega^{\otimes k})$$
$$= H^* \left\{ \left(\bigwedge{}^\bullet \mathfrak{n}^\vee \otimes \mathcal{A}^2(G,\Gamma) \otimes \mathbb{C}_{-k} \right)^K \right\},$$

where $\bigwedge{}^\bullet \mathfrak{n}^\vee$ corresponds to antiholomorphic differentials in the Dolbeault complex, K acts on $\mathbb{C}_{-k}$ through the character $e^{-ik\theta}$, $(\;)^K$ denotes K-invariants, and $\mathcal{A}^2$ denotes L^2 automorphic forms. Nothing is lost here by replacing the latter by cuspidal forms,[8] and so the last displayed expression

$$= H^*(\mathfrak{n}, {}^\circ\mathcal{A}(G,\Gamma))_k \tag{6.5}$$
$$= \bigoplus_{\pi \in \widehat{G}} H^*(\mathfrak{n}, V_\pi)_k^{\oplus m_\pi(\Gamma)}$$

using (6.3). By Proposition 6.1, $H^*(\mathfrak{n}, V_\pi)_k$ is trivial or 1-dimensional. Reasoning as in the proof of Theorem 5.6 brings us to the essentially tautological result

$$S^0(Y_\Gamma, \omega^{\otimes m}) = Hom_{(\mathfrak{g},K)} \left(D^+_{m-1}, {}^\circ\mathcal{A}(G,\Gamma) \right) \cong S_m(\Gamma), \tag{6.6}$$

along with (noting $1-(2-m) = m-1$) the more interesting statement

$$S^1(Y_\Gamma, K_{Y_\Gamma} \otimes (\omega^\vee)^{\otimes m}) = S^1(Y_\Gamma, \omega^{\otimes 2-m}) = Hom_{(\mathfrak{g},K)} \left(D^-_{m-1}, {}^\circ\mathcal{A}(G,\Gamma) \right) \underset{c}{\cong} S_m(\Gamma), \tag{6.7}$$

[8]One way to see this is that only discrete series representations contribute, and for each pair $D^+_{m-1} \oplus D^-_{m-1}$ in $\mathcal{A}^2(G,\Gamma)$, the lowest weight vector for the D^+_{m-1} is an L^2 holomorphic automorphic form in the sense of [Bo] hence cuspidal by Cor. 7.10 of [op. cit.].

where "$\underset{c}{\cong}$" indicates a conjugate linear isomorphism.

Together, (6.6) and (6.7) lead to an "explicit" proof of Serre duality for $\overline{Y_\Gamma}$. Recalling $dg = \frac{dx\wedge dy\wedge d\theta}{2\pi y^2}$, the positive-definite inner product

$$\langle \Phi_1, \Phi_2 \rangle = \int_{\Gamma\backslash G} \Phi_1 \overline{\Phi_2} dg$$

on ${}^\circ\mathcal{A}(G,\Gamma)$ becomes the *Petersson inner product*

$$\langle f_1, f_2 \rangle_m = \int_{\Gamma\backslash\mathfrak{H}} f_1 \overline{f_2} y^m \frac{dx\wedge dy}{y^2}$$

on $S_m(\Gamma)$. Altogether, we get a composite isomorphism

$$\begin{tikzcd} S^0(Y_\Gamma, \omega^{\otimes m})^\vee \arrow[rr, bend right, "\mathrm{SD}", "\cong"'] & S^0(Y_\Gamma, \omega^{\otimes m}) \arrow[l, "\langle\cdot, f\rangle_m \mapsfrom f"', "\underset{c}{\cong}"] \arrow[r, "\underset{c}{\cong}"'] & S^1(Y_\Gamma, K_{Y_\Gamma} \otimes (\omega^\vee)^{\otimes m}) \end{tikzcd} \tag{6.8}$$

recovering Serre duality in this case:

Exercise 6.4. Check $S^0(Y_\Gamma, L) = H^0(\overline{Y_\Gamma}, L(-D))$ and $S^1(Y_\Gamma, L) = H^1(\overline{Y_\Gamma}, L)$; then apply (6.4).

