# CUP PRODUCTS IN AUTOMORPHIC COHOMOLOGY: THE CASE OF $Sp_4$

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Abstract. We study limits of discrete series with infinitesimal character zero for $Sp_4$: their $\mathfrak{n}$-cohomology and their contribution to "nonclassical" automorphic cohomology of the period domain for Hodge structures of mirror quintic type. As an application, we obtain the first generalization beyond $SU(2,1)$ of a result of [C1], showing that this cohomology can be reached by cup products of pairs of "classical" automorphic cohomology classes.

## 1. Introduction

In three very interesting and suggestive papers [C1, C2, C3], H. Carayol introduced new aspects of complex geometry and Hodge theory into the study of "nonclassical" automorphic representations — in particular, those involving the *totally degenerate limits of discrete series* (TDLDS). Focusing on test cases corresponding to anisotropic [C1, C2] resp. isotropic [C3] $\mathbb{Q}$-forms of $SU(2,1)$, the overriding theme of these works is the goal of producing arithmetic structures on the cohomology of non-algebraic generalizations of Shimura varieties. These generalizations, which he calls Griffiths-Schmid varieties (and which for us shall be "arithmetic quotients of Mumford-Tate domains"), together with their coherent cohomology, are the central object of study in this article.

In the rich history of results associating Galois representations to automorphic representations of reductive $\mathbb{Q}$-algebraic groups $G$, the technology has so far been mostly limited to representations whose archimedean component is in the discrete series (DS) of $G(\mathbb{R})$. For a brief discussion and references, see the introduction to [Go1]. A recent breakthrough of W. Goldring [op. cit.] extends its scope to holomorphic limits of DS; while for non-holomorphic nondegenerate limits, their occurrence in the coherent cohomology of Shimura varieties

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at least yields the "partial result" of algebraicity of their Hecke eigenvalues [Go2]. For degenerate limits, establishing such a partial result is especially important: for example, Langlands functoriality would "transfer" its validity for the $SU(2,1)$ TDLDS over to the principal series generated by Maass forms of eigenvalue $\frac{1}{4}$ [Go3]. Moreover, Carayol's goal may be the best current idea for proving the algebraicity of eigenvalues for degenerate limits, since their functorial stability [Go2] prevents any direct transfer of this sort from the nondegenerate to the degenerate case.

Our perspective on the (archimedean components of) automorphic representations we shall consider is heavily informed by Hodge theory. In complex algebraic geometry, Hodge (and mixed Hodge) structures are the linear-algebra objects that record the periods of integrals of differential forms on a variety. Mumford-Tate groups are the reductive linear-algebraic groups that describe the symmetries of these periods, and their orbits provide homogeneous classifying spaces for Hodge structures with given symmetries, generalizing period domains. The theory bifurcates into

• the classical case, where the classifying spaces are holomorphically fibered over Hermitian symmetric domains, and their arithmetic quotients are quasiprojective algebraic varieties fibered over Shimura varieties (and defined over a number field by work of Deligne, Milne, and others); and

• the nonclassical case, where the quotients are in general *not* algebraic varieties — in particular, powers of the canonical bundle have no sections. On the other hand, they do have nontrivial higher (Dolbeault) cohomology.

In either setting, the coherent cohomology of the quotients can be computed by a Matsushima-like formula in terms of automorphic representations, typically the discrete series and their limits. A result of Mirkovič [Mi] says that degenerate limits do not contribute in the classical case, which looks pretty bad at first if one wishes to understand arithmetic properties of automorphic representations like TDLDS. However, it turns out that there is a cornucopia of hidden arithmetic in nonclassical Mumford-Tate domains and their quotients, in spite of the non-algebraicity of the latter. Just at first glance, the fact that one has (as for Shimura varieties) a reflex field, an adelic description, and Hecke operators is highly suggestive. Moreover, they are endowed with a rich web of classical subdomains, such as Shimura curves.

H. Carayol discovered, in the special case of $SU(2,1)$, that there are at least three possible approaches to putting $\bar{\mathbb{Q}}$-structures on the coherent cohomology of the quotients. The first two use correspondences, resp. correspondences and cup products, to relate cohomology of Shimura varieties to nonclassical automorphic cohomology. The third approach computes, in the non-co-compact case – so that one has boundary components in the sense of [KU] – a new kind of Fourier coefficient in the cohomology of the boundary component quotients (which must be algebraic). Generalizing the latter beyond $SU(2,1)$ is one of the subjects of the author's joint work with G. Pearlstein [KP1, KP2].

In the present paper we shall focus on the co-compact setting. Though the pure correspondence (Penrose transform) approach above "succeeds" in producing an arithmetic structure, the only representations involved are discrete series. It is in conjunction with cup products that Penrose transforms allow us to describe, by the key result of [C1], all of the TDLDS-related Dolbeault cohomology in terms of spaces of Picard modular forms. Unfortunately, in order for this to provide a $\bar{\mathbb{Q}}$-structure on the former (and establish the algebraicity of Hecke eigenvalues for the TDLDS as well as Maass forms!), there remains the quite significant hurdle of checking that the cup-product kernels are defined over $\bar{\mathbb{Q}}$. For example, restricting the cup-product mechanism to Shimura curves does not work in any immediately obvious way, since the cup-product target is an $H^2$.

Anticipating the eventual removal of this obstruction, one naturally wonders how far Carayol's cup-product result generalizes. Using the framework developed in [GGK2], below we shall study Penrose transforms and cup products in the cohomology of arithmetic quotients of orbits of $Sp_4(\mathbb{R})$. Our principal focus is on the period domain $D = Sp_4(\mathbb{R})/H$ ($H$ a compact Cartan subgroup) for polarized Hodge structures of weight 3 and Hodge numbers $(1,1,1,1)$. The main result is a surjectivity statement for cup-products in the automorphic cohomology of $D$, in analogy to Carayol's result for $SU(2,1)$, cf. Theorem 5.4. Several technical lemmas, for instance on $\mathfrak{n}$-cohomology of TDLDS, are required, and we shall indicate methods of proof which appear likely to generalize to other groups. The better part of §§2-4 consists of a review of results from [GGK1] and [GGK2] for the reader's convenience and to establish notation, again sticking mainly to the co-compact case for simplicity.

*Acknowledgments:* It is a pleasure to thank Phillip Griffiths and Wushi Goldring for generously sharing their ideas and introducing me to Carayol's work and its context in the Langlands program. I am indebted to Henri Carayol and Anthony Knapp for their generous help with the argument in Lemma 5.5, to Griffiths for sharing his correspondence [Sc3] with Schmid (which helped me write §6.1), and to Schmid for his insights and advice. I would also like to express my appreciation to Peter Sarnak for his invitation to the IAS and to him and Richard Taylor for helpful discussions. Finally, the author gratefully acknowledges partial support from NSF Standard Grant DMS-1068974.

## 2. Review of Mumford-Tate groups and domains

Begin by fixing a finite-dimensional $\mathbb{Q}$-vector space $V$, an integer $n$, and a sequence $\underline{h} = \{h^{p,q}\}_{p+q=n}$ satisfying $|\underline{h}| := \sum h^{p,q} = \dim V$ and $h^{p,q} = h^{q,p}$.

**Definition 2.1.** A weight $n$ Hodge structure on $V$ with Hodge numbers $\underline{h}$, is a homomorphism

$$\varphi : S^1 \to SL(V_{\mathbb{R}})$$

with $\varphi(-1) = (-1)^n \mathrm{id}_V$, such that

$$V_\varphi^{p,q} := \left\{ z^{p-q}\text{-eigenspace of } \varphi(z) \right\} \subset V_{\mathbb{C}}$$

satisfies $\dim_{\mathbb{C}} V_\varphi^{p,q} = h^{p,q}$ $(\forall\, p, q)$.

This yields a Hodge decomposition

$$V_{\mathbb{C}} = \oplus_{p+q=n} V_\varphi^{p,q} \;\text{ with }\; \overline{V_\varphi^{p,q}} = V_\varphi^{q,p}$$

and (decreasing) Hodge filtration

$$F_\varphi^\bullet V_{\mathbb{C}} := \oplus_{p \geq \bullet} V_\varphi^{p,n-p}$$

with $\dim F_\varphi^i = f^i := \sum_{p\geq i} h^{p,n-p}$.

**Example 2.2.** The simplest nontrivial example is "$H^1(E_\tau)$", of type $\underline{h} = (1,1)$ where $E_\tau = \mathbb{C}/\langle 1, \tau\rangle$. Taking $V = H^1(E_\tau, \mathbb{Q}) = \mathbb{Q}\langle \alpha, \beta\rangle$, the Hodge decomposition is $V_\mathbb{C} = \mathbb{C}\langle\omega\rangle \oplus \mathbb{C}\langle\bar{\omega}\rangle$, where $\omega = \alpha + \tau\beta$.

Given $\varphi$, the tensor spaces $T^{k,\ell} := V^{\otimes k} \otimes \check{V}^{\otimes \ell}$ inherit HS of weight $n(k-\ell)$, giving rise to spaces of Hodge tensors

$$Hg^{k,\ell} := T^{k,\ell} \cap \left(T^{k,\ell}_\mathbb{C}\right)^\varphi$$

including the endomorphisms

$$T^{1,1} = End(V), \quad Hg^{1,1} = End(V, \varphi).$$

**Example 2.3.** Continuing Ex. 2.2, we have $\alpha \wedge \beta = \text{const.} \times \omega \wedge \bar{\omega} \in Hg^{2,0}$. If $\tau = i$, we also have$\alpha \otimes \alpha + \beta \otimes \beta = \frac{1}{2}(\omega \otimes \bar{\omega} + \omega \bar{\otimes} \omega) \in Hg^{2,0}$. In fact, there is an extra Hodge tensor if and only if $[\mathbb{Q}(\tau) : \mathbb{Q}] = 2$.

Next, let $Q : V \otimes V \to \mathbb{Q}$ be a $(-1)^n$-symmetric, nondegenerate bilinear form.

**Definition 2.4.** $\varphi$ is polarized by $Q$ if the Hodge-Riemann bilinear relations

$$\begin{array}{ll} \text{(HR I)} & \varphi(S^1) \subset Aut(V, Q) \\ \text{(HR II)} & Q(v, \varphi(i)\bar{v}) > 0 \ (\forall v \neq 0) \end{array}$$

hold.

Evidently we have $Q \in Hg^{0,2}$ and that HS of type $(1,1)$ are polarizable, though already type $(2,2)$ need not be.

Mumford-Tate groups were introduced [Mu] for abelian varieties and [De] for Hodge structures. Let $(V, Q, \varphi)$ be a polarized HS.

**Definition 2.5.** The MT group $G$ of $\varphi$ is (equivalently):

(i) the (largest) algebraic subgroup of $Aut(V, Q)$ fixing all $Hg^{a,b}$ pointwise;

(ii) the smallest $\mathbb{Q}$-algebraic subgroup of $Aut(V, Q)$ with real points containing $\varphi(S^1)$.

$G$ is reductive and absolutely connected. We single out two cases:

**Example 2.6.** (a) [simple groups] A simple real Lie group $\mathcal{G}$ arises as $G(\mathbb{R})$ with $G$ a MT group $\iff$ $\mathcal{G}$ has a compact maximal torus $\iff$ $\mathcal{G}$ has discrete series representations. [GGK1, H-C]

(b) [tori] A CM Hodge structure is one with abelian MT group: e.g., for "$H^1(E_\tau)$" we have

$$G \cong \begin{cases} U_{\mathbb{Q}(\tau)} & \text{if} \quad [\mathbb{Q}(\tau :)\mathbb{Q}] = 2 \\ SL_2 & \text{if} \quad [\mathbb{Q}(\tau) : \mathbb{Q}] > 2 \end{cases} .$$

We can use MT groups to construct homogeneous complex manifolds classifying HS with given Hodge tensors.

**Definition 2.7.** Let $(V, Q, \varphi)$ be a PHS with Hodge numbers $\underline{h}$ and MT group $G$. The corresponding MT domain is the orbit

$$D := G(\mathbb{R})^+.\varphi = \left\{ (V, Q, g\varphi g^{-1}) \middle| g \in G(\mathbb{R})^+ \right\} \cong G(\mathbb{R})^+/H_\varphi$$

(where the isotropy group $H_\varphi$ is compact).

From [GGK1] one knows that $D$ is the connected component through $\varphi$ of the MT Noether-Lefschetz locus

$$NL_\varphi := \left\{ \begin{array}{c} Q\text{-polarized HS on } V \text{ with} \\ \text{Hodge numbers } \underline{h} \text{ and MT group } G \end{array} \right\},$$

and also an analytic open subset in its compact dual

$$\check{D} := G(\mathbb{C}).F^\bullet_\varphi \cong G(\mathbb{C})/P_\varphi$$

(where $P_\varphi$ is parabolic), which is a projective variety defined over a number field.

**Example 2.8.** (a) CM Hodge structures are the 0-dimensional MT domains, and are always dense in $D$.

(b) Period domains $D_{\underline{h}}$, classifying HS with given $(V, Q, \underline{h})$, are MT domains with $G = Aut(V, Q)$. All other MT domains yield subdomains of period domains.

In the special case (related to the theory of Shimura varieties) where $D$ is Hermitian symmetric, $H_\varphi$ is *maximal* compact. More generally, we call $D$ classical if it fibers holomorphically or anti-holomorphically over a HSD.

**Example 2.9.** $D_{(2,2)}$ ($\cong$ Siegel's upper half space $\mathfrak{H}_2$ ) is Hermitian symmetric. $D_{(1,0,1,1,0,1)}$, which fibers holomorphically over $D_{(2,2)}$ by defining $V^{(1,0)}_{\text{new}} :=$

$V^{(5,0)}_{\text{old}} \oplus V^{(3,2)}_{\text{old}}$, is classical; while $D_{(1,1,1,1)}$ is non-classical. It fibers over $D_{(2,2)}$ via $V^{(1,0)}_{\text{new}} := V^{(2,1)}_{\text{old}} \oplus V^{(0,3)}_{\text{old}}$, but this mapping is non-holomorphic (i.e. neither holo nor anti).

Now let $S$ be a complex algebraic manifold.

**Definition 2.10.** A polarized variation of HS $\mathcal{V}$ of type $\underline{h}$ over $S$ comprises:

$$\begin{cases} V = \mathbb{Q}\text{-local system of rank } |\underline{h}| \text{ over } S, \\ Q : \mathbb{V} \times \mathbb{V} \to \mathbb{Q} \ ((-1)^n\text{-symmetric, nondegenerate), and} \\ \mathcal{F}^\bullet \subset \mathcal{V} := \mathbb{V} \otimes \mathcal{O}_S \text{ filtration by holomorphic sub-bundles,} \end{cases}$$

such that fibers are PHS of type $\underline{h}$, and the flat connection $\nabla$ annihilating $\mathbb{V}$ has $\nabla \mathcal{F}^\bullet \subset \mathcal{F}^{\bullet - 1} \otimes \Omega^1_S$.

Fixing $s_0 \in S$ and $V := \mathbb{V}_{s_0}$, we have the monodromy group

$$\Gamma := \text{image}\,\{\rho_{\mathbb{V}} : \pi_1(S) \to Aut(V, Q)(\mathbb{Q})\}$$

and the MT group $G \leq Aut(V, Q)$ of $\mathcal{V}$ [An]. The period map induced by $\mathcal{V}$

$$\Phi : S \to \Gamma \backslash D := \Gamma \backslash G(\mathbb{R})^+ / H$$

is locally liftable, holomorphic, and horizontal.

**Example 2.11.** [DR] There exists a family $\mathcal{X} \xrightarrow{\pi} \mathbb{P}^1 \backslash \{0, 1, \infty\} = S$ of smooth projective 6-folds and a sub-VHS $\mathcal{V} \subset R^6 \pi_* \mathbb{Q} \otimes \mathcal{O}_S$ with $\underline{h} = (1, 1, 1, 1, 1, 1, 1)$ and MT group $G$ with $G(\mathbb{R}) = \text{split } G_2$.

If $D$ is classical and $\Gamma$ is a neat congruence subgroup, then $\Gamma \backslash D$ is algebraic and fibers over a Shimura variety with compact fibers. If $D$ is non-classical, then it is expected in general that $\Gamma \backslash D$ is non-algebraic.

**Example 2.12.** If $\Gamma$ is co-compact in $G(\mathbb{R})$ and $D$ is non-classical, then it is known that $H^0(\Gamma \backslash D, \mathcal{O}(\mathcal{L})) = \{0\}$ for any nontrivial homogeneous holomorphic line bundle $\mathcal{L}$, with the consequence that there are no holomorphic automorphic forms.

In §§3-6 we shall assume that $\Gamma$ is co-compact.

Henceforth we restrict to the setting

$$D = G(\mathbb{R})/H \subset G(\mathbb{C})/P = \check{D}$$

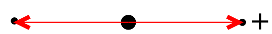
with $G(\mathbb{R})$ semisimple and $H$ a maximal torus. In this case, $P$ is a (general) Borel subgroup $B$, and $\check{D}$ is the flag variety of

$$\mathfrak{g} := \mathrm{Lie}(G(\mathbb{C})) = \underbrace{\mathfrak{h} \oplus \mathfrak{n}}_{\mathfrak{b}} \oplus \mathfrak{n}^+ = \mathfrak{h} \oplus \left( \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}^{-\alpha} \right) \oplus \left( \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}^{\alpha} \right).$$

Open $G(\mathbb{R})$-orbits $D$ in $\check{D}$, called flag domains, correspond to choices of $\Phi^+$ (or Weyl chamber) up to the action of $W_K$.

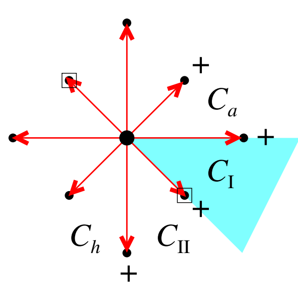
The key examples of Mumford-Tate domains in this paper are the following simple flag domains:

**Example 2.13.** ($\mathfrak{g} = sl_2$) $D_{(1,1)} \cong \mathfrak{H}$ and $\overline{\mathfrak{H}}$ are both of the form $SL_2(\mathbb{R})/U(1)$. $W_K$ is trivial and there are two inequivalent chambers:

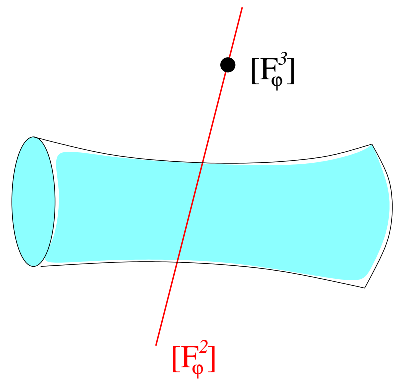



with $\mathfrak{H}$ corresponding to the positive one.

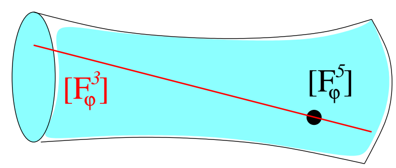
**Example 2.14.** ($\mathfrak{g} = sp_4$) $W_K \cong \mathbb{Z}_2$ and there are 4 equivalence classes of chambers:




$D_{\mathrm{I}} \cong D_{(1,1,1,1)}$ (corr. to the positive chamber), $D_h \cong D_{(1,0,1,1,0,1)}$, $D_{\mathrm{II}} \cong \overline{D_{\mathrm{I}}}$ and $D_a \cong \overline{D_h}$ (with $D_a$ and $D_h$ classical) are all of the form $Sp_4(\mathbb{R})/U(1)^{\times 2}$. They each parametrize polarized HS on a $\mathbb{Q}$-vector space $V$ of dimension 4 equipped with an alternating form $Q$. Points $\varphi$ of $D_{\mathrm{I}}$ correspond to indefinite Lagrange (i.e. $Q$-isotropic) flags in $\mathbb{P}V$:




where the solid denotes the locus where $-iQ(v,\bar{v}) < 0$. The points $\varphi \in D_h$ correspond to definite Lagrange flags:




**Example 2.15.** [C1, C2, C3] ($\mathfrak{g} = su(2,1)$) Let $V$ be a 6-dimensional $\mathbb{Q}$-vector space equipped with an alternating nondegenerate form $Q : V \times V \to \mathbb{Q}$ and an action

$$\mu : \mathbb{F} := \mathbb{Q}(\sqrt{-d}) \hookrightarrow End_{\mathbb{Q}}(V)$$

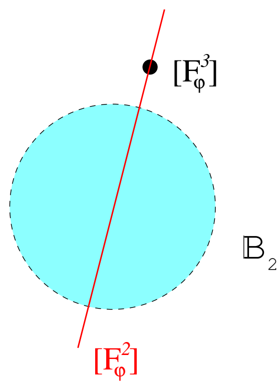
of a quadratic field, such that the eigenspaces of $\mu$ are $Q$-isotropic: writing $V_{\mathbb{F}} = V_+ \oplus V_-$, we have $Q(V_+, V_+) = 0$. Consider $Q$-polarized Hodge structures $\varphi$ on $V$ with Hodge numbers $\underline{h} = (1,2,2,1)$, subject to the constraints that $\mu(\mathbb{F}) \subset End(V,\varphi)$ and $\dim(V_+^{3,0}) = \dim(V_+^{2,1}) = \dim(V_+^{1,2}) = 1$. The full picture of the Hodge decomposition is then given by

| | $(3,0)$ | $(2,1)$ | $(1,2)$ | $(0,3)$ |
|---|---|---|---|---|
| $V_+$ | 1 | 1 | 1 | 0 |
| $V_-$ | 0 | 1 | 1 | 1 |

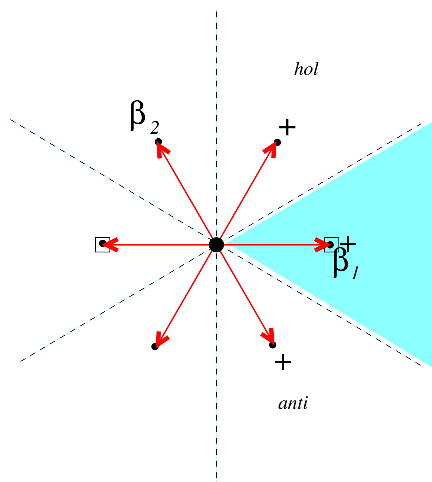
and the $\mathbb{F}$-Hermitian form $\langle v,w \rangle := -\sqrt{-d}Q(v,w)$ has signature $(2,1)$. Hence $G := Aut(V,Q) \cap Res_{\mathbb{F}/\mathbb{Q}}(GL(V_+))$ has $G(\mathbb{R}) \cong U(2,1)$, and the domain parametrizing all such Hodge structures is given by

$$D = U(2,1).\varphi \cong SU(2,1)/U(2).$$

The points $\varphi \in D$ are in 1-to-1 correspondence with flags in $\mathbb{P}V_+$ of the form




where $\mathbb{B}_2 := \{[v] | \langle v, v\rangle < 0\}$ (with closure $\mathbb{B}_2^-$), $[F^3_{\varphi,+}]$ is outside $\mathbb{B}_2^-$ and $[F^2_{\varphi,+}]$ intersects $\mathbb{B}_2$. In the root diagram




there are 3 chamber-classes, and $D$ correlates to the shaded one. Thinking of $\mathbb{B}_2$ as $\mathbb{F}$-multiplication HS of weight 1 with $\dim(V_+^{1,0}) = 1$, $\dim(V_+^{0,1}) = 2$, taking $V^{1,0}_{\text{new},+} := V^{2,1}_{\text{old},+}$ maps $D$ non-holomorphically onto $\mathbb{B}_2$. A domain $D_{\text{hol}}$ mapping holomorphically to the ball, and accordant with the top chamber, is obtained by repeating the construction with weight 5 and $1 = \dim(V_+^{3,2}) = \dim(V_+^{2,3}) = \dim(V_+^{0,5})$.

It is not yet clear whether the main result of this paper should extend to exceptional (especially non-Hermitian) groups, but there is at least a setting where one could try:

**Example 2.16.** [GGK1] ($\mathfrak{g} = g_2$) Consider a 7-dimensional $\mathbb{Q}$-vector space, with a symmetric bilinear form $Q$ of signature $(3,4)$ and a 3-tensor $\mathfrak{t} \in V^{\otimes 3}$ such that $G := Aut(V, Q, \mathfrak{t})$ is (a $\mathbb{Q}$-form of) split $G_2$. The possible gap-free Hodge numbers of a $Q$-polarized HS on $V$ with MT group $G$, are

$$\underline{h} = (2,3,2),\ (1,2,1,2,1),\ \text{and}\ (1,1,1,1,1,1,1).$$

The latter type of Hodge structure is parametrized by a simple flag domain, with TDLDS contributing to the automorphic cohomology.

## 3. Cohomology and automorphic cohomology

We continue to consider a MT domain $D = G(\mathbb{R})/H$, with $G$ semisimple, $H$ a compact Cartan subgroup, and $d := \dim_{\mathbb{C}} D$. Fix a compact submanifold $Z = K/H$, where $K \leq G(\mathbb{R})$ is a maximal compact subgroup containing $H$, and set $s := \dim_{\mathbb{C}} Z$. The weight lattice

$$\Lambda = \{\lambda \in \mathfrak{h}_{\mathbb{C}}^* \,|\, \langle \lambda, h \rangle \in 2\pi i \mathbb{Z}\ \forall h \in \ker(\exp) \subset \mathfrak{h}\} \xrightarrow[\cong]{\exp} X^*(H)$$

contains the roots $\Phi$, and we shall assume that it also contains

$$\rho := \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha,$$

i.e. that $G(\mathbb{R})$ is acceptable. (For $G = SL_2$, we will write "1" for $\rho$.) $W$ [resp. $W_K$] will denote the Weyl [resp. compact Weyl] group.

To each weight $\mu \in \Lambda$ we associate a $G(\mathbb{R})$-equivariant holomorphic line bundle

$$\mathcal{L}_\mu := G(\mathbb{R}) \times_H \mathbb{C}_\mu = \frac{G(\mathbb{R}) \times \mathbb{C}}{(ge^h, z) \sim (g, e^{\langle \mu, h \rangle} z)} \longrightarrow D.$$

Writing $\check{D} = G(\mathbb{C})/B$, we have $T^{1,0}_{F^\bullet} D \cong \mathfrak{g}/\mathfrak{b} \cong \mathfrak{n}^+ \implies K_D|_{F^\bullet} \cong \bigwedge^d \check{\mathfrak{n}}^+$ hence

$$\mathcal{L}_{-2\rho} \cong K_D.$$

The homogeneous square root $\mathcal{L}_{-\rho}$ of the canonical bundle, whose existence is guaranteed by our assumption, plays a key supporting role in the appearance of TDLDS in the automorphic cohomology of MT domains.

In the present setting, the contrast between classical and non-classical is quite vivid:

**Example 3.1.** Let $k > 0$.

(a) For $G = SL_2$ (so $D = \mathfrak{H}$), $H^0(D, \mathcal{O}(\mathcal{L}_{-k}))^\Gamma$ is the space $M_k(\mathfrak{H}, \Gamma)$ of (holomorphic) modular forms of weight $k$.

(b) For $G$ not a product of copies of $SL_2$, $H^0(D, \mathcal{O}(\mathcal{L}_{-k\rho})) = \{0\}$.

However, in case (b) there are nontrivial higher cohomologies. The simplest statement (Theorem 3.2 below), conjectured by Langlands and then proved by Schmid, is actually obtained by looking at $L_2$ cohomology (with respect to the $G(\mathbb{R})$-invariant Hermitian metrics on $D$ and $\mathcal{L}_\mu$ provided by the Killing form $(\ ,\ )$) as a representation of $G(\mathbb{R})$ under left translation.

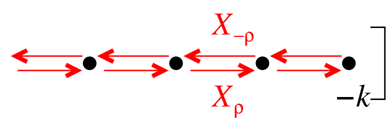
The discrete series $\{\mathfrak{V}_\lambda\}$ (indexed by $\lambda \in \frac{\Lambda^{\text{reg}}}{W_K}$) are the (unitary, infinite-dimensional) irreducible representations of $G(\mathbb{R})$ appearing in the discrete spectrum of its left-regular representation on $L^2(G(\mathbb{R}))$. For $\lambda \in \Lambda^{\text{reg}}$, write

$$q(\lambda) := \left|\left\{\alpha \in \Phi_c^+ \,|\, (\lambda, \alpha) < 0\right\}\right| + \left|\left\{\beta \in \Phi_n^+ \,|\, (\lambda, \beta) > 0\right\}\right|$$

where $\Phi_c$ denotes the compact roots and $\Phi_n$ the noncompact ones. (We also set $\rho_c := \frac{1}{2}\sum_{\alpha \in \Phi_c^+} \alpha$ and $\rho_n := \frac{1}{2}\sum_{\beta \in \Phi_n^+} \beta$.)

**Theorem 3.2.** [Sc2] *$H^p_{(2)}(D, \mathcal{O}(\mathcal{L}_\mu))$ is isomorphic (as a $G(\mathbb{R})$-representation) to $\mathfrak{V}_{\mu+\rho}$ if $\mu + \rho$ is a regular weight and $p = q(\mu + \rho)$; otherwise, it is zero.*

**Example 3.3.** ($G = SL_2$) We can easily visualize $\mathfrak{D}^-_{k-1} := \mathfrak{V}_{-k+1}$ as an $sl_2$-representation:




By Theorem 3.2, it is isomorphic to $H^0_{(2)}(\mathfrak{H}, \mathcal{O}(\mathcal{L}_{-k}))$. To make the latter more concrete, set

$$L^2_{\text{hol}}(\mathfrak{H}, \sigma_k) := \left\{ f \in \mathcal{O}(\mathfrak{H}) \,\middle|\, \int_{\mathfrak{H}} |f|^2 y^{k-2} dx \wedge dy < \infty \right\}$$

and define

$$\pi_k : SL_2(\mathbb{R}) \to Aut\left(L^2_{\text{hol}}(\mathfrak{H}, \sigma_k)\right)$$

by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = g \longmapsto \left\{ f(\tau) \mapsto \frac{f(\frac{a\tau+b}{c\tau+d})}{(c\tau+d)^k} \right\}.$$

Dropping the $L^2$ condition gives the Frechét space $MG(D^-_{k-1})$, the (non-unitary) maximal globalization of this (unitary) discrete series representation.

**Theorem 3.4.** [Sc1] *$H^p(D, \mathcal{O}(\mathcal{L}_\mu))$ is always the maximal globalization of its underlying Harish-Chandra (algebraic $(\mathfrak{g}, K)$-) module. This is also the MG of $H^p_{(2)}(D, \mathcal{O}(\mathcal{L}_\mu))$ $(= \mathfrak{V}_{\mu+\rho})$ if $\mu + \rho$ is regular and*

*(i) $\mu + \rho$ is antidominant (and $\rho = s$), or*

*(ii) [holo. discrete series case] $p = q(\mu + \rho) = 0$ and $D$ is classical.*

*In both cases, $H^p_{(2)}$ is dense in $H^p$. For $p > s$, $H^p(D, \mathcal{O}(\mathcal{L}_\mu)) = \{0\}$.*

We shall write $V_\lambda$ for the underlying Harish-Chandra ($(\mathfrak{g}, K)$-)module of $\mathfrak{V}_\lambda$ (and of its MG), i.e. the space $(\mathfrak{V}_\lambda)_{K\text{-fin}}$ of $K$-finite vectors. The infinitesimal character $\chi_V : Z(\mathfrak{g}) \to \mathbb{C}$ describes the action on a HC module $V$ of the center of $U(\mathfrak{g})$, which may be viewed as invariant differential operators. For any $\lambda \in \mathfrak{h}^*_{\mathbb{C}}$, $\chi_\lambda$ is defined by composing the natural projection $Z(\mathfrak{g}) \twoheadrightarrow Z(\mathfrak{h}_{\mathbb{C}}) = \text{Sym}(\mathfrak{t}_{\mathbb{C}})$ with evaluation at $\lambda - \rho$. We note (in the discrete series case) $\chi_{V_\lambda} = \chi_\lambda$ for $\lambda \in \Lambda^{\text{reg}}$, and (in general) that $\chi_\lambda = \chi_{\lambda'} \iff \lambda = w(\lambda')$ for some $w \in W$. There are finitely many unitary irreps with any given infinitesimal character. See [Kn1] as a good general reference, and [Sc1, GGK2] for a discussion of infinitesimal characters in our context.