But in fact we have much more: on the level of representatives, the isomorphism (6.7) is given by complex conjugation

$$\begin{array}{ccccc} S_m(\Gamma) & \ni & f & & \\ & & \downarrow & & \\ e^{im\theta} y^{\frac{m}{2}} f & = & \Phi_f & \in & D^+_{m-1} \\ & & \downarrow & & c.c. \downarrow \underset{c}{\cong} \\ e^{-im\theta} y^{-\frac{m}{2}} (y^m \bar{f}) & = & \overline{\Phi_f} & \in & D^-_{m-1} \\ & & \downarrow & & \\ S^1(Y_\Gamma, K_{Y_\Gamma} \otimes (\omega^\vee)^{\otimes m}) & \ni & y^m \bar{f} \frac{d\bar{\tau}}{y^2} (\otimes d\tau) & & \end{array}$$

(where we think of $\frac{d\bar{\tau}}{y^2}$ as having formal weight $+2$). This does three things:

- makes "SD" (6.8) in some sense the identity map;
- gives us explicit representatives of $S^1(Y_\Gamma, \cdots)$; and
- anticipates a special case of the Penrose transforms used in [Ca, GGK, Ke].[9]

[9] cf. §5 of [Ke] for example.

Remark 6.5. If one takes instead $\mathfrak{n} = \mathbb{C}\left\langle\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\rangle$, Frobenius reciprocity tells us that $H^*(\mathfrak{n}, V_\pi)$ is essentially always nonzero for admissible irreducible representations V_π, as it has to recover the original representation of the Levi we induced from. While less interesting from our point of view, this observation plays a crucial role in the classification of admissible representations (cf. for example [Wa] or, for a shortcut, the very nice review [Co]).

Remark 6.6. In the more general setting of [GGK, sec. IV] where the analogue of Y_Γ is non-algebraic, we cannot define $S^*(\cdots)$ as in Definition 6.3. Hence (6.5) itself becomes the definition of cuspidal automorphic cohomology; a challenging problem would be to find some kind of equivalent geometric definition, perhaps via L^2 cohomology, or – even better – in the context of the partial compactifications [KU] of Kato and Usui.

7. Appendix I: $L^2(G)$ and the $\{D_m^\pm\}$

Discrete series are, by definition, the irreducible representations occurring in the discrete spectrum of $L^2(G)$. This is obviously a more natural "proof" of their unitaricity than the approach taken (for different reasons) in §5, so we would be remiss not to briefly treat this story.

With $G = SL_2(\mathbb{R})$ and dg as above, $L^2(G)$ is the completion of

$$\left\{\phi \in C^\infty(G) \,\middle|\, \int_G |\phi|^2 dg < \infty\right\};$$

G acts via π (right translation) and $\tilde{\pi}$ (left translation: $(\tilde{\pi}(g_0).\phi)(g) := \phi(g_0^{-1}g)$). Write

$$L^2(G)^m := \left\{\phi \in L^2(G) \,\middle|\, \pi(k_\theta).\phi = e^{im\theta}\phi\right\},$$
$$L^2(G)_n := \left\{\phi \in L^2(G) \,\middle|\, \tilde{\pi}(k_\theta).\phi = e^{in\theta}\phi\right\},$$
$$L^2(G)_n^m := L^2(G)^m \cap L^2(G)_n$$

for the weight spaces of $L^2(G)$. We shall also be interested in the spaces

$$L^2(\mathfrak{H}, \mu_m) := \overline{\left\{\mathfrak{f} \in C^\infty(\mathfrak{H}) \,\middle|\, \int_{\mathfrak{H}} |\mathfrak{f}|^2 d\mu_m\right\}}$$

where $d\mu_m := y^{m-2}dx \wedge dy$, and $L^2_{\mathrm{hol}}(\mathfrak{H}, \mu_m) := \ker(\bar{\partial}) \subset L^2(\mathfrak{H}, \mu_m)$, which carry an action of G by

$$\tilde{\pi}_m(g).\mathfrak{f} := \mathfrak{f}|^m_{g^{-1}}.$$

Exercise 7.1. Check that $\tilde{\pi}_m$ is unitary.