The description of representations as Dolbeault cohomology groups in Theorem 3.4 facilitates their decomposition into $K$-types. Writing $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{k} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$, we have for the (co)normal bundle along $Z = K/H \subset D$

$$\mathcal{N}_{Z/D} \cong K \times_H \mathfrak{p}^+ \text{ and } \mathcal{N}^\vee_{Z/D} \cong K \times_H \check{\mathfrak{p}}^+.$$

In case $(i)$ of Theorem 3.4 (and assuming[1] $(\Phi_n^-, \Phi_c^+) \leq 0$), by Borel-Weil-Bott for $Z$ we have

$$H^{s-1}(Z, Sym^n \mathcal{N}_{Z/D}^{\vee} \otimes \mathcal{O}_Z(\mathcal{L}_\mu)) = \{0\} \quad (\forall n \geq 0).$$

Together with the Theorem, this yields short exact sequences

$$\begin{array}{c} 0 \to H^s(D, \mathcal{I}_Z(\mathcal{L}_\mu)) \to H^s(D, \mathcal{O}_D(\mathcal{L}_\mu)) \to H^s(Z, \mathcal{O}_Z(\mathcal{L}_\mu)) \to 0 \\ 0 \to H^s(D, \mathcal{I}_Z^2(\mathcal{L}_\mu)) \to H^s(D, \mathcal{I}_Z(\mathcal{L}_\mu)) \to H^s(Z, \mathcal{N}_{Z/D}^{\vee} \otimes \mathcal{O}_Z(\mathcal{L}_\mu)) \to 0 \\ \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \end{array}$$

and hence an isomorphism of representations of $K$

$$H^s(D, \mathcal{O}(\mathcal{L}_\mu))_{K\text{-fin}} \cong \bigoplus_{n \geq 0} \underbrace{H^s(Z, Sym^n \mathcal{N}_{Z/D}^{\vee} \otimes \mathcal{O}_Z(\mathcal{L}_\mu))}_{=: \mathsf{W}_n} \tag{3.1}$$

with "minimal" $K$-type $\mathsf{W}_0$. (Note that for $n > 0$ the $\{\mathsf{W}_n\}$ may be reducible as representations of $K$.)

Next, consider the setting of case $(ii)$ of Theorem 3.4, with $\mu + \rho$ regular, $q(\mu + \rho) = 0$, and a holomorphic fibration

$$\begin{array}{ccccc} Z & \hookrightarrow & D & = & G(\mathbb{R})/H \\ \downarrow & & \downarrow \pi & & \\ \{x\} & \hookrightarrow & X & = & G(\mathbb{R})/K. \end{array}$$

Writing $\mathcal{W}$ for the holomorphic vector bundle with fibers

$$\mathcal{W}_x = H^0\left(\pi^{-1}(x), \mathcal{O}_{\pi^{-1}(x)}(\mathcal{L}_\mu)\right),$$

the above argument with $s = 0$ and $Z \subset D$ replaced by $\{x\} \subset X$ yields

$$H^0(D, \mathcal{O}(\mathcal{L}_\mu))_{K\text{-fin}} \underset{\pi^*}{\overset{\cong}{\leftarrow}} H^0(X, \mathcal{O}(\mathcal{W}))_{K\text{-fin}} \cong \bigoplus_{n \geq 0} \underbrace{Sym^n \check{\mathfrak{p}}^+ \otimes \mathcal{W}_x}_{=: \mathsf{W}_n} \tag{3.2}$$

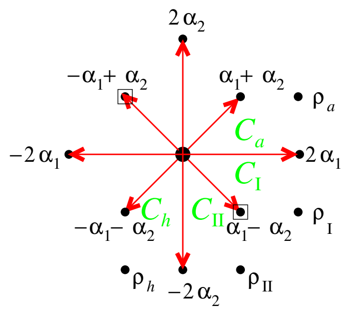
The action of $\mathfrak{g}$ on the right-hand side of (3.1) resp. (3.2) can be read off from the "differentiation" map

$$\mathfrak{p}^+ \otimes Sym^n \check{\mathfrak{p}}^+ \to Sym^{n-1} \check{\mathfrak{p}}^+$$

[1] This is satisfied for $Sp_4$ and $SU(2,1)$, but not (for example) for $G_2$; while convenient here, this assumption is too strong in general and we expect there is a way around it.

and duality under the Killing form, recovering the full $(\mathfrak{g}, K)$-module structure, see [GG].

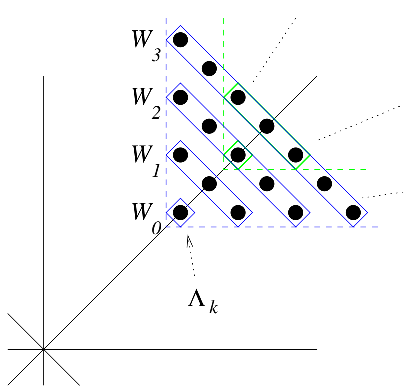
Either approach $(i)$ or $(ii)$ recovers Example 3.3 (for $SL_2$), and (3.2) is applied to $SU(2,1)$ in §IV.F of [GGK2]. Here we shall describe two examples for $G = Sp_4$ which are essential in §§4-6, using the root diagram




in which $\rho_I = 2\alpha_1 - \alpha_2$, etc. It is easy to identify irreducible representations of $K = U(2)$ by their pullback to $U(1) \times SU(2)$. Let $\mathcal{S}_j(i)$ denote the $j^{\text{th}}$ standard irrep of $SU(2)$ twisted by the character of weight $\frac{i}{2}(\alpha_1 + \alpha_2)$; this is a pullback if and only if $2|i + j$.

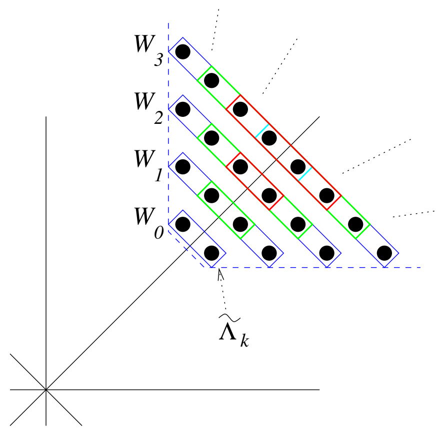
**Example 3.5.** We first use (3.2) to decompose two holomorphic discrete series. Writing $\Lambda_k := k(\alpha_1 + \alpha_2)$ and $\tilde{\Lambda}_k := k(\alpha_1 + \alpha_2) + \alpha_1$ ("Blattner parameters"), we have $V_{(k)} :=$

$$V_{\Lambda_k + \rho_h} = H^0(D_h, \mathcal{O}(\mathcal{L}_{\Lambda_k}))_{K\text{-fin}} \cong \bigoplus_{n \geq 0} \left( \bigoplus_{m=0}^{\lfloor \frac{n}{2} \rfloor} \mathcal{S}_{2(n-2m)}(2n + 2k) \right)$$



and $\tilde{V}_{(k)} :=$

$$V_{\tilde{\Lambda}_k+\rho_h} = H^0(D_h, \mathcal{O}(\mathcal{L}_{\tilde{\Lambda}_k}))_{K\text{-fin}} \cong \bigoplus_{n\geq 0}\left(\bigoplus_{m=0}^{n} \mathcal{S}_{2(n-m)+1}(2n+2k+1)\right).$$




For later use we introduce the filtration

$$\mathsf{F}_a V_{(k)} := \bigoplus_{n\geq 0}\left(\bigoplus_{m=0}^{\min\{\lfloor \frac{n}{2}\rfloor, a\}} \mathcal{S}_{2(n-2m)}(2n+2k)\right) \tag{3.3}$$

by sub-representations of $K$.

The next example concerns limits of discrete series (LDS). These are tempered irreps parametrized by pairs $(\lambda, C)$, with $\lambda$ in the (closed) chamber $C$ and not orthogonal to any $C$-simple compact root, modulo the action of $W_K$ [CK]. To briefly summarize their construction (cf. [Kn2] for details), note that the HC module of any finitely generated admissible representation has decomposition of the form

$$V = \bigoplus_{i=1}^{n} p_{\lambda_i} V := \bigoplus_{i=1}^{n} \ker\left((z - \chi_{\lambda_i}(z))^d\right),$$

where we regard each $p_\lambda$ as an endofunctor on the category of HC modules. This implies, by considering (for each $i$) quotients of successive $\ker\left((z - \chi_{\lambda_i}(z))^e\right)$ ($e \leq d$), a composition series for $V$. Combining the projections with the operation of tensoring with the finite dimensional representation of lowest weight $-\nu$, Zuckerman [Zu] defined exact tensor endofunctors

$$\psi_\lambda^{\lambda+\nu} := p_\lambda[(\cdot)\otimes \mathfrak{W}_{-\nu}]p_{\lambda+\nu}$$

"translating" the infinitesimal character by $-\nu$. If $\lambda$ and $\nu$ are ($C$-)dominant integral weights, with $\lambda$ singular and $\lambda+\nu$ regular, the HC module

$$\psi_\lambda^{\lambda+\nu}\left(V_{\lambda+\nu}\right) := V_{(\lambda,C)}$$

underlies a tempered irrep (independent of $\nu$) with infinitesimal character $\chi_\lambda$. In particular, the TDLDS are given by the "formula"

$$V_{(0,C)} := p_0\left[V_\rho \otimes \mathfrak{W}_{-\rho}\right].$$

A limit of DS $V_{(\lambda,C)}$ is called "degenerate" if $\lambda$ is orthogonal to one or more (necessarily non-simple) compact roots. "Totally degenerate" simply means[2] $\lambda=0$, and $Sp_4$ has two distinct TDLDS, with HC modules $V_{(0,C_{\mathrm{I}})}$ and $V_{(0,C_{\mathrm{II}})}$ (as the reader can check). In general, it is fairly easy to show that if no compact root is $C$-simple, then the domain corresponding to $C$ is non-classical.[3]

**Example 3.6.** Specializing back to $Sp_4$, we shall prove that Theorem 3.4 "extends to the limit", allowing us to obtain $K$-type decompositions for $V_{(0,C_{\mathrm{II}})}$ and $V_{(0,C_{\mathrm{I}})}$. First note that in the calculation (3.1), we really only need $\mu+\rho_c$ regular antidominant (and only with respect to $\Phi_c^+$ at that). Hence it yields a $K$-type decomposition for $H^1(D_I,\mathcal{O}(\mathcal{L}_{-\rho_{\mathrm{I}}}))_{K\text{-fin}}$ with $\mathsf{W}_0=\mathcal{S}_1(-1)$, which in particular implies its irreducibility. Moreover, by [Ag] (cf. [Sc1, sec. 5]) it has infinitesimal character $\chi_0$.

Now $V_{(0,C_{\mathrm{II}})}$ is the $\chi_0$-summand of $\mathfrak{W}_{-\rho_{\mathrm{II}}}\otimes V_{\rho_{\mathrm{II}}}$, or equivalently (using $w\in W_K$) of

$$\mathfrak{W}^{\rho_{\mathrm{I}}}\otimes V_{-\rho_{\mathrm{I}}} \cong H^0(\check{D},\mathcal{O}(\mathcal{L}_{\rho_{\mathrm{I}}}))\otimes H^1(D_{\mathrm{I}},\mathcal{O}(\mathcal{L}_{-2\rho_{\mathrm{I}}})),$$

where the isomorphism is by Theorem 3.4 and the Borel-Weil theorem [Sc1]. Restricting to $\mathbb{P}^1\cong Z\subset D_{\mathrm{I}}$ produces a diagram

$$\begin{array}{ccc} H^0(\check{D},\mathcal{O}(\mathcal{L}_{\rho_{\mathrm{I}}}))\otimes H^1(D_{\mathrm{I}},\mathcal{O}(\mathcal{L}_{-2\rho_{\mathrm{I}}}))_{K\text{-fin}} & \longrightarrow & H^1(D_{\mathrm{I}},\mathcal{O}(\mathcal{L}_{-\rho_{\mathrm{I}}}))_{K\text{-fin}} \\ \downarrow & & \downarrow \\ H^0(\mathbb{P}^1,\mathcal{O}(3)))\otimes H^1(\mathbb{P}^1,\mathcal{O}(-6)) & \longrightarrow & H^1(\mathbb{P}^1,\mathcal{O}(-3)) \end{array} \tag{3.4}$$

[2]For $SL_2(\mathbb{R})$, owing to the lack of a compact root, this creates the terminological peculiarity that the two LDS $D_0^\pm$ are both "nondegenerate" and "totally degenerate".

[3]Indeed, this follows at once from the observation that $[\mathfrak{p}^-,\mathfrak{p}^-]=\mathfrak{n}_c \implies [\mathfrak{p}^-,\mathfrak{p}^-]\neq 0 \implies$ $D$ cannot fiber holomorphically over a Hermitian symmetric domain.

in which the top row is an HC-module morphism and (using Serre duality and writing $\mathbb{C}[X,Y]=\oplus_{d\geq 0}S^d$) the bottom row is the obvious map $S^3\otimes\check{S}^4\to\check{S}^1$. Since the multiplication $S^3\otimes S^1\to S^4$ has no right kernel, this map is surjective, and so the top row of (3.4) is nonzero. From the Schur lemma it is now clear that its composition with the inclusion of $V_{(0,C_{\mathrm{II}})}$ is an isomorphism.

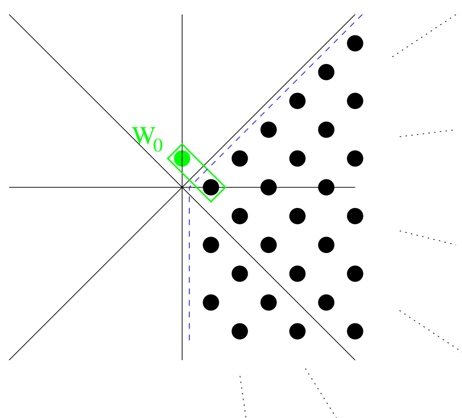
We conclude that

$$V_{(0,C_{\mathrm{II}})}\cong H^1(D_{\mathrm{I}},\mathcal{O}(\mathcal{L}_{-\rho_{\mathrm{I}}}))_{K\text{-fin}}\cong \underbrace{\mathcal{S}_1(-1)}_{\mathsf{W}_0^{\mathrm{II}}}\oplus\left(\bigoplus_{n\geq 1}\mathsf{W}_n^{\mathrm{II}}\right)$$

and, by a symmetric argument,

$$V_{(0,C_{\mathrm{I}})}\cong H^1(D_{\mathrm{II}},\mathcal{O}(\mathcal{L}_{-\rho_{\mathrm{II}}}))_{K\text{-fin}}\cong \underbrace{\mathcal{S}_1(1)}_{\mathsf{W}_0^{\mathrm{I}}}\oplus\left(\bigoplus_{n\geq 1}\mathsf{W}_n^{\mathrm{I}}\right)$$

Less cluttered than the full diagram of $K$-types is this picture of their highest weights, which for $V_{(0,C_{\mathrm{I}})}$ is:




The diagram for $V_{(0,C_{\mathrm{II}})}$ is exactly the reflection of this one in $\alpha_1+\alpha_2$. Notice that between them, the two TDLDS have as their $K$-types all of the irreps of $K$ of even dimension (and none of odd dimension).

Here as for $V_{(k)}$, we introduce a filtration by taking $\mathsf{F}_aV_{(0,C_{\mathrm{I}})}$ to be the sum of all $K$-types with highest weight $m\alpha_1+n\alpha_2$ with $m\leq a+1$. We note only that the above extension shows that

$$\mathsf{F}_0V_{(0,C_{\mathrm{I}})}=\bigoplus_{n\geq 0}\mathcal{S}_{2n+1}(1-2n), \tag{3.5}$$

where in particular each $K$-type appears with multiplicity one.

*Remark* 3.7. In general, $H^*(D, \mathcal{O}(\mathcal{L}_\mu))$ is a right $G(\mathbb{R})$-module, whereas below we shall be interested in the Lie algebra cohomology of irreducible summands $V_\pi$ of $\mathcal{A}(G(\mathbb{R}), \Gamma)$, a left $G(\mathbb{R})$-module. With this caveat firmly in mind, we can still apply the $K$-type decompositions derived by the above methods to the $\{V_\pi\}$.