With $f^{[m]}$ and $\Phi^{[m]}$ given by the formulas (5.1)-(5.2), we have isomorphisms of representations

$$\begin{tikzcd} (L^2(\mathfrak{H}, \mu_m), \tilde{\pi}_m) \arrow[r, bend left, "\Phi^{[m]}"] \arrow[r, phantom, "\cong"] & (L^2(G)^m, \tilde{\pi}) \arrow[l, bend left, "f^{[m]}"] \end{tikzcd}. \tag{7.1}$$

since

$$\Phi^{[m]}_{(\tilde{\pi}_m(\sigma).\mathfrak{f})}(g) = \Phi^{[m]}_{\mathfrak{f}|^m_{\sigma^{-1}}}(g) \overset{(5.3)}{=} \mathfrak{f}|^m_{\sigma^{-1}}|^m_g(i) = \mathfrak{f}|^m_{\sigma^{-1}g}(i) = \Phi^{[m]}_{\mathfrak{f}}(\sigma^{-1}g)$$
$$= (\tilde{\pi}(\sigma).\Phi^{[m]}_{\mathfrak{f}})(g).$$

Let ϕ be a lowest weight vector for a copy of D^+_{m-1} in $(L^2(G), \pi)$. Consider the diagram

$$\begin{tikzcd} (L^2(\mathfrak{H}, \mu_m), \tilde{\pi}_m) \arrow[r, "\Phi^{[m]}", "\cong"'] \arrow[d, "-2iy^2\partial_{\bar{\tau}}"', "=L^{[m]}_{E_-}"] & (L^2(G)^m, \tilde{\pi}) \arrow[d, "\mathcal{L}_{E_-}"] \\ (L^2(\mathfrak{H}, \mu_{m-2}), \tilde{\pi}_{m-2}) \arrow[r, "\Phi^{[m-2]}", "\cong"'] & (L^2(G)^{m-2}, \tilde{\pi}) \end{tikzcd} \tag{7.2}$$

which commutes essentially by (5.5) (and the fact that left and right translations commute). Since $\phi \in \ker(\mathcal{L}_{E_-}) \cap L^2(G)^m$, (7.2) makes it clear that $f^{[m]}_\phi \in L^2_{\mathrm{hol}}(\mathfrak{H}, \mu_m)$. Conversely, it is clear that given any $\mathfrak{f} \in L^2_{\mathrm{hol}}(\mathfrak{H}, \mu_m)$, $\Phi^{[m]}_{\mathfrak{f}}$ generates a copy of D^+_{m-1} in $(L^2(G), \pi)$. We claim that

Proposition 7.2. *$(L^2_{hol}(\mathfrak{H}, \mu_m), \tilde{\pi}_m) \cong D^-_{m-1}(\cong (D^+_{m-1})^*)$ for each $m > 1$. (For $m \leq 1$ it is zero.)*

Assuming this, we have constructed an embedding of $D^+_{m-1} \otimes D^-_{m-1}$ in $L^2(G)$, which is bi-equivariant in the sense that π restricts to the action on D^+_{m-1} and $\tilde{\pi}$ to the action on D^-_{m-1}. Going through a similar process for every D^+_k and

D_k^-, we recover the full discrete spectrum

$$\left(\hat{\bigoplus_{m>1}} D_{m-1}^+ \hat{\otimes} (D_{m-1}^+)^*\right) \oplus \left(\hat{\bigoplus_{m>1}} D_{m-1}^- \hat{\otimes} (D_{m-1}^-)^*\right)$$

in $L^2(G)$.

Proof. To verify Proposition 7.2, it will suffice to show (for $m \geq 2$) that

$$L^2_{\text{hol}}(\mathfrak{H}, \mu_m)_k := \left\{\mathfrak{f} \in L^2_{\text{hol}}(\mathfrak{H}, \mu_m) \mid \tilde{\pi}_m(k_\theta).\mathfrak{f} = e^{ik\theta}\mathfrak{f}\right\}$$

is 1-dimensional for $k = -m, -m-2, -m-4, \ldots$ and trivial otherwise. (Look at the list of unitary representations of G.)