Turning to automorphic cohomology, let $\Gamma$ be an arithmetic subgroup in $G(\mathbb{Q})$, co-compact in $G(\mathbb{R})$. Recalling that $d = \dim_{\mathbb{C}}(D)$, there is the basic

**Theorem 3.8.** [GS] *For $\mu$ regular and $k \gg 0$,*

$$H^p(\Gamma\backslash D, \mathcal{O}(\mathcal{L}_{k\mu})) \begin{cases} = \quad 0 & , \text{ for } p \neq q(\mu) \\ \\ \sim C_\mu vol(\Gamma\backslash D) \cdot k^d & , \text{ for } p = q(\mu) \end{cases} .$$

For a more precise result, introduce *automorphic forms*

$$\mathcal{A}(G(\mathbb{R}), \Gamma) := C^\infty(\Gamma\backslash G(\mathbb{R}))_{K, Z(\mathfrak{g})\text{-fin}} \cong \bigoplus_{\pi \in \widehat{G(\mathbb{R})}} V_\pi^{\oplus m_\pi(\Gamma)},$$

where the direct sum may include more than discrete series, and the action of $G(\mathbb{R})$ is now by *right* translation. Denoting the unitary dual of $G(\mathbb{R})$ by $\widehat{G(\mathbb{R})}$ and the infinitesimal character of $V_\pi$ by $\chi_\pi$, we have

**Theorem 3.9.** [Wi1] *For any weight $\mu$, we have*

$$H^p(\Gamma\backslash D, \mathcal{O}(\mathcal{L}_\mu)) \cong \bigoplus_{\left\{ \begin{array}{c} \pi \in \widehat{G(\mathbb{R})} : \\ \chi_\pi = \chi_{-(\mu+\rho)} \end{array} \right.} H^p(\mathfrak{n}, V_\pi)_{-\mu}^{\oplus m_\pi(\Gamma)},$$

*where the (finitely many) direct summands of the right-hand side are (the $-\mu$ weight-subspaces of) Lie-algebra cohomology groups.*

*Proof.* (Sketch) Expand the left-hand side as Dolbeault cohomology

$$H^p \left\{ A^{0,\bullet}\left(\Gamma\backslash D, \mathcal{L}_\mu\right), \bar{\partial} \right\}.$$

Since $\mathfrak{n}(= \mathfrak{n}^-) \cong T^{0,1}_{F^\bullet} D$ and $D = G(\mathbb{R})/H$, this may be recast (as in [C1]) as

$$H^p \left\{ Hom_H \left( \bigwedge\nolimits^{\bullet} \mathfrak{n}, C^\infty\left(\Gamma\backslash G(\mathbb{R})\right) \otimes \mathbb{C}_\mu \right) \right\}$$

$$\cong H^p\left(\mathfrak{n}, C^\infty(\Gamma\backslash G(\mathbb{R}))\right)_{-\mu}$$
$$\cong \bigoplus_{\pi\in\widehat{G(\mathbb{R})}} H^p(\mathfrak{n}, V_\pi)_{-\mu}^{\oplus m_\pi(\Gamma)}.$$

By the main result of [CO], this now equals the right-hand side. □

Note that the weight of a class in $H^p(\mathfrak{n}, V_\pi)$ is the weight of the representative in $\bigwedge^p \check{\mathfrak{n}} \otimes V_\pi$.

*Remark* 3.10. If we assume $\lambda := -(\mu+\rho)$ is regular and $V_\lambda$ integrable, then the right-hand side of Theorem 3.9 becomes

$$H^p(\mathfrak{n}, V_\lambda)_{-\mu}^{\oplus m_\pi(\Gamma)},$$

and $\dim\left(H^p(\mathfrak{n}, V_\lambda)_{-\mu}\right) = 1$. In fact, the integrability assumption can be relaxed due to [Wi1]: we only need *Property W*, which is the statement (of "sufficient regularity") that for each $\alpha \in \Phi_{nc}$ with $(\lambda,\alpha) < 0$, and writing $\mathcal{B} := \sum_{\beta\in\Phi\colon(\lambda,\beta)<0}\beta$, we have $(\lambda + \frac{1}{2}\mathcal{B}, \alpha) < 0$. (In fact, [Wi2] relaxes the conditions even further to include holomorphic limits of discrete series.)

For our purposes, it is also essential to know which irreps contribute to the direct sum in Theorem 3.9 if $\mu + \rho = 0$. It is clear for $SL_2$, and shown in [C1] for $SU(2,1)$, that only TDLDS appear; and this will be verified below for $Sp_4$. For now, we will show how Theorem 3.9 may be used to deduce

**Proposition 3.11.** *Define $d_I^{(k)}$ by $\dim\left(H^k(\mathfrak{n}, V_{(0,C_I)})\right)$ and $d_{II}^{(k)}$ similarly. For every $k$, these are 0 or 1. In particular, $d_I^{(k)} = 0$ for $k = 0,3,4$, while $d_{II}^{(k)} = 0$ for $k = 0,1,4$.*

*Proof.* Let $V := V_{(0,C_{\rm I})} \oplus V_{(0,C_{\rm II})}$. By [CO] we know that $H^k(\mathfrak{n}, V) = H^k(\mathfrak{n}, V)_{\rho_{\rm I}}$, where we recall that $\rho_I = 2\alpha_1 - \alpha_2$. The $E_1$ page of the Hochschild-Serre spectral sequence converging to this group is $H^j(\mathfrak{n}_K, V \otimes \bigwedge^i \check{\mathfrak{p}}^-)_{\rho_{\rm I}}$, which is automatically zero for $j \neq 0,1$ or $i \neq 0,1,2,3$. (Note that $\mathfrak{n}_K = \langle X_{-\alpha_1+\alpha_2}\rangle$ and $\mathfrak{p}^- = \langle X_{2\alpha_2}, X_{-2\alpha_1}, X_{-\alpha_1-\alpha_2}\rangle$, and that $H^0$ resp. $H^1$ are the kernel resp. cokernel of $X_{-\alpha_1+\alpha_2}$.) The reader will easily verify, using the $K$-type decompositions in 3.6, that this reduces to

(3.6)

| | | | |
|---|---|---|---|
| $H^1(\mathfrak{n}_K, V_{(0,C_{\rm I})})_{\rho_{\rm I}}$ | $H^1(\mathfrak{n}_K, V_{(0,C_{\rm II})} \otimes \check{\mathfrak{p}}^-)_{\rho_{\rm I}}$ | $0$ | $0$ |
| $0$ | $0$ | $H^0(\mathfrak{n}_K, V_{(0,C_{\rm I})} \otimes \bigwedge^2 \check{\mathfrak{p}}^-)_{\rho_{\rm I}}$ | $H^0(\mathfrak{n}_K, V_{(0,C_{\rm II})} \otimes \bigwedge^3 \check{\mathfrak{p}}^-)_{\rho_{\rm I}}$ |

and that all four spaces remaining are 1-dimensional. □

In fact – and this is the key point – the spectral sequence degenerates at $E_2$ so that the remaining four dimensions $d_{\mathrm{I}}^{(1)}, d_{\mathrm{I}}^{(2)}, d_{\mathrm{II}}^{(2)}, d_{\mathrm{II}}^{(3)}$ are one. This is proved in §6.1.

## 4. Penrose transforms and arithmeticity

In this section we consider consequences of the existence[4] of certain holomorphic correspondences between the the different flag domains $D$ in a flag variety $\check{D}$. When $G$ is of Hermitian type, this is of particular significance as it mediates between the classical and non-classical M-T domains ($G(\mathbb{R})$-orbits) in $\check{D}$.

Fixing a general Borel subgroup $B$ of $G(\mathbb{C})$, for $w \in W/W_K$ we set

$$D_w := G(\mathbb{R}).\underbrace{wBw^{-1}}_{=:B_w} \subset G(\mathbb{C}).B \cong G(\mathbb{C})/B = \check{D},$$

where the action (on $B_w$ resp. $B$) is by conjugation. Writing $H_{\mathbb{C}}$ for the complexification of $H = B \cap G(\mathbb{R})$, we introduce the correspondence space

$$\mathcal{W} := \left( \bigcap_{w \in W} G(\mathbb{R}) B_w \right) \Big/ H_{\mathbb{C}} \underset{\text{open}}{\subset} G(\mathbb{C})/H_{\mathbb{C}} =: \check{\mathcal{W}}. \tag{4.1}$$

For each $w \in W/W_K$ there is a holomorphic projection

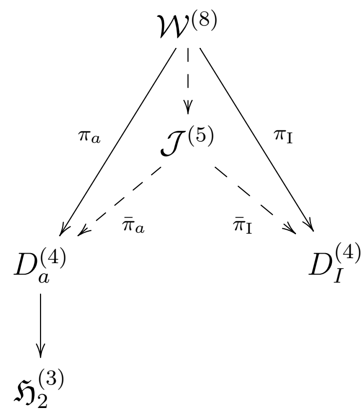
$$\begin{array}{ccc} \mathcal{W} & \hookrightarrow & \check{\mathcal{W}} \\ {\scriptstyle \pi_w}\downarrow & & \downarrow \\ D_w & \hookrightarrow & \check{D}, \end{array}$$

obtained by letting the "numerator" in $\mathcal{W}$ act on $B_w$. There are various ways of thinking of $\mathcal{W}$ as a subset of a product of the $\{D_w\}$. The *CM points* $P \in \mathcal{W}$ are those with $\pi_w(P) \in D_w$ CM for each $w$.

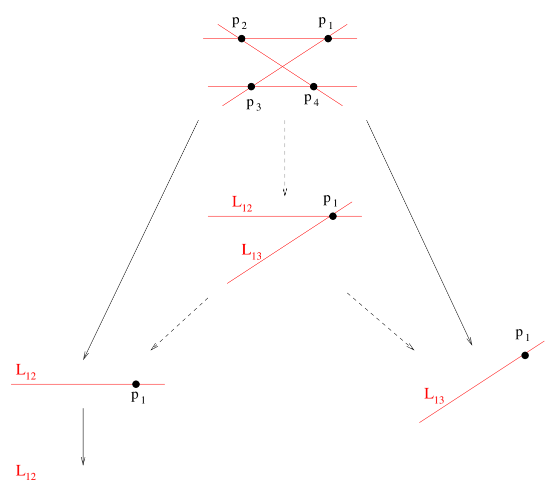
[4]The general proof uses Matsuki duality and can be found in [GG]. The *ad hoc* construction we use here will suffice for the examples we consider.

**Example 4.1.** ($G = Sp_4$) Denoting complex dimension by a superscript, the piece of the correspondence picture we shall need is

(4.2)




where $\mathfrak{H}_2$ is Siegel upper half space. Referring to Example 2.14, the points of $\mathcal{W}$ may be represented by compatible 4-tuples of Lagrange flags in $\mathbb{P}V$ called *Lagrange quadrilaterals* [GGK2]. Let $\{\tilde{p}_{1,0}, \tilde{p}_{2,0}, \tilde{p}_{3,0}, \tilde{p}_{4,0}\} \subset V$ be a (Hodge) basis compatible with $B \subset G(\mathbb{C})$, and write $\tilde{p}_i := g.\tilde{p}_{i,0}$, $p_i := [\tilde{p}_i]$ for the moving linear resp. projective frames parametrized by $G(\mathbb{C})$. The maps in (4.2) are then described by




(4.3)

in which $\overline{p_1 p_2} \perp \overline{p_3 p_4}$. In addition, we note that $\mathcal{W}$ can be viewed as an open subset of $D_{\mathrm{I}} \times D_{\mathrm{I}}$, with the inclusion given by mapping the quadrilateral to $((p_1, L_{13}), (p_2, L_{24}))$.

Correspondence spaces allow us to present the Dolbeault ($\bar{\partial}$-) cohomology groups of interest (on $D$) as relative holomorphic de Rham cohomology (on $\mathcal{W}$). Writing $\pi : \mathcal{W} \to D$ for one of the $\{\pi_w\}$ and denoting the relative differentials $\Omega^\bullet_{\mathcal{W}}/\mathrm{im}\{\pi^*\Omega^1_D \otimes \Omega^\bullet_{\mathcal{W}}[-1]\}$ by $\Omega^\bullet_\pi$, we have

**Theorem 4.2.** [EGW]$H^p\left(D, \mathcal{O}(\mathcal{L}_\mu)\right) \cong H^p\left\{H^0\left(\mathcal{W}, \Omega^\bullet_\pi(\pi^*\mathcal{L}_\mu)\right), d_\pi\right\}$.

*Proof.* Applying the hypercohomology spectral sequence to the resolution

$$\pi^{-1}\mathcal{O}_D(\mathcal{L}_\mu) \xrightarrow{\simeq} \Omega^\bullet_\pi(\pi^*\mathcal{L}_\mu) \left[:= \mathcal{O}_{\mathcal{W}}(\pi^*\mathcal{L}_\mu) \to \Omega^1_\pi(\pi^*\mathcal{L}_\mu) \to \cdots\right]$$

yields

$$H^a(\mathcal{W}, \Omega^b_\pi(\pi^*\mathcal{L}_\mu)) \implies H^{a+b}(\mathcal{W}, \pi^{-1}\mathcal{O}_D(\mathcal{L}_\mu)). \tag{4.4}$$

On the other hand, applying the Leray spectral sequence and

$$R^q\pi_*\pi^{-1}\mathcal{O}(\mathcal{L}_\mu) = R^q\pi_*\mathbb{C} \otimes \mathcal{O}(\mathcal{L}_\mu)$$

gives

$$H^p(D, R^q\pi_*\mathbb{C} \otimes \mathcal{O}(\mathcal{L}_\mu)) \implies H^{p+q}(\mathcal{W}, \pi^{-1}\mathcal{O}(\mathcal{L}_\mu)). \tag{4.5}$$

Results of [FHW] now imply that (i) $\mathcal{W}$ is Stein and (ii) fibers of $\pi$ are contractible. From (i) resp. (ii) , we get $a = 0$ in (4.4) resp. $q = 0$ in (4.5), from which we conclude that both terms in the Theorem are isomorphic to $H^p(\mathcal{W}, \pi^{-1}(\mathcal{O}(\mathcal{L}_\mu)))$. □

*Remark* 4.3. The same argument goes through for the quotient of $\mathcal{W}$ by a neat, co-compact arithmetic subgroup $\Gamma \leq G(\mathbb{Q})$ [GG], which gives

$$H^p(\Gamma\backslash D, \mathcal{O}(\mathcal{L}_\mu)) \cong H^p\left\{H^0(\Gamma\backslash\mathcal{W}, \Omega^\bullet_\pi(\pi^*\mathcal{L}_\mu))\right\}. \tag{4.6}$$

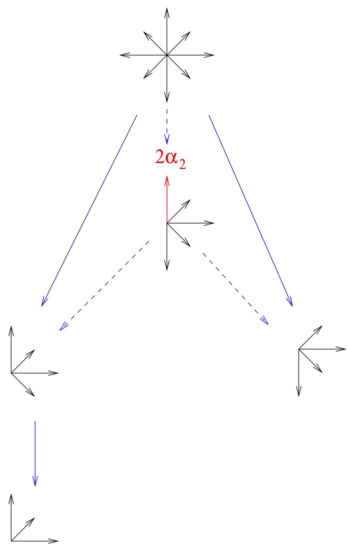
Confining ourselves to the $G = Sp_4$ case, we now explain how to use (4.2) and Theorem 4.2 to move cohomology from a classical to a non-classical flag domain. It would apparently be convenient to use $\mathcal{J}$ in place of $\mathcal{W}$; while we cannot do this directly, as $\mathcal{J}$ is not Stein, this motivates how we shall proceed. (To reduce notational clutter, we will drop the subscript "I" on $\pi_{\mathrm{I}}$.)

If $g^{-1}dg =: \{\omega^i_j\}$ is the matrix of left-invariant 1-forms with respect to the $\{\tilde{p}_i\}$, then we have

$$d\tilde{p}_j = \sum_i \omega^i_j \tilde{p}_i.$$

In particular, $\omega_2^3$ spans the pullback of $\Omega^1_{\bar{\pi}}$ to $G(\mathbb{C})$ since (referring to (4.3)) it records the motion of $L_{12}$ with $p_1$ and $L_{13}$ held fixed. By picturing the tangent spaces and differentials to the maps in (4.2) in terms of root vectors

(4.7)




we immediately read off that the weight of $\Omega^1_{\bar{\pi}}$ is $-2\alpha_2 = \rho_{\mathrm{I}} - \rho_a$.