Indeed, writing $\ell = -k$, we have

$$\frac{\mathfrak{f}(k_\theta^{-1}\langle\tau\rangle)}{(\tau \sin\theta + \cos\theta)^m} = e^{-i\ell\theta}\mathfrak{f}(\tau) \quad (\forall\theta, \tau) \tag{7.3}$$

hence in particular (noting $k_\theta^{-1}\langle\tau\rangle = \frac{\tau\cos\theta - \sin\theta}{\tau\sin\theta + \cos\theta} \implies k_\theta^{-1}\langle i\rangle = i$)

$$\frac{\mathfrak{f}(i)}{e^{im\theta}} = e^{-i\ell\theta}\mathfrak{f}(i) \quad (\forall\theta).$$

This leaves us with the two possibilities $\ell = m$ or $\mathfrak{f}(i) = 0$.

Consider the first case. If Q is the quotient of two holomorphic functions satisfying (7.3) with $\ell = m$, then

$$Q(k_\theta^{-1}\langle\tau\rangle) = Q(\tau) \quad (\forall\theta),$$

which implies (by basic complex analysis) that Q is constant. Further, setting $\mathfrak{f}_{m,0}(\tau) := \frac{1}{(\tau+i)^m}$ we have

$$\mathfrak{f}_{m,0}|^m_{k_\theta^{-1}}(\tau) = \frac{1}{\left(\frac{\tau\cos\theta-\sin\theta}{\tau\sin\theta+\cos\theta} + i\right)^m (\tau\sin\theta + \cos\theta)^m} = \frac{e^{-im\theta}}{(\tau+i)^m}.$$

Therefore, $\mathfrak{f}_{m,0}$ is the generator of $L^2_{\text{hol}}(\mathfrak{H}, \mu_m)_{-m}$ *if* the latter is nonzero.

Turning to the second case, suppose $\mathfrak{f}(i) = 0$ (and ℓ possibly different from m). Then $F(\tau) := \frac{\mathfrak{f}(\tau)}{\mathfrak{f}_{m,0}(\tau)}$ satisfies

$$F(k_\theta^{-1}\langle\tau\rangle) = e^{-i(\ell-m)\theta}F(\tau).$$

This is only possible if $F(\tau)$ is a power of $\frac{\tau-i}{\tau+i}$. Since this function transforms under k_θ^{-1} by $e^{-2i\theta}$, $\ell-m$ must also be even. So we define

$$\mathfrak{f}_{m,\alpha}(\tau) := \left(\frac{\tau-i}{\tau+i}\right)^{\alpha}\frac{1}{(\tau+i)^m}. \tag{7.4}$$

It remains to consider the question of when $\mathfrak{f}_{m,\alpha}$ is L^2 with respect to $d\mu_m$. For $\mathfrak{f}_{m,0}$, we have

$$\int_{\mathfrak{H}}\frac{y^{m-2}}{|\tau+i|^{2m}}dx\wedge dy<\infty \quad\iff\quad m>1,$$

so that $L^2_{\mathrm{hol}}(\mathfrak{H},\mu_m)=\{0\}\iff m\leq 1$. For $\mathfrak{f}_{m,\alpha}$,

$$\int_{\mathfrak{H}}\left|\frac{\tau-i}{\tau+i}\right|^{2\alpha}\frac{y^{m-2}}{|\tau+i|^{2m}}dx\wedge dy<\infty \quad\iff\quad m>1 \text{ and } \alpha>-1,$$

i.e. for our purposes $m\geq 2$ and $\alpha\geq 0$. □

The proof gives at once the following

Corollary 7.3. *For $m\geq 2$, we have*

$$L^2_{hol}(\mathfrak{H},\mu_m)=\hat{\bigoplus}_{\alpha\geq 0}L^2_{hol}(\mathfrak{H},\mu_m)_{-(m+2\alpha)},$$

where the weight spaces on the RHS are 1-dimensional, with generator $\mathfrak{f}_{m,\alpha}$.

Exercise 7.4. (For readers who wish to check explicitly that the $\mathfrak{f}_{m,\alpha}$ give a basis for D^-_{m-1}.)

(a) Prove that composing $\Phi^{[m]}$ with complex conjugation and inversion $\Phi(g)\mapsto\Phi(g^{-1})$ gives an embedding $\left(\overline{L^2_{\mathrm{hol}}(\mathfrak{H},\mu_m)},\tilde{\pi}_m\right)\hookrightarrow(L^2(G),\pi)$.