Let $\mu, \mu' \in \Lambda$ be weights with $\mu + \rho_{\mathrm{I}} = \mu' + \rho_a$ and

$$(\mu, \alpha_1 - \alpha_2) < 0. \tag{4.8}$$

(For instance, referring to Example 3.5, we could take $\mu' = \Lambda_{-k}$.) Then we have that $\omega_2^3$ descends to a section

$$\omega \in H^0(\mathcal{J}, \underbrace{\Omega^1_{\bar{\pi}}(\bar{\pi}^*\mathcal{L}_{\rho_a - \rho_{\mathrm{I}}})}_{\text{weight } 0}) \cong H^0\left(\mathcal{J}, \Omega^1_{\bar{\pi}}(\bar{\pi}^*\mathcal{L}_{\mu} \otimes \bar{\pi}_a^*\check{\mathcal{L}}'_{\mu'})\right).$$

Given $F \in H^0(D_a, \mathcal{O}(\mathcal{L}'_{\mu'}))$, the product

$$(\bar{\pi}^*F)\omega \in H^0\left(\mathcal{J}, \Omega^1_{\bar{\pi}}(\bar{\pi}^*\mathcal{L}_{\mu})\right) \tag{4.9}$$

is then necessarily $d_{\bar{\pi}}$-closed, as the relative dimension of $\mathcal{J} \to D_{\mathrm{I}}$ is one. In fact, its pullback to $H^0(\mathcal{W}, \Omega^1_{\pi}(\pi^*\mathcal{L}_{\mu}))$ is $d_{\pi}$-closed since $\mathcal{J} \subset \mathcal{W} \to D_{\mathrm{I}}$ is an invariant sub-bundle.[5] Writing $\tilde{\omega}$ for the pullback of $\omega$ to $H^0\left(\mathcal{W}, \Omega^1_{\pi}(\pi^*\mathcal{L}_{\rho_a - \rho_{\mathrm{I}}})\right)$, and using Theorem 4.2, we can now make the

[5]This follows from (4.7), since $2\alpha_2$ is not bracket-generated by the root vectors corresponding to $T\mathcal{W}/\pi^*TD$.

**Definition 4.4.** The *Penrose transform* for $Sp_4$ is the map

$$\mathcal{P} : H^0\left(D_a, \mathcal{O}(\mathcal{L}_{\mu'})\right) \to H^1\left\{H^0\left(\mathcal{W}, \Omega^{\bullet}_{\pi}(\pi^*\mathcal{L}_{\mu})\right), d_\pi\right\} \cong H^1\left(D_{\mathrm{I}}, \mathcal{O}(\mathcal{L}_\mu)\right)$$

given by $F \mapsto (\pi^* F)\tilde{\omega}$.

Now $\mathcal{J}$ is covered by copies of $\mathbb{P}^1$, lying over the translates of $Z \subset D_{\mathrm{I}}$, with $\mathcal{O}(1)$ given by the restriction of $\bar{\pi}^*\mathcal{L}_{\alpha_1-\alpha_2}$ to each copy. If the pull-back $(\pi^*F)\tilde{\omega}$ of (4.9) to $\mathcal{W}$ is a $d_\pi$-coboundary, then $(\bar{\pi}^*F)\omega = d_{\bar{\pi}}G$ for some $G \in H^0\left(\mathcal{J}, \mathcal{O}(\bar{\pi}^*\mathcal{L}_\mu)\right)$. But the restriction of this $G$ to each $\mathbb{P}^1$ lies in $H^0\left(\mathbb{P}^1, \mathcal{O}\left((\mu, \alpha_1 - \alpha_2)\right)\right)$, which is zero under our assumption (4.8); and so $F = 0$. This establishes

**Proposition 4.5.** *Assuming* (4.8)*, the Penrose transform for $Sp_4$ is injective.*

In order to use this to study automorphic cohomology, we shall need a "mod $\Gamma$" version. By the Cartan-Leray spectral sequence,

$$H^1(\Gamma, H^0(D_{\mathrm{I}}, \mathcal{O}(\mathcal{L}_\mu))) \to H^1(\Gamma\backslash D_{\mathrm{I}}, \mathcal{O}(\mathcal{L}_\mu)) \xrightarrow{\epsilon} H^1(D_{\mathrm{I}}, \mathcal{O}(\mathcal{L}_\mu))^\Gamma$$
$$\to H^2(\Gamma, H^0(D_{\mathrm{I}}, \mathcal{O}(\mathcal{L}_\mu)))$$

is exact. On the compact fibers $Z \cong \mathbb{P}^1$ of the (non-holomorphic) projection $D_{\mathrm{I}} \twoheadrightarrow \mathfrak{H}_2$, we have $\mathcal{L}_\mu|_Z \cong \mathcal{O}_{\mathbb{P}^1}\left((\mu, \alpha_1 - \alpha_2)\right)$; together with (4.8), this implies $H^0(D, \mathcal{O}(\mathcal{L}_\mu)) = \{0\}$ as above, making $\varepsilon$ an isomorphism. Taking $\Gamma$-invariants in Definition 4.4 and composing with $\epsilon^{-1}$ gives the top row of

(4.10)
$$\begin{array}{ccc} H^0\left(\Gamma \setminus D_a, \mathcal{O}(\mathcal{L}'_{\mu'})\right) & \overset{\mathcal{P}_\Gamma}{\hookrightarrow} & H^1\left(\Gamma \setminus D_{\mathrm{I}}, \mathcal{O}(\mathcal{L}_\mu)\right) \\ \uparrow & & \uparrow \\ H^0\left(\mathfrak{n}_a, V_\lambda\right)^{\oplus m(\Gamma)}_{-\mu'} & \overset{??}{\dashrightarrow} & H^1\left(\mathfrak{n}_{\mathrm{I}}, V_\lambda\right)^{\oplus m(\Gamma)}_{-\mu}, \end{array}$$

where $\lambda := -(\mu + \rho_{\mathrm{I}}) = -(\mu' + \rho_a)$. According to Remark 3.10, when $\lambda$ satisfies "Property W", the vertical arrows are isomorphisms and the bottom spaces are both of dimension $m(\Gamma)$.

**Example 4.6.** In particular, taking $\mu' = \Lambda_{-k}$ and[6] $\mu =$

(4.11)
$$\mu_k^{(1)} := \Lambda_{-k} + 2\alpha_2$$

[6]The meaning of the notation "$\mu_k^{(1)}$" will becoe clear in §6.2.

with $k \geq 4$, "Property W" holds for $\lambda$ and we arrive at part (a) of

**Theorem 4.7.** *(a) For $k \geq 4$, the Penrose transform induces an isomorphism*

$$\mathcal{P}_\Gamma : \overline{M_k^{Sie}(\mathfrak{H}_2, \Gamma)} \cong H^0(\Gamma\backslash D_a, \mathcal{O}(\mathcal{L}_{\Lambda_{-k}}) \stackrel{\cong}{\to} H^1(\Gamma\backslash D_I, \mathcal{O}(\mathcal{L}_{\mu_k^{(1)}}))$$

*from (complex conjugates of scalar-valued, weight $k$) Siegel modular forms to non-classical automorphic cohomology.*

*(b) Similarly, there is an isomorphism*

$$\mathcal{P}_\Gamma^{(2)} : M_{W_{(k)}}^{Sie}(\mathfrak{H}_2, \Gamma) \cong H^0(\Gamma\backslash D_h, \mathcal{O}(\mathcal{L}_{\tilde{\Lambda}_k})) \stackrel{\cong}{\to} H^2(\Gamma\backslash D_I, \mathcal{O}(\mathcal{L}_{\tilde{\mu}_k^*}))$$

*where $\tilde{\mu}_k^* := \Lambda_k - 2\alpha_1 - \alpha_2$, $W_{(k)} = \mathcal{S}_1(2k+1)$ is the rank-2 irrep of $K$ with highest weight $\tilde{\Lambda}_k$, and $M_{W_{(k)}}^{Sie}(\mathfrak{H}_2, \Gamma)$ is the corresponding space of 2-vector-valued modular forms.*[7]

In each case, the first isomorphism comes from taking $\Gamma$-invariants of the first isomorphism in (3.2).

For our purposes below, it is instructive to see how an "infinitesimal" (i.e. Lie algebra cohomology) version of the Penrose transform fills in the bottom arrow of (4.10). First observe that $\mathfrak{n}_a$ is spanned by $\{X_{-2\alpha_2}, X_{-\alpha_1-\alpha_2}, X_{-2\alpha_1}, X_{-\alpha_1+\alpha_2}\}$, and $\mathfrak{n}_{\mathrm{I}}$ by $\{X_{-\alpha_1-\alpha_2}, X_{-2\alpha_1}, X_{-\alpha_1+\alpha_2}, X_{2\alpha_2}\}$, viz.


Recalling that $-2\alpha_2 = \rho_{\mathrm{I}} - \rho_a$, and writing $\omega^\alpha := X_\alpha^*$ for the duals (of weight $-\alpha$) to the root vectors, we have

**Definition 4.8.** The *Penrosette transform* for $sp_4$ is the map

$$p : H^0(\mathfrak{n}_a, V_\lambda)_{-\mu'} \to H^1(\mathfrak{n}_{\mathrm{I}}, V_\lambda)_{-\mu}$$

induced by tensoring with $\omega^{2\alpha_2}$, where $\lambda = -(\mu + \rho_{\mathrm{I}}) = -(\mu' + \rho_a)$.

In [GGK2] it is shown that $p_\Gamma^{\oplus m(\Gamma)}$ makes diagram (4.10) commute.

We conclude the (largely) expository sections of this paper with a brief discussion of arithmeticity of automorphic cohomology in the co-compact setting.

[7]for instance, see [Gh]

Classically the standard way to put a $\bar{\mathbb{Q}}$-structure on automorphic forms is to use the canonical model:

**Example 4.9.** (a) The tautological family of abelian surfaces $\mathcal{A} \to \mathcal{M}(\Gamma) := \Gamma\backslash\mathfrak{H}_2$ over the Siegel modular variety is defined over $\bar{\mathbb{Q}}$. Via algebraic de Rham cohomology, the Hodge bundles $\left(\bigwedge^2 \mathcal{V}^{1,0}_{\mathcal{A}/\mathcal{M}(\Gamma)}\right)^{\otimes k} \to \Gamma\backslash\mathfrak{H}_2$ are defined over $\bar{\mathbb{Q}}$. As they identify with the homogeneous bundle $\mathcal{L}_{\Lambda_k}$, this puts a $\bar{\mathbb{Q}}$-vector space structure on $M_k^{\mathrm{Sie}}(\mathfrak{H}_2, \Gamma) \cong H^0(\Gamma\backslash\mathfrak{H}_2, \mathcal{O}(\mathcal{L}_{\Lambda_k}))$.

(b) Over the Picard modular variety $\Gamma\backslash\mathbb{B}_2 =: \tilde{\mathcal{M}}(\Gamma)$, the tautological family $\tilde{\mathcal{A}}$ consists of abelian threefolds with endomorphisms by some $\mathcal{O}_{\mathbb{Q}(\sqrt{-d})}$. The relation $\mathcal{L}_{\varepsilon_{-k}} \cong \left(\mathcal{V}^{1,0}_{\tilde{\mathcal{A}}/\tilde{\mathcal{M}}(\Gamma),+}\right)^{\otimes k}$, where

$$\varepsilon_k := \frac{k}{3}(2\beta_2 + \beta_1),$$

then implies a $\bar{\mathbb{Q}}$-structure on the space of Picard modular forms of degree $k$.

In the presence of a Penrose transform $\mathcal{P}_\Gamma$, (a) or (b) leads to a $\bar{\mathbb{Q}}$-structure on the subspace $\mathrm{im}(\mathcal{P}_\Gamma)$ of some automorphic cohomology group. The simplest example is apparent from Theorem 4.7(a), giving meaning (for $k \geq 4$) to $H^1(\Gamma\backslash D_{\mathrm{I}}, \mathcal{O}(\mathcal{L}_{\mu_k^{(1)}}))_{\bar{\mathbb{Q}}}$, and we note that this is despite the *non-algebraicity* of $\Gamma\backslash D_{\mathrm{I}}$.

In the absence of boundary components, an important alternate arithmeticity criterion is obtained by considering values of automorphic forms at CM points [Sh], after pulling them back to the relevant classical domain (here $\mathfrak{H}_2$ or $\mathbb{B}_2$). What about automorphic cohomology, say $H^1$, especially in the nonclassical case? There are in fact some natural generalizations of the "CM criterion":

(1) Are the pullbacks to (classical) Shimura curves in (nonclassical) $\Gamma\backslash D$ defined over $\bar{\mathbb{Q}}$? More generally,
(2) Given a maximal VHS arising from a family of varieties defined over $\bar{\mathbb{Q}}$, is the pullback along the corresponding period map defined over $\bar{\mathbb{Q}}$?
(3) Representing $H^1$ via (4.6) as elements of $H^0(\Gamma\backslash\mathcal{W}, \Omega^\bullet_\pi(\pi^*\mathcal{L}_\mu))$, we can consider "evaluating" their pullback at CM points $P \in \mathcal{W}$.

How (1) interacts with the Penrose transform is already a deep and apparently difficult problem, and (2) would be harder still. The interaction of (3) with

$\mathcal{P}_\Gamma$, while far less consequential, is completely clear. Since $\check{\mathcal{W}} \to \check{D}$ and $P$ are defined over $\bar{\mathbb{Q}}$, so are the $\mathfrak{F}_k := \Omega^1_\pi \otimes \pi^* \mathcal{L}_{\mu_k^{(1)}}$ and $\mathfrak{F}_k|_P$.

**Theorem 4.10.** [GGK2] *Let $\alpha \in H^1(\Gamma\backslash D_I, \mathcal{O}(\mathcal{L}_{\mu_k}))_{\bar{\mathbb{Q}}}$ ($k \geq 4$) with "pullback" $\tilde{\alpha} \in \ker(d_\pi) \subset H^0(\mathcal{W}, \Omega^1_\pi(\pi^*\mathcal{L}_{\mu_k^{(1)}}))$, and $\varphi_P \in \mathfrak{H}_2$ be the image of $P$ (via $D'$). Then $\Delta^{-k}\tilde{\alpha}(P) \in (\mathfrak{F}_k|_P)_{\bar{\mathbb{Q}}}$, where $\Delta \in \mathbb{C}^*/\mathbb{Q}^*$ depends only on the CM type of $\varphi_P$.*

## 5. Cup products and TDLDS

We now turn to the statements of three key results, and their concrete interpretations in terms of automorphic forms. In each case, they illustrate how cup-products of cohomology classes related to modular forms (and hence to holomorphic discrete series) produce classes related, by Theorem 3.9, to totally degenerate limits of discrete series. We remind the reader that (the Lie algebra cohomology of) such representations do not appear in the coherent cohomology of any Shimura variety. $\Gamma \leq G(\mathbb{Q})$ will denote a neat and co-compact arithmetic subgroup throughout.

The first result, for $G = SU(2,1)$, is Carayol's:

**Theorem 5.1.** [C1] *Let $D$ be as in Example 2.15, and $k \geq 5$. Then the cup product*

$$\theta_\Gamma : H^1(\Gamma\backslash D, \mathcal{O}(\mathcal{L}_{\eta_k^{(1)}})) \otimes H^1(\Gamma\backslash D, \mathcal{O}(\mathcal{L}_{-\beta_1-\eta_k^{(1)}})) \to H^2(\Gamma\backslash D, \mathcal{O}(\mathcal{L}_{-\beta_1})) \tag{5.1}$$

*is virtually surjective, where $\eta_k^{(1)} := \varepsilon_k - \beta_1 - \beta_2$.*

**Definition 5.2.** In the theorem, and in later statements to come, *virtual surjectivity* means that for each $\Gamma$, there exists $\Gamma_0 \subset \Gamma$ such that the composite of $\theta_{\Gamma_0}$ with the push-forward

$$H^*(\Gamma_0\backslash D, \mathcal{O}(\mathcal{L})) \twoheadrightarrow H^*(\Gamma\backslash D, \mathcal{O}(\mathcal{L}))$$

is surjective. Obviously this is really a statement about the *family* of cup-product maps, and will henceforth be denoted by the symbol $\widetilde{\twoheadrightarrow}$. Likewise, a cup product (family) is *virtually right-injective*, written

$$H^p(\Gamma\backslash D, \mathcal{O}(\mathcal{L}_1)) \otimes H^q(\Gamma\backslash D, \mathcal{O}(\mathcal{L}_2)) \widetilde{\hookrightarrow} H^{p+q}(\Gamma\backslash D, \mathcal{O}(\mathcal{L}_1 \otimes \mathcal{L}_2)),$$

if given any nonzero class in $H^q(\Gamma\backslash D, \mathcal{O}(\mathcal{L}_2))$, its pullback to $H^q(\Gamma_0\backslash D, \mathcal{O}(\mathcal{L}_2))$ (for some $\Gamma_0$) has nontrivial product with a class in $H^p(\Gamma_0\backslash D, \mathcal{O}(\mathcal{L}_1))$.