(b) Show by explicit computation (using (5.4)) that $\mathcal{L}_{E_+}\overline{\Phi^{[m]}_{\mathfrak{f}_{m,\alpha}}(g^{-1})}=(m+\alpha)\overline{\Phi^{[m]}_{\mathfrak{f}_{m,\alpha+1}}(g^{-1})}$ and $\mathcal{L}_{E_-}\overline{\Phi^{[m]}_{\mathfrak{f}_{m,0}}(g^{-1})}$.

Now set

$$\phi_{m,\alpha,0}(g_{\tau,\theta}):=\Phi^{[m]}_{\mathfrak{f}_{m,\alpha}}(g_{\tau,\theta})=e^{im\theta}y^{\frac{m}{2}}\frac{(\tau-i)^\alpha}{(\tau+i)^{m+\alpha}}$$

and

$$\phi_{m,\alpha,\beta}:=\left(\mathcal{L}_{E_+}\right)^{\beta}\phi_{m,\alpha,0} \tag{7.5}$$

for all $\alpha,\beta\in\mathbb{Z}_{\geq 0}$.

Corollary 7.5. (7.5) *gives a countable collection of copies of* D^+_{m-1} *in* $(L^2(G),\pi)$ *indexed by* α. *More precisely,* $\widehat{span}\left\langle\{\phi_{m,\alpha,\beta}\}_{\beta\in\mathbb{Z}_{\geq 0}}\right\rangle$ *is the unique copy in* $L^2(G)_{-(m+2\alpha)}$.

Example 7.6. To see what some of these functions are, write

$$\mathfrak{f}_{m,\alpha,\beta} := (L_{E_+})^\beta \mathfrak{f}_{m,\alpha} \, (= f^{[m+2\beta]}\phi_{m,\alpha,\beta}).$$

Show that $\mathfrak{f}_{m,0,\beta}(\tau) = \frac{(m+\beta-1)!}{(m-1)!y^\beta}\frac{(\bar{\tau}+i)^\beta}{(\tau+i)^{m+\beta}}$. (You could also compute $\mathfrak{f}_{m,\alpha,1}$, in which case you will see that they get ugly rather quickly.)

A nice reference for the L^2 theory for SL_2 specifically, which includes (unlike these notes) a discussion of the continuous spectrum, is [Bo].

8. Appendix II: Poincaré series

The natural question at this point is whether we can link the story in §7 up with that in §5: how do we pass from the $\{\phi_{m,\alpha,\beta}\}$ to cuspidal automorphic forms? The answer, as in the construction of Eisenstein series in §3, is found in an averaging procedure. Again we take $\Gamma := SL_2(\mathbb{Z})$ for simplicity. Given a complex-valued function ϕ on G, the associated *Poincaré series* is the function on $\Gamma\backslash G$ defined by

$$\mathcal{P}_\phi := \sum_{\gamma\in\Gamma}\tilde{\pi}(\gamma)\phi$$

(i.e. $\mathcal{P}_\phi(g) := \sum\phi(\gamma^{-1}g)$), if this converges. We shall require some preliminary results before applying this to the functions in §7.

Lemma 8.1. *Assume* ϕ *is* $Z(\mathfrak{g})$*-, left-*K*-, and right-*K*-finite, and belongs to* $L^1(G)$. *Then:*

(i) $\mathcal{P}_\phi$ *converges absolutely and uniformly on compact sets;*
(ii) $\mathcal{P}_\phi$ *is bounded and smooth;*
(iii) $\mathcal{P}_\phi$ *defines an automorphic form for* Γ *(cf.* (3.1)*); and*
(iv) $\mathcal{P}_\phi\in L^1(\Gamma\backslash G)$.

Proof. (Sketch[10]) *(i)* is a consequence of discreteness of Γ (essentially topology); *(iv)* is because $\int_{\Gamma\backslash G}|\mathcal{P}_\phi|dg = \int_G|\phi|dg < \infty$; and (assuming *(ii)*) *(iii)* is

[10]complete details may be found in [Bo], §§2, 6.1.

trivial, as the sum doesn't interfere with the assumed $Z(\mathfrak{g})$- resp. right-K-finiteness.

The interesting bit is *(ii)*. First, the smoothness is a consequence of being $Z(\mathfrak{g})$- and either left- or right-K-finite (via a regularity result). Moreover, as a consequence of left-K-finiteness, one has an open neighborhood $U \subset G$ of id_G satisfying $U\gamma \cap U\sigma = \emptyset$ ($\forall$ distinct $\gamma, \sigma \in \Gamma$), and $\alpha \in C_c^\infty(G)$ supported on U, such that the convolution $\alpha * \phi = \phi$. Hence (writing $\tilde{g} = \gamma g h$) we have

$$\begin{aligned}\phi(\gamma g) = \int_G \alpha(\gamma g h)\phi(h^{-1})dh &= \int_G \alpha(\tilde{g})\phi(\tilde{g}^{-1}\gamma g)d\tilde{g} \\ \implies \ |\phi(\gamma g)| &\le \|\alpha\|_\infty \int_{U\gamma g} |\phi(h)|dh \\ \implies \ \sum_{\gamma\in\Gamma} |\phi(\gamma g)| &\le \|\alpha\|_\infty \int_G |\phi(h)|dh < \infty\end{aligned}$$

for every $g \in G$. ☐

Lemma 8.2. *(i) Under the same assumptions as in Lemma 8.1,*

$$\beta(g) := \int_{-\infty}^{\infty} \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx$$

is identically zero on G.

(ii) The same result holds with ϕ replaced by $\tilde{\pi}(\gamma)\phi$ for any $\gamma \in \Gamma$.

Proof. (Sketch) We will not need the left-K-finiteness. Since the other hypotheses are left-Γ-invariant, we need only prove *(i)* (following [Bo], §8.8). By an argument similar to that in the proof of Lemma 8.1, these remaining properties imply that ϕ is smooth and bounded; so β is at least well-defined.

As β is left-N-invariant and right-K-finite, it suffices to show $\beta|_A = 0$, where we recall

$$A = \left\{ \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} \middle| y \in \mathbb{R}_+ \right\} = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \middle| t \in \mathbb{R} \right\}.$$

We have $-2\Omega|_A = 2y^2\partial_y^2 = \frac{1}{2}\partial_t^2 - \partial_t$, and so $Z(\mathfrak{g})$-finiteness implies that $\beta|_A$ is a sum of terms which are (a) annihilated by operators of the form $\left(\frac{1}{2}\partial_t^2 - \partial_t - \lambda\right)^m =: D_{\lambda,m}$. Furthermore, the L^1 hypothesis on ϕ implies that these terms are (b) integrable with respect to $\frac{dy}{y^2} = -2e^{-2t}dt$; they are also (c) smooth and bounded (essentially because ϕ is).

Now if $s := \sqrt{1+2\lambda}$ then the functions solving $D_{\lambda,m}(\cdot) = 0$ are of the form

$$\begin{cases} P(t)e^{t(1+s)} + Q(t)e^{t(1-s)}, & s \neq 0 \\ P(t)e^t, & s = 0 \end{cases}$$

with $\deg P, \deg Q \leq m$. So (a) $\implies$ $\beta|_A$ is of this form, while the boundedness in (c) $\implies$

$$\begin{cases} \Re(s) = -1 \text{ and } P, Q \text{ constant}, & s \neq 0 \\ P \equiv 0, & s = 0. \end{cases}$$

But $\int_{-\infty}^{\infty} \text{const.} \cdot e^{it\theta} \cdot \frac{dt}{e^{2t}}$ does not converge (contradicting (b)) unless, of course, the constant is zero. □

Theorem 8.3. *Under the assumptions in Lemma 8.1, $\mathcal{P}_\phi \in {}^\circ\mathcal{A}(G,\Gamma)$.*

Proof. ([Bo], §8.9) Lemma 8.1 says that $\mathcal{P}_\phi \in \mathcal{A}(G,\Gamma)$, so it remains to check the cusp condition:

$$\begin{aligned} \int_0^1 \mathcal{P}_\phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx &= \int_{\{N\cap\Gamma\}\backslash N} \left(\sum_{\gamma\in\Gamma} \phi\left(\gamma^{-1}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right)\right) dx \\ &= \int_{-\infty}^{\infty} \left(\sum_{\gamma\in\{N\cap\Gamma\}\backslash\Gamma} \phi\left(\gamma^{-1}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right)\right) dx \\ &= \sum_{\gamma\in\{N\cap\Gamma\}\backslash\Gamma} \underbrace{\int_{-\infty}^{\infty} \phi\left(\gamma^{-1}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx}_{\dagger}. \end{aligned}$$