Carayol proved Theorem 5.1 by showing the Serre-dual statement

$$H^1(\Gamma\backslash D, \mathcal{O}(\mathcal{L}_{\eta_k^{(1)}})) \otimes H^1(\Gamma\backslash D, \mathcal{O}(\mathcal{L}_{-\beta_1})) \hookrightarrow\!\!\!\!\!\!\rightarrow H^2(\Gamma\backslash D, \mathcal{O}(\mathcal{L}_{\eta_k^{(1)}-\beta_1})).$$

(Here we are using that $\mathcal{L}_{-2\beta_1} \cong K_{\Gamma\backslash D}$.) We will apply the same general approach to $Sp_4$, essentially because "finding a copy of a TDLDS inside a cup product of DS" turns out to be technically less forbidding than "finding a DS inside the cup-product of a DS with a TDLDS".

The left-hand side of (5.1) is accessible by Penrose transforms of Picard modular forms, and thus has a $\bar{\mathbb{Q}}$-structure, while the right-hand side is not. More precisely, writing $\mathrm{pr} : D_{\mathrm{hol}} \twoheadrightarrow \mathbb{B}_2$ (cf. Example 2.15), there are isomorphisms [C1, GGK2]

$$\begin{aligned} M_k^{\mathrm{Pic}}(\mathbb{B}_2, \Gamma) &\cong H^0(\Gamma\backslash\mathbb{B}_2, \omega_{\Gamma\backslash\mathbb{B}_2}^{\otimes k/3}) \\ &\xrightarrow[\mathrm{pr}^*]{\cong} H^0(\Gamma\backslash D_{\mathrm{hol}}, \mathcal{O}(\mathcal{L}_{\varepsilon_{-k}})) \\ &\xrightarrow[\mathcal{P}_\Gamma]{\cong} H^1(\Gamma\backslash D, \mathcal{O}(\mathcal{L}_{-\beta_1-\eta_k^{(1)}})), \end{aligned}$$

and similarly (using $D_{\mathrm{anti}}$)

$$\overline{M_k^{\mathrm{Pic}}(\mathbb{B}_2, \Gamma)} \xrightarrow{\cong} H^1(\Gamma\backslash D, \mathcal{O}(\mathcal{L}_{\eta_k^{(1)}}))$$

where the bar denotes complex conjugation.

To interpret the right-hand side of (5.1), we shall use basic results on the unique TDLDS $V_0$ of $SU(2,1)$, for which we refer to [GGK2, sec. IV.F]. In particular, we recall that $V_0$ is the only unitary irrep with trivial infinitesimal character and nontrivial $H^2(\mathfrak{n}, \cdot)$. Reasoning as in the proof of Theorem 3.9, we have

$$H^2(\Gamma\backslash D, \mathcal{O}(\mathcal{L}_{-\beta_1})) \cong H^2 \left\{ Hom_H \left( \textstyle\bigwedge^{\bullet}\mathfrak{n}, \mathcal{A}(G(\mathbb{R}), \Gamma) \otimes \mathbb{C}_{-\beta_1} \right) \right\}, \tag{5.2}$$

which (using [CO]) consists of finitely many copies of $H^2(\mathfrak{n}, V_0)$. From (IV.F.6) of [op. cit.], the latter is generated by the homomorphism that sends $X_{\beta_2} \wedge X_{-(\beta_1+\beta_2)}$ to a generator of the (1-dimensional, pure weight 0) lowest $K$-type of $V_0$. As $X_{\beta_2} \wedge X_{-(\beta_1+\beta_2)}$ already has weight $-\beta_1$, the right-hand side of (5.2) is

$$\cong \left\{ f \in \mathcal{A}(G(\mathbb{R}),\Gamma) \;\middle|\; \begin{array}{c} f \text{ has weight } 0\text{, i.e. is } H\text{-invariant,} \\ \text{and belongs to (the lowest } K\text{-type of)} \\ \text{a copy of } V_0 \end{array} \right\}$$

$$\cong \left\{ f \in C^\infty(\Gamma\backslash D) \;\middle|\; \Omega(f) = -\frac{1}{3}f \right\}$$

where $\Omega \in Z(\mathfrak{g})$ is the Casimir operator (normalized as in [Kn1]) regarded as a second-order differential operator. One may regard this as a space of "nonclassical Maass forms".

Altogether then, Theorem 5.1 amounts to the surjectivity statement

$$\overline{M_k^{\mathrm{Pic}}(\Gamma)} \otimes M_k^{\mathrm{Pic}}(\Gamma) \widetilde{\twoheadrightarrow} C^\infty(\Gamma\backslash D)^{\Omega+\frac{1}{3}}, \tag{5.3}$$

providing a (rather indirect) "formula" for $SU(2,1)$ Maass forms in terms of holomorphic and antiholomorphic modular forms.

Before turning to the $Sp_4$ result which shall occupy us for the remainder of the paper, we consider the "toy model" case of $SL_2$,. Since the irreps generated by Maass forms have no $\mathfrak{n}(=\langle X_{-1}\rangle)$-cohomology, the closest analogue of what Carayol did for $SU(2,1)$ involves limits of discrete series $D_0^\pm$ in place of $V_0$, and hence modular forms of weight one. In this case, it is obvious that

$$H^0(\Gamma\backslash\mathfrak{H}, \mathcal{O}(\mathcal{L}_{-k})) \otimes H^0(\Gamma\backslash\mathfrak{H}, \mathcal{O}(\mathcal{L}_{-1})) \hookrightarrow H^0(\Gamma\backslash\mathfrak{H}, \mathcal{O}(\mathcal{L}_{-(k+1)})$$

without any "integrability" assumption on $k$: we are merely asserting that for any element of $f \in M_1(\Gamma)\backslash\{0\}$, there is $g \in M_k(\Gamma)$ such that $fg \in M_{k+1}(\Gamma)$ is not identically zero. (Indeed, we can simply take $g = f^k$, and there is no need for $\Gamma_0 \leq \Gamma$ hence none for "$\widetilde{\hookrightarrow}$".) By duality this yields the

**Theorem 5.3.** *For any $k \geq 0$, we have*

$$H^0(\Gamma\backslash\mathfrak{H}, \mathcal{O}(\mathcal{L}_{-k})) \otimes H^1(\Gamma\backslash\mathfrak{H}, \mathcal{O}(\mathcal{L}_{k-1})) \twoheadrightarrow H^1(\Gamma\backslash\mathfrak{H}, \mathcal{O}(\mathcal{L}_{-1})).$$

The interpretation is much more interesting. The theory of automorphic forms on $SL_2(\mathbb{R})$ (cf. [Ke]) suggests a map

$$M_{k+1}(\Gamma) \cong H^0(\Gamma\backslash\mathfrak{H}, \mathcal{O}(\mathcal{L}_{-(k+1)})) \overset{\cong}{\to} H^1(\Gamma\backslash\mathfrak{H}, \mathcal{O}(\mathcal{L}_{k-1})) \tag{5.4}$$

defined by

$$G(\tau) \longmapsto \overline{G(\tau)}\Im(\tau)^{k-1}d\bar{\tau},$$

where we think of the right-hand side as Dolbeault cohomology – that is, as $C^\infty$ modular $(0,1)$-forms of weight $-(k-1)$, modulo the image of $\bar{\partial}$. That $\overline{G(\tau)}y^{k-1}d\bar{\tau}$ defines such an object is seen from writing, for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $\gamma^*\left(\overline{G(\tau)}y^{k-1}d\bar{\tau}\right) =$

$$\overline{(c\tau+d)^{k+1}} \cdot \overline{G(\tau)} \cdot \frac{y^{k-1}}{|c\tau+d|^{2(k-1)}} \cdot \frac{d\bar{\tau}}{\overline{(c\tau+d)}^2} = \frac{\overline{G(\tau)}y^{k-1}d\bar{\tau}}{(c\tau+d)^{k-1}}.$$

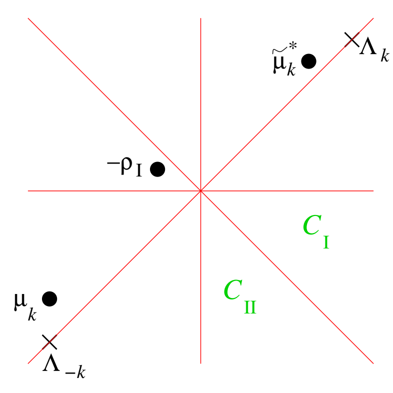
In fact, (5.4) turns out to be the composition of Penrose transform with de Rham conjugation.

Putting Theorem 5.3 and (5.4) together, we have

$$M_k(\Gamma)\otimes M_{k+1}(\Gamma) \twoheadrightarrow H^1(\Gamma\backslash\mathfrak{H},\mathcal{O}(\mathcal{L}_{-1})) \xleftarrow{\cong} H^0(\Gamma\backslash\mathfrak{H},\mathcal{O}(\mathcal{L}_{-1})) \cong M_1(\Gamma)$$
$$F\otimes G \longmapsto F\bar{G}y^{k-1}d\bar{\tau} \equiv (\bar{f}/y)d\bar{\tau} \longleftarrow f$$

where the "$\equiv$" is up to $\bar{\partial}$-coboundary. The target $M_1(\Gamma)$ is the group of sections of the particular theta characteristic (namely, $\mathcal{L}_{-1}$) of $\Gamma\backslash\mathfrak{H}$ given by the Hodge bundle associated to the elliptic modular surface over it.

Apart from its intrinsic interest, the reason for briefly treating $SL_2$ here is its greater analogy to our main result, Theorem 5.4 below. Like $SL_2(\mathbb{R})$, $Sp_4(\mathbb{R})$ has two TDLDS, and as we shall see the target group for the cup-product is a certain space of (non-holomorphic) sections of Hodge bundles. We remind the reader that $\mu_k^{(1)} = \Lambda_{-k}+2\alpha_2 = -k\alpha_1+(2-k)\alpha_2$, $\tilde{\mu}_k^* = \Lambda_k - 2\alpha_1 - \alpha_2 = (k-2)\alpha_1+(k-1)\alpha_2$, and $\rho_{\mathrm{I}} = 2\alpha_1+\alpha_2$:



**Theorem 5.4.** *Let $D_I = D_{(1,1,1,1)}$ as in previous sections, and $k \geq 5$; then we have*

$$H^1(\Gamma\backslash D_I, \mathcal{O}(\mathcal{L}_{\mu_k^{(1)}})) \otimes H^2(\Gamma\backslash D_I, \mathcal{O}(\mathcal{L}_{\tilde{\mu}_k^*})) \widetilde{\twoheadrightarrow} H^3(\Gamma\backslash D_I, \mathcal{O}(\mathcal{L}_{-\rho_I})). \tag{5.5}$$

By Theorem 4.7, the left-hand side of (5.5) is "classically accessible" by Penrose transforms, and isomorphic to $\overline{M_k^{\mathrm{Sie}}(\mathfrak{H}_2,\Gamma)} \otimes M^{\mathrm{Sie}}_{W_{(k)}}(\mathfrak{H}_2,\Gamma)$. Of course, the right-hand side (apart from our main result) is not, but we can expand

$$H^3(\Gamma\backslash D_{\mathrm{I}}, \mathcal{O}(\mathcal{L}_{-\rho_{\mathrm{I}}})) \cong \bigoplus_{\left\{\begin{array}{c}\pi\in\widehat{G(\mathbb{R})}:\\ \chi_\pi=\chi_0\end{array}\right.} H^p(\mathfrak{n}_{\mathrm{I}}, V_\pi)_{\rho_{\mathrm{I}}}^{\oplus m_\pi(\Gamma)}. \tag{5.6}$$

Unlike the situation for $SU(2,1)$, there are besides the TDLDS other unitary irreps of infinitesimal character zero. In order to proceed, we need to know which ones have $\mathfrak{n}$-cohomology.

**Lemma 5.5.** *$V_{(0,C_I)}$ and $V_{(0,C_{II})}$ are the only unitary irreps of infinitesimal character zero and even-dimensional $K$-types.*

*Proof.* If $P = MAN$ is a minimal parabolic subgroup of $Sp_4(\mathbb{R})$, we have $M \cong \mathbb{Z}_2 \times \mathbb{Z}_2 = \{\mathrm{id}, -\mathrm{id}, \mu, -\mu\}$, with characters $\sigma_0$ (trivial), $\sigma_1$ ($-1$ on $-\mathrm{id}$ and $\mu$), $\sigma_2$ ($+1$ on $-\mathrm{id}$ and $-1$ on $\mu$), and $\sigma_3 = \sigma_1\sigma_2$. Writing $P_0 := AN$, the irreducible constituents of

$$Ind_{P_0}^G(1\otimes 1) \cong \bigoplus_{i=0}^3 Ind_P^G(\sigma_i \otimes 1 \otimes 1) =: \bigoplus_{i=0}^3 (V_{[i]}, \pi_{[i]})$$

are precisely the unitary irreps of $Sp_4(\mathbb{R})$ with infinitesimal character 0 [Vo]. By the theory of the $R$-group [KZ], one deduces that $V_{[0]}$ is irreducible, while each of $V_{[1]}$, $V_{[2]}$, $V_{[3]}$ splits into two irreps [Kn3]. Since $W_K$ exchanges $\sigma_1$ and $\sigma_3$, we have $V_{[1]} \cong V_{[3]}$ which by the main result of [CK] equals the direct sum of the TDLDS. Finally, $\sigma_i(-\mathrm{id}) = -1 \implies \pi_{[i]}(-\mathrm{id}) = -1$; and so by the representation theory of $U(2)$, the dimensions of $K$-types appearing in $V_{[i]}$ are all of parity opposite to that of $i$. □

**Proposition 5.6.** *Let $V$ be a unitary irrep with $\chi_V = \chi_0$. Then*

$$H^*(\mathfrak{n}_I, V) \neq \{0\} \implies V = V_{(0,C_I)} \text{ or } V_{(0,C_{II})}.$$

*Consequently, only $V_{(0,C_I)}$ or $V_{(0,C_{II})}$ can contribute to $H^*(\Gamma\backslash D_I, \mathcal{O}(\mathcal{L}_{-\rho_I}))$.*

*Proof.* Let $(\bigwedge_{j=1}^n \omega^{\gamma_j}) \otimes v$ be a nonzero monomial term in a representative of a class in $H^n(\mathfrak{n}_{\rm I}, V)$. By [CO], $H^n(\mathfrak{n}_{\rm I}, V) = H^n(\mathfrak{n}_{\rm I}, V)_{\rho_{\rm I}}$. Writing $\mathcal{R} \subset \Lambda$ for the root lattice, we therefore have $\text{weight}(v) - \sum \gamma_j = \rho_{\rm I} \notin \mathcal{R} \implies \text{weight}(v) \notin \mathcal{R} \implies v$ cannot belong to an odd-dimensional $K$-type. Now apply Lemma 5.5. □

Combining this result with Proposition 3.11, the expansion in equation (5.6) simplifies to

$$\left\{H^3(\mathfrak{n}_{\rm I}, V_{(0,C_{\rm II})})_{\rho_{\rm I}}\right\}^{\oplus m_{(0,C_{\rm II})}(\Gamma)}$$

where (anticipating §6.1) the $H^3$ has dimension 1. In fact, it is generated by a representative of the form $v\omega^{2\alpha_2} \wedge \omega^{-2\alpha_1} \wedge \omega^{-(\alpha_1+\alpha_2)} \in \bigwedge^3 \check{\mathfrak{n}}_{\rm I} \otimes V_{(0,C_{\rm II})}$, where $v \in \ker(X_{-\alpha_1+\alpha_2}) \subset \mathsf{W}_0^{\rm II}$ has weight $-\alpha_1$. (Indeed, this has weight $-\alpha_1 - 2\alpha_2 + 2\alpha_1 + (\alpha_1 + \alpha_2) = 2\alpha_1 - \alpha_2 = \rho$, and is clearly closed: higher degree $\mathfrak{n}$-cohomology is often easier to compute.) Reasoning as for $SU(2,1)$, we conclude that the right-hand side of (5.5) is isomorphic to

$$\left\{ F \in C^\infty(\Gamma\backslash Sp_4(\mathbb{R})) \;\middle|\; \begin{array}{c} F \in \ker(X_{-\alpha_1+\alpha_2}),\ \Omega(F) = \frac{-5}{12}F, \\ F \text{ transforms with weight } -\alpha_1 \\ \text{under the right action of } H \end{array} \right\},$$

in which the transformation law is exactly that for a section $\sigma_F$ of the tautological Hodge bundle $\mathcal{V}^{3,0}$ over $\Gamma\backslash D$. Furthermore, if $\tilde{\sigma}_F$ is its lift to $D$, $F \in \ker(X_{-\alpha_1+\alpha_2})$ is equivalent to holomorphicity of the restrictions $\tilde{\sigma}_F|_{gZ}$ to the translates($\cong \mathbb{P}^1$) of $K/H$. Certainly these functions merit further study, as do the spaces $H^*(D_{\rm I}, \mathcal{O}(\mathcal{L}_{-\rho_{\rm I}}))$ and their concrete geometric interpretation in general.