By Lemma 8.2, † is zero for each γ. □

Now the $\{\phi_{m,\alpha,\beta}\}$ are $Z(\mathfrak{g})$-, left-K, and right-K-finite. They are L^1 iff

$$\int_G |\phi_{m,\alpha,\beta}| dg = \int_G \left| y^{\frac{m+2\beta}{2}} e^{i(m+2\beta)\theta} \mathfrak{f}_{m,\alpha,\beta}(\tau) \right| \frac{dx\wedge dy\wedge d\theta}{2\pi y^2}$$

$$= \int_{\mathfrak{H}} y^{\frac{m}{2}+\beta-2} \left|\mathfrak{f}_{m,\alpha,\beta}(\tau)\right| dx\wedge dy \tag{8.1}$$

is finite. For $\beta = 0$, (8.1) is

$$\int_{\mathfrak{H}} y^{\frac{m}{2}-2} \left|\frac{\tau - i}{\tau + i}\right|^{\alpha} \frac{dx \wedge dy}{|\tau + i|^m} < \int_{\mathfrak{H}} y^{\frac{m}{2}-2} \frac{dx \wedge dy}{|\tau + i|^m}. \tag{8.2}$$

Exercise 8.4. Check that the right-hand side of (8.2) is finite $\iff$ $m \geq 3$.

Since infinitesimal right translation must preserve integrability, we have that $\phi_{m,\alpha,\beta} = \mathcal{L}_{E_+}^{\beta} \phi_{m,\alpha,0} \in L^1(G)$ for $m \geq 3$. Setting $\mathfrak{P}_{m,\alpha,\beta} := \mathcal{P}_{\phi_{m,\alpha,\beta}}$, we conclude the

Corollary 8.5. *The Poincaré series* $\mathfrak{P}_{m,\alpha,\beta}$ *belongs to* ${}^{\circ}\mathcal{A}(G, \Gamma)$ *for every* $m \geq 3$ *and* $\alpha, \beta \geq 0$.

We have had to do things this way because Poincaré series do not interact well with the L^2 condition.

The upshot is that $\mathcal{P}$ yields intertwining maps (for each $m \geq 3$) from the right regular sub-representations

$$\left(\left\{\widehat{\text{span}}\langle\{\phi_{m,\alpha,\beta}\}_{\alpha,\beta\geq 0}\rangle \subset L^2(G)\right\}, \pi\right) = \begin{matrix}\text{infinitely many copies} \\ \text{of } D_{m-1}^{+}\end{matrix}$$

to the Casimir eigenspaces

$$\left({}^{\circ}\mathcal{A}(G, \Gamma)\left[\frac{m}{2}\left(1 - \frac{m}{2}\right)\right]^{+}, \pi\right) = \begin{matrix}\text{finitely many copies} \\ \text{of } D_{m-1}^{+}\end{matrix}$$

(where the "+" singles out the D_{m-1}^{+} isotypical component). Note that the target space is (being generated by $\mathcal{L}_{E_+}^{\beta\geq 0} \circ \Phi^{[m]}$ of weight m cusp forms) actually *trivial* for m odd or less than 12, so in that case all the Poincaré series vanish identically. (This seems rather difficult to check by hand!) Moreover, a standard result is that these intertwining maps are surjective. We shall explain one way to see this at the end.

Using the correspondence $\tilde{\pi}_m \longleftrightarrow \tilde{\pi}$ (under $\Phi^{[m]}$ resp. $f^{[m]}$), we may view these maps on the level of functions on $\mathfrak{H}$. This yields intertwining maps

$$\left(\left\{\widehat{\text{span}}\langle\{\mathfrak{f}_{m,\alpha,\beta}\}_{\alpha,\beta\geq 0}\rangle \subset \hat{\bigoplus_{m\geq 0}} L^2(\mathfrak{H}, \mu_{m+2\beta})\right\}, \pi\right) \overset{\mathsf{p}}{\twoheadrightarrow}$$