## 6. Proof of Theorem 5.4

In the first two subsections below, which are of a preliminary nature, we compute the $\mathfrak{n}$-cohomology classes that we shall need, as well as representatives of those classes. Then we turn (5.5) into a virtual injectivity statement, and reduce that to an assertion about cup-products in Lie-algebra cohomology. We see no reason why these methods should not generalize (with some computational effort of course) to arbitrary reductive groups, or at least those of Hermitian type.

Henceforth $\mathfrak{n}$ shall denote $\mathfrak{n}_{\mathrm{I}}$.

6.1. **Degeneration of the Hochschild-Serre spectral sequence.** We prove that the sequences computing $H^*(\mathfrak{n}, V_{(0,C_{\mathrm{I}})})$ and $H^*(\mathfrak{n}, V_{(0,C_{\mathrm{II}})})$ (whose direct sum has $E_2$ page (3.6)) have $E_2 = E_\infty$. The idea for the argument, which is more convenient for our purposes than using [So][8], is due to [Sc3].

By [CO] and the results in §3, writing $\mathsf{V} := V_{\rho_{\mathrm{I}}} \otimes \mathfrak{W}_{-\rho_{\mathrm{I}}}$, we have

$$H^*(\mathfrak{n}, \mathsf{V})_{\rho_{\mathrm{I}}} = H^*(\mathfrak{n}, p_0\mathsf{V})_{\rho_{\mathrm{I}}} = H^*(\mathfrak{n}, V_{(0,C_{\mathrm{I}})}). \tag{6.1}$$

Let $\mathfrak{F}^\bullet$ be the decreasing filtration of $\mathfrak{W}_{-\rho_{\mathrm{I}}}$ by $\mathfrak{b}(= \mathfrak{n} + \mathfrak{h})$-submodules, with $Gr^i_{\mathfrak{F}}\mathfrak{W}_{-\rho_{\mathrm{I}}} \cong \mathbb{C}_{\lambda_i}$ ($\lambda_i$ a weight of $\mathfrak{W}_{-\rho_{\mathrm{I}}}$) and top filtrand $\mathfrak{F}^0\mathfrak{W}_{-\rho_{\mathrm{I}}} \cong \mathbb{C}_{-\rho_{\mathrm{I}}}$. (We can choose this filtration so that, say, $\mathbb{C}_{\rho_{\mathrm{II}}} = Gr^{-10}_{\mathfrak{F}}\mathfrak{W}_{-\rho_{\mathrm{I}}}$.) This produces a filtration by $\mathfrak{b}$-modules on the entire complex $\bigwedge^\bullet \check{\mathfrak{n}} \otimes \mathsf{V}$, with associated spectral sequence

$$\mathcal{E}_1^{i,j} = H^{i+j}(\mathfrak{n}, V_{\rho_{\mathrm{I}}} \otimes Gr^i_{\mathfrak{F}}\mathfrak{W}_{-\rho_{\mathrm{I}}})_{\rho_{\mathrm{I}}} = H^{i+j}(\mathfrak{n}, V_{\rho_{\mathrm{I}}})_{\rho_{\mathrm{I}}-\lambda_i} \otimes \mathbb{C}_{\lambda_i} \tag{6.2}$$

converging to (6.1).

We will need the next result, which follows from the proof of Theorem 3.2(cf. Lemmas 2 and 4 in [Sc1, sec. 3]):

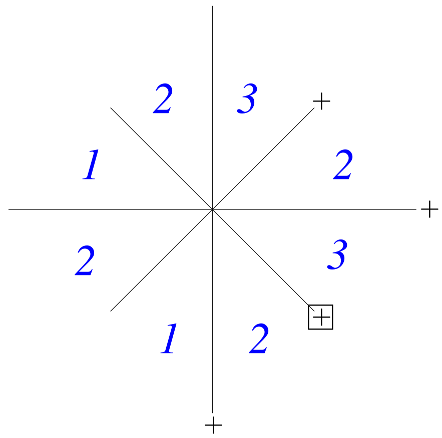
**Lemma 6.1.** *Given $\lambda, \mu + \rho \in \Lambda$, with $\lambda$ regular (so that $V_\lambda$ is in the discrete series),*

$$\dim H^p(\mathfrak{n}, V_\lambda)_{-\mu} = \begin{cases} 1, & \text{if } p = q(\mu+\rho) \quad \text{and } -(\mu+\rho) \in W_K.\lambda \\ 0, & \text{otherwise} \end{cases}.$$

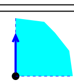
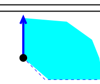
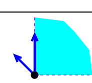
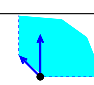
By [CO] (or the Lemma), the only possible weights of $H^*(\mathfrak{n}, V_{\rho_{\mathrm{I}}})$ are $2\rho_{\mathrm{I}}$ and $\rho_{\mathrm{I}} - \rho_{\mathrm{II}}$; the Lemma also tells us that only $H^1$ [resp. $H^2$] is nonzero in weight $2\rho_{\mathrm{I}}$ [resp. $\rho_{\mathrm{I}} - \rho_{\mathrm{II}}$]. Therefore, in (6.2), only $\mathcal{E}_1^{-10,12}$ and $\mathcal{E}_1^{0,1}$ are nontrivial, of dimension 1. But $d_k$ maps $\mathcal{E}_k^{i,j} \to \mathcal{E}^{i+k,j-k+1}$, hence fails (for any $k$) to connect the nonzero terms, and $\mathcal{E}_1^{\bullet,\bullet} = \mathcal{E}_\infty^{\bullet,\bullet}$. It is now clear (referring to (6.1) and Prop. 3.11) that $d_{\mathrm{I}}^{(1)} = d_{\mathrm{I}}^{(2)} = 1$; a similar argument works for $V_{(0,C_{\mathrm{II}})}$, and we conclude that (3.6) collapses at $E_2$.

[8] Moreover, it has been brought to our attention that the arguments there may not be complete.

6.2. **Representatives of $\mathfrak{n}$-cohomology classes.** We begin with the DS $V_{(k)}$ and $\tilde{V}_{(k)}$ from Example 3.5. Fixing the $\Phi^+$ corresponding to $\mathfrak{n}(=\mathfrak{n}^-)$, we have $q(\cdot)\ =$




According to Lemma 6.1, $V_{(k)} = V_{\lambda_k}$ $(\lambda_k = \Lambda_k - \alpha_1 - 2\alpha_2 \underset{W_K}{\equiv} \Lambda_k - 2\alpha_1 - \alpha_2)$ has nonzero cohomologies $H^1(\mathfrak{n}, V_{(k)})_{-\mu_k^{(1)}}$ and $H^2(\mathfrak{n}, V_{(k)})_{-\mu_k^{(2)}}$ with $-\mu_k^{(1)} = \Lambda_k - 2\alpha_2$ and $-\mu_k^{(2)} = \Lambda_k + \alpha_1 - 3\alpha_2$. For $\tilde{V}_{(k)} = V_{\tilde{\lambda}_k}$ $(\tilde{\lambda}_k = \tilde{\Lambda}_k - \alpha_1 - 2\alpha_2 \underset{W_K}{\equiv} \tilde{\Lambda}_k - 3\alpha_1)$, $H^1(\mathfrak{n}, \tilde{V}_{(k)})_{-\tilde{\mu}_k^{(1)}}$ and $H^2(\mathfrak{n}, \tilde{V}_{(k)})_{-\tilde{\mu}_k^{(2)}}$ are the nontrivial spaces, with $-\tilde{\mu}_k^{(2)} = \tilde{\Lambda}_k + \alpha_1 - 3\alpha_2$ and $-\tilde{\mu}_k^{(1)} = \tilde{\Lambda}_k - \alpha_1 - \alpha_2$. Knowing the weights makes it easy to guess the representatives, which are summarized in the table:

| | $V_{(k)}$ | $\tilde{V}_{(k)}$ |
|---|---|---|
| $H^1(\mathfrak{n}, \cdots)$ | $v_k \omega^{2\alpha_2}$  | $X_{-\alpha_1+\alpha_2}(w_k)\omega^{2\alpha_2}$  |
| $H^2(\mathfrak{n}, \cdots)$ | $v_k \omega^{2\alpha_2} \wedge \omega^{-\alpha_1+\alpha_2}$  | $w_k \omega^{2\alpha_2} \wedge \omega^{-\alpha_1+\alpha_2}$  |

Here $v_k$ (of weight $\Lambda_k$) generates the $0^{\text{th}}$ $K$-type $\mathcal{S}_0(2k)$ of $V_{(k)}$, while the two vectors $\{w_k, X_{-\alpha_1+\alpha_2}(w_k)\}$ (with weights $\tilde{\Lambda}_k$, $\tilde{\Lambda}_k - \alpha_1 + \alpha_2$) span the $0^{\text{th}}$ $K$-type $\mathcal{S}_1(2k+1)$ of $\tilde{V}_{(k)}$. The representatives are trivially closed and (as the reader may check) of the correct weights.[9]

For $V_{(0,C_{\text{I}})}$ we shall only compute a representative of $H^1(\mathfrak{n}, V_{(0,C_{\text{I}})})_{\rho_{\text{I}}}$. Let $\{u, X_{-\alpha_1+\alpha_2}(u)\}$ span $W_0^{\text{I}} = \mathcal{S}_1(1)$, with $u$ of weight $\alpha_1$. Since (3.6) degenerates

[9]We remind the reader that $\omega^{\alpha}$ has weight $-\alpha$.

by §6.1, the "Kostant element"

$$u\omega^{-\alpha_1+\alpha_2} \in H^1(\mathfrak{n}_K, V_{(0,C_{\mathrm{I}})})_\rho = E_2^{0,1}$$

survives in that spectral sequence. This produces a generator of the form

$$u\omega^{-\alpha_1+\alpha_2} - A\omega^{-(\alpha_1+\alpha_2)} - B\omega^{-2\alpha_1} - C\omega^{2\alpha_2} \tag{6.3}$$

where $A$, $B$, $C$ have respective weights $\alpha_1 - 2\alpha_2$, $-\alpha_2$, $2\alpha_1 + \alpha_2$ (so that the weight of each monomial is $\rho_{\mathrm{I}}$). This is crucial to the argument in §6.5.

6.3. **Reduction of the Theorem to representation theory.** Using Serre duality and $K_{\Gamma\backslash D_{\mathrm{I}}} \cong \mathcal{L}_{-2\rho_{\mathrm{I}}}$, and noting that $\mu_k^{(1)} - \rho_{\mathrm{I}} = \tilde{\mu}_k^{(2)}$, virtual surjectivity in (5.5) is equivalent to

(6.4)

$$\begin{array}{ccccc} H^1(\Gamma\backslash D_{\mathrm{I}}, \mathcal{O}(\mathcal{L}_{\mu_k^{(1)}})) & \otimes & H^1(\Gamma\backslash D_{\mathrm{I}}, \mathcal{O}(\mathcal{L}_{-\rho_{\mathrm{I}}})) & \tilde{\hookrightarrow} & H^2(\Gamma\backslash D_{\mathrm{I}}, \mathcal{O}(\mathcal{L}_{\tilde{\mu}_k^{(2)}})) \\ \| & & \| & & \| \\ H^1(\mathfrak{n}, V_{(k)})^{\oplus m_k(\Gamma)}_{-\mu_k^{(1)}} & & H^1(\mathfrak{n}, V_{(0,C_{\mathrm{I}})})^{\oplus m_0(\Gamma)}_{\rho} & & H^2(\mathfrak{n}, \tilde{V}_{(k)})^{\oplus \tilde{m}_k(\Gamma)}_{-\tilde{\mu}_k^{(2)}} \end{array}.$$

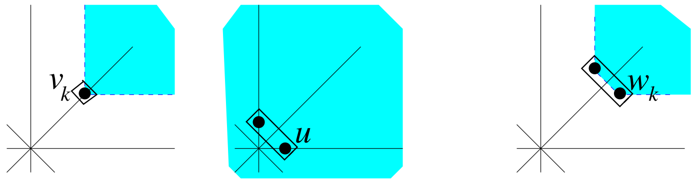
(The vertical equalities follow from §§6.1, 2 and Theorem 3.9.) Since $Sp_4$ is Hermitian and (for $k \geq 5$) $\tilde{V}_{(k)}$ is integrable, the general automorphic machinery of [Ha, sec. 7] and [C1, sec. 5] applies as in [GGK2, sec. IV.F Step 5] essentially without change. We do not reproduce the argument here, which reduces (6.4) (hence Theorem 5.4) to the following analogue of Théorème 4.2 in [C1]:

**Proposition 6.2.** *(i) $\tilde{\mathfrak{V}}_{(k)}$ appears with multiplicity one in $\mathfrak{V}_k \hat{\otimes} \mathfrak{V}_{(0,C_I)}$ (as irreducible sub-representation);*
*(ii) The corresponding projection $V_{(k)} \otimes V_{(0,C_I)} \overset{p}{\twoheadrightarrow} \tilde{V}_{(k)}$ of H-C modules induces (by restriction) a bijection between the tensor product of minimal $K$-types on the left and minimal $K$-types on the right; and*
*(iii) The composition of cup-product with this projection yields an isomorphism*

$$H^1(\mathfrak{n}, V_{(k)})_{-\mu_k^{(1)}} \otimes H^1(\mathfrak{n}, V_{(0,C_I)})_{\rho_I} \overset{\cong}{\to} H^2(\mathfrak{n}, \tilde{V}_{(k)})_{-\tilde{\mu}_k^{(2)}}.$$

The "multiplicity $\geq 1$" part of $(i)$ is dealt with exactly as in [C1, sec. 4] (also see [GGK2, sec. IV.F Step 4]) using Prop. 1.2.3 of [HL]: beyond noting the integrability of $V_{(k)}$, we only need to check that the minimal $K$-types of

$$\mathfrak{V}_{(k)} \hat{\otimes} \mathfrak{V}_{(0,C_{\mathrm{I}})} \quad \text{and} \quad \tilde{\mathfrak{V}}_{(k)} \;-$$




namely $\mathcal{S}_0(2k) \otimes \mathcal{S}_1(1)$ and $\mathcal{S}_1(2k+1)$ – agree, which they evidently do.

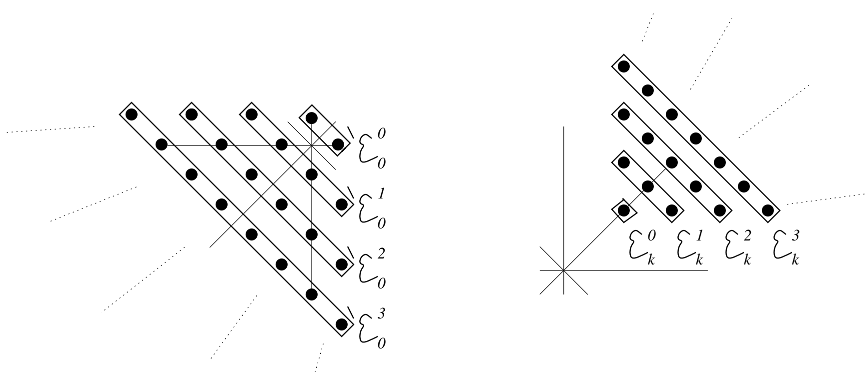
6.4. **Establishing the multiplicity one result.** At this point the hardest remaining step is the "multiplicity $\leq 1$" part of $(i)$ above. Our approach generalizes Carayol's in [C1, sec. 4].