$$\left({}^{\circ}C_{m+2\beta}^{\infty}(\mathfrak{H}, \Gamma)\left[\frac{m}{2}\left(1 - \frac{m}{2}\right)\right], \pi\right),$$

where p is computed *on each summand* by

$$\mathsf{p}_{\mathfrak{f}} := \sum_{\gamma\in\Gamma} \tilde{\pi}_{m+2\beta}(\gamma).\mathfrak{f},$$

i.e. $\mathsf{p}_{\mathfrak{f}}(\tau) = \sum f|_{\gamma^{-1}}^{m+2\beta}(\tau) = \sum \frac{f(\gamma^{-1}\langle\tau\rangle)}{(-c\tau+a)^{m+2\beta}}$ (where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$). By the absolute and uniform convergence on compact sets, we see that the $\mathfrak{f}_{m,\alpha,0} \in L^2_{\mathrm{hol}}(\mathfrak{H}, \mu_m)$ must go to holomorphic functions. (This is also clear from the fact that ${}^{\circ}C^{\infty}_m(\mathfrak{H},\Gamma)\left[\frac{m}{2}\left(1-\frac{m}{2}\right)\right] = S_m(\Gamma)$.) In fact, writing

$$P_{m,\alpha,\beta} := \mathsf{p}_{\mathfrak{f}_{m,\alpha,\beta}},$$

we have in particular

$$P_{m,\alpha,0}(\tau) = \sum_{\gamma\in\Gamma} \frac{\mathfrak{f}_{m,\alpha}\left(\gamma\langle\tau\rangle\right)}{(c\tau+d)^m} = \sum_{\gamma\in\Gamma} \frac{((a-ic)\tau+(b-id))^{\alpha}}{((a+ic)\tau+(b+id))^{\alpha+m}}.$$

Corollary 8.6. *The* $\{P_{m,\alpha,0}\}_{\alpha\geq 0}$ *span*[11] $S_m(\Gamma)$ *for each* $m \geq 3$.

Computing further, we break the sum into two stages: first, writing $\Gamma_0 = \Gamma \cap N$, we set

$$\mathcal{F}_{m,\alpha}(\tau) := \sum_{\gamma_0\in\Gamma_0} \mathfrak{f}_{m,\alpha}(\gamma_0\langle\tau\rangle) = \sum_{n\in\mathbb{Z}} \frac{(\tau+n-i)^{\alpha}}{(\tau+n+i)^{m+\alpha}};$$

then we average over cosets to obtain

$$P_{m,\alpha,0}(\tau) = \sum_{\gamma\in\Gamma_0\backslash\Gamma} \frac{\mathcal{F}_{m,\alpha}(\gamma\langle\tau\rangle)}{(c\tau+d)^m}.$$

Exercise 8.7. Use complex analysis to show that

$$\mathcal{F}_{m,\alpha}(\tau) = (-1)^{m-1}\sum_{k=0}^{\alpha} \frac{(2i)^k\binom{\alpha}{k}\pi^{m+k}}{(m+k-1)!}\cot^{(m+k-1)}(\pi(\tau+i)).$$

Compute the q-expansions of the $\cot^{(a)}(z) := \frac{d^a}{dz^a}\cot(z)$.

A more classical approach to Poincaré series may be found in [Gu, Chap. III] (and many other sources). This bypasses all the L^1 business and writes

[11] Of course, these are zero for $m < 12$.

down (for $a \geq 1$ rather than[12] $\alpha \geq 0$)

$$Q_{m,a}(\tau) := \sum_{\gamma \in \Gamma_0 \backslash \Gamma} \frac{\exp\left(2\pi i a \gamma\langle\tau\rangle\right)}{(c\tau+d)^m}.$$

That these span $S_m(\Gamma)$ is seen rather easily using the Petersson inner product.

Exercise 8.8. Using Exercise 8.7 above, deduce that after replacing $\{q^a = \exp(2\pi i a\tau)\}_{a\geq 1}$ by $\{\mathcal{F}_{m,\alpha}\}_{\alpha\geq 0}$, the result still spans the cusp forms; the surjectivity statements for the intertwining maps above follow.

As was the case with §7, [Bo] is a good reference for (some of) the approach we have taken here.

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