We begin with the initial

**Observation 6.3.** *$\mathcal{S}_a(\alpha) \otimes \mathcal{S}_b(\beta)$ only contains a copy of $\mathcal{S}_1(2k+1)$ if $|a-b| = 1$ and $\alpha + \beta = 2k+1$ (and then contains exactly one).*

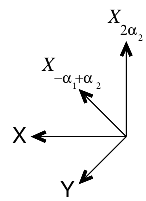
Here we remind the reader that $\mathcal{S}_1(2k+1)$ is the lowest $K$-type of $\tilde{V}_{(k)}$. Referring to Examples 3.5 and 3.6, particularly (3.3) and (3.5), we find that $V_{(0,C_{\mathrm{I}})}$ has no $K$-types "compatible" (in the sense of the Observation) with any of the $K$-types in $V_{(k)}/\mathsf{F}_0 V_{(k)}$; and it has exactly one $K$-type $'\mathcal{E}_0^n \cong \mathcal{S}_{2n+1}(1-2n)$ compatible with each $K$-type $\mathcal{E}_k^n \cong \mathcal{S}_{2n}(2n+2k)$ of $\mathsf{F}_0 V_{(k)}$. In fact, we have

$$\mathsf{F}_0 V_{(0,C_{\mathrm{I}})} = \bigoplus_{n \geq 0} {}'\mathcal{E}_0^n \quad \text{and} \quad \mathsf{F}_0 V_{(k)} = \bigoplus_{n \geq 0} \mathcal{E}_k^n$$




so that any copy of $\mathcal{S}_1(2k+1)$ in $\mathfrak{V}_{(k)} \otimes \mathfrak{V}_{(0,C_{\mathrm{I}})}$ is contained in the closure of $\mathcal{A}_k := \mathsf{F}_0 V_{(k)} \otimes \mathsf{F}_0 V_{(0,C_{\mathrm{I}})}$.

We shall need to understand the action of (some of) $\mathfrak{n}$ on $\mathcal{A}_k$. Begin by relabeling $X_{-2\alpha_1} =: \mathsf{X}$, $X_{-\alpha_1-\alpha_2} =: \mathsf{Y}$




as shown. By looking at weights, it is clear that

$$\mathsf{X}\mathcal{E}_k^0 = \{0\} = \mathsf{Y}\mathcal{E}_k^0. \tag{6.5}$$

The following will be proved in §6.6:

**Lemma 6.4.** *$\mathsf{F}_0 V_{(0,C_I)}$ and $\mathsf{F}_0 V_{(k)}$ are closed under the action of $\mathsf{X} := X_{-2\alpha_1}$ and $\mathsf{Y} := X_{-(\alpha_1+\alpha_2)}$.*

By the Lemma, $\mathcal{A}_k$ is closed under the action of $x := \mathsf{X} \otimes \mathrm{id}$, $y := \mathsf{Y} \otimes \mathrm{id}$, $x' := \mathrm{id} \otimes \mathsf{X}$, $y' := \mathrm{id} \otimes \mathsf{Y}$ on $V_{(k)} \otimes V_{(0,C_{\mathrm{I}})}$; it is evident from the pictures that

$$\left\{ \begin{array}{l} x \text{ and } y \text{ map } \mathcal{E}_k^n \otimes {}'\mathcal{E}_0^n \to \mathcal{E}_k^{n-1} \otimes {}'\mathcal{E}_0^n \\ x' \text{ and } y' \text{ map } \mathcal{E}_k^n \otimes {}'\mathcal{E}_0^n \to \mathcal{E}_k^n \otimes {}'\mathcal{E}_0^{n+1} \end{array} \right. ,$$

and of course $\mathsf{X}$ acting on the tensor product is $x + x'$ (likewise for $\mathsf{Y}$).

Now let $\mathsf{W} \cong \mathcal{S}_1(2k+1)$ be the minimal $K$-type of a copy of $\tilde{\mathfrak{V}}_{(k)}$ inside $\mathfrak{V}_{(k)} \otimes \mathfrak{V}_{(0,C_{\mathrm{I}})}$. We have just seen that

$$\mathsf{W} \subset \ker(\mathsf{X}) \cap \ker(\mathsf{Y}) \cap \overline{\bigoplus_{n \geq 0} \mathcal{E}_k^n \otimes {}'\mathcal{E}_0^n} \subset \overline{\mathsf{F}_0 V_{(k)} \otimes \mathsf{F}_0 V_{(0,C_{\mathrm{I}})}}, \tag{6.6}$$

where the bar denotes closure. Write $\xi \in \mathsf{W}$ for a generator in weight $\tilde{\Lambda}_k$, so that $\mathsf{W} = \mathbb{C}\langle \xi, X_{-\alpha_1+\alpha_2}(\xi)\rangle$, and decompose $\xi = \xi_0 + \xi_1 + \xi_2 + \cdots$, with each $\xi_n \in \mathcal{E}_k^n \otimes {}'\mathcal{E}_0^n$. Using (6.5), we then have

$$\left\{ \begin{array}{l} 0 = \mathsf{X}\xi = (x'\xi_0 + x\xi_1) + (x'\xi_1 + x\xi_2) + \cdots \\ 0 = \mathsf{Y}\xi = (y'\xi_0 + y\xi_1) + (y'\xi_1 + y\xi_2) + \cdots \end{array} \right. \tag{6.7}$$

where parentheses group terms by tensor products of $K$-types.

Suppose $\xi_0 = 0$. Then from (6.7), $x\xi_1 = 0 = y\xi_1$. This locates $\xi_1$ in

$$\left\{ \begin{array}{c} \text{lowest weight} \\ \text{part of } \mathcal{E}_k^1 \end{array} \right\} \otimes {}'\mathcal{E}_0^1,$$

which has trivial intersection with the copy of $\mathcal{S}_1(2k+1)$ in $\mathcal{E}_k^1 \otimes {}'\mathcal{E}_0^1$. So $\xi_1 = 0$. Using (6.7) again, we have $x\xi_2 = 0 = y\xi_2$, and so $\xi_2$ belongs to

$$\left\{ \begin{array}{c} \text{lowest weight} \\ \text{part of } \mathcal{E}_k^2 \end{array} \right\} \otimes {}'\mathcal{E}_0^2,$$

which forces $\xi_2 = 0$. Continuing in this manner, we find that $\xi = 0$, a contradiction.

We conclude that there is only one copy of $\tilde{\mathfrak{V}}_{(k)}$ in $\mathfrak{V}_{(k)} \otimes \mathfrak{V}_{(0,C_{\mathrm{I}})}$: otherwise, we would have a direct sum of (at least 2) $K$-sub-representations $\mathsf{W} \cong \mathcal{S}_1(2k+1)$ satisfying (6.6). This would force the existence of a nontrivial solution to (6.7) with $\xi_0 = 0$, which as we have just seen is impossible.

6.5. **Conclusion of the proof.** With Proposition 6.2($i$) established, let $p : V_{(k)} \otimes V_{(0,C_{\mathrm{I}})} \twoheadrightarrow \tilde{V}_{(k)}$ be the restriction of the resulting orthogonal projection, and $i : \tilde{V}_{(k)} \hookrightarrow \mathfrak{V}_{(k)} \otimes \mathfrak{V}_{(0,C_{\mathrm{I}})}$ the inclusion. We consider $v_k \in \mathcal{E}_k^0$ and $u \in {}'\mathcal{E}_0^0$ in the lowest $K$-types of the tensor factors. Denote the lowest $K$-type of $\tilde{V}_{(k)}$ by $\mathsf{W}_0$.

Suppose $p(v_k \otimes u) = 0$. Then $v_k \otimes u$ is orthogonal to $i(\mathsf{W}_0)$, and in particular to $\xi := i(w_k) = \xi_0 + \xi_1 + \cdots$, where $\xi_0 \in \mathcal{E}_k^0 \otimes {}'\mathcal{E}_0^0$ is nonzero (by §6.4) hence some nonzero multiple of $v_k \otimes u$. But this is impossible, since (by virtue of the orthogonality of the $\{\mathcal{E}_k^n\}$) the $\xi_i$ are orthogonal. Hence $p(v_k \otimes u) \neq 0$, and the only possibility (due to weights) is that

$$p(v_k \otimes u) = c w_k, \; c \neq 0, \tag{6.8}$$

establishing ($ii$).

To show ($iii$), we need to use the representatives constructed in §6.2: referring to the table and (6.3), we have the generators

(1) $v_k \omega^{2\alpha_2}$ for $H^1(\mathfrak{n}, V_{(k)})_{-\mu_k^{(1)}}$
(2) $u\omega^{-\alpha_1+\alpha_2} - A\omega^{-(\alpha_1+\alpha_2)} - B\omega^{-2\alpha_1} - C\omega^{2\alpha_2}$ for $H^1(\mathfrak{n}, V_{(0,C_{\mathrm{I}})})$
(3) $w_k \omega^{2\alpha_2} \wedge \omega^{-\alpha_1+\alpha_2}$.

Wedging the first two together gives

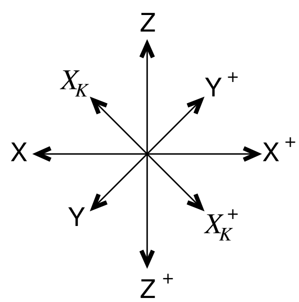
$$(v_k \otimes u)\omega^{2\alpha_2} \wedge \omega^{-\alpha_1+\alpha_2} - (v_k \otimes A)\omega^{2\alpha_2} \wedge \omega^{-(\alpha_1+\alpha_2)} - (v_k \otimes B)\omega^{2\alpha_2} \wedge \omega^{-2\alpha_1},$$

where we note that the $C$ term disappears. Applying $p$ to the coefficient vectors (and using (6.8)) yields

$$c w_k \omega^{2\alpha_2} \wedge \omega^{-\alpha_1+\alpha_2},$$

because the weights $\Lambda_k + \alpha_1 - 2\alpha_2$ (of $v_k \otimes A$) and $\Lambda_k - \alpha_2$ (of $v_k \otimes B$) live outside the region where the weights of $\tilde{V}_{(k)}$ are located. Thus we have finished off $(i)$-$(iii)$ modulo the

6.6. **Proof of Lemma 6.4.** For this it will be convenient to record the bracket relations for the root vectors of $sp_4$. We rename these as follows to minimize notation:




where $\mathsf{X}$ and $\mathsf{Y}$ are as above, and $X_K, X_K^+$ generate $Lie(K)$. Adjacent root vectors commute as do the "long root" vectors, since their weights do not sum to roots. Otherwise we have

$$[\mathsf{X}^+, X_K] = -\mathsf{Y}^+,\ [\mathsf{X}^+, \mathsf{Y}] = X_K^+,\ [\mathsf{X}, \mathsf{Y}^+] = -X_K,\ [\mathsf{X}, X_K^+] = \mathsf{Y},$$

$$[\mathsf{Z}, \mathsf{Y}] = X_K,\ [\mathsf{Z}, X_K] = -\mathsf{Y}^+,\ [\mathsf{Z}^+, X_K] = \mathsf{Y},\ [\mathsf{Z}^+, \mathsf{Y}^+] = X_K^+,$$

$$[X_K^+, \mathsf{Y}] = 2\mathsf{X}^+,\ [X_K, \mathsf{Y}^+] = -2\mathsf{X},\ [\mathsf{Y}^+, X_K] = -2\mathsf{Z},\ [\mathsf{Y}, X_K^+] = 2\mathsf{Z}^+,$$

and writing $H_1, H_2$ for generators of $\mathfrak{h}$,

$$[\mathsf{X}^+, \mathsf{X}] = H_1,\ [\mathsf{Y}^+, \mathsf{Y}] = H_1 + H_2,\ [\mathsf{Z}^+, \mathsf{Z}] = -H_2,\ [X_K^+, X_K] = H_1 - H_2.$$

For the Lemma, we deal with $V_{(0,C_1)}$ first. Given $\mu \in {}'\mathcal{E}_0^n$ of pure weight $\lambda_k^n := \alpha_2 - 2n\alpha_1 + k(\alpha_1 - \alpha_2)$, $(X_K)^k \mu$ is a lowest weight vector in ${}'\mathcal{E}_0^n$ and

$(X_K)^{k+1}\mu = 0$. Since $X_K$ and $\mathsf{X}$ are adjacent we have

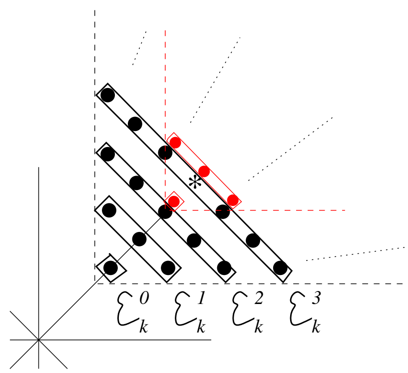
$$(X_K)^{k+1}\mathsf{X}\mu = \mathsf{X}(X_K)^{k+1}\mu = 0,$$

which establishes that $\mathsf{X}\mu$ lands in ${}'\mathcal{E}_0^{n+1}$ and (referring to the picture in Example 3.6) not a "wider" $K$-type. It follows that $(X_K)^{k+2}X_K^+\mathsf{X}\mu = 0$ as well. Now clearly $X_K^+\mu$ is pure of weight $\lambda_{k+1}^n$ and in ${}'\mathcal{E}_0^n$, so that $0 = (X_K)^{k+2}X_K^+\mu$. Using $\mathsf{Y} = [\mathsf{X}, X_K^+]$ gives

$$(X_K)^{k+2}\mathsf{Y}\mu = (X_K)^{k+2}\mathsf{X}X_K^+\mu - (X_K)^{k+2}X_K^+\mathsf{X}\mu = 0,$$

proving $\mathsf{Y}\mu \in {}'\mathcal{E}_0^{n+1}$.

Turning to $V_{(k)}$, in the simplest case the issue is that in the picture



we want to show that $\mathsf{Y}$ of the vector "*" in $\mathcal{E}_k^3$ cannot go into the 1-dimensional $K$-irrep to its lower left, and instead belongs to $\mathcal{E}_k^2$. So let $z_n \in \mathcal{E}_k^n$ be the lowest weight vector and consider $(X_K^+)^\ell z_n (\neq 0)$ for any $\ell \in \mathbb{Z} \cap [0, 2n]$. We want to show that this is sent by $\mathsf{X}$ and $\mathsf{Y}$ to something in $\mathcal{E}_k^{n-1}$. First, since $[\mathsf{Z}^+, X_K^+] = 0$ and $\mathsf{Z}^+ z_n \in \mathcal{E}_k^{n-1}$, we have

$$\mathsf{Z}^+(X_K^+)^\ell z_n = (X_K^+)^\ell \mathsf{Z}^+ z_n \in \mathcal{E}_k^{n-1}.$$

Hence $\oplus_{n\geq 0}\mathcal{E}_k^n$ is closed under $X_K^+, X_K, \mathsf{Z}^+$ and

$$\mathsf{Y}(X_K^+)^\ell z_n = [\mathsf{Z}^+, X_K](X_K^+)^\ell z_n \in \mathcal{E}_k^{n-1}$$

shows it is also closed under $\mathsf{Y}$ so that

$$\mathsf{X}(X_K^+)^\ell z_n = -\frac{1}{2}[X_K, \mathsf{Y}](X_K^+)^\ell z_n \in \mathcal{E}_k^{n-1},$$

which completes the proof.

It is remarkable that precise formulas for the action of $\mathfrak{g} = sp_4$ on $V_{(k)}$ and $V_{(0,C_{\rm I})}$ (in terms of a set of generators of the $K$-types, as given in sections 1 and 4 of [C1] for $su(2,1)$) turn out to be unnecessary in proving the main Theorem 5.4. What we did need were: (a) the statement that $V_{(0,C_{\rm I})}$ is given by $H^1(D_{\rm II}, \mathcal{O}(\mathcal{L}_{-\rho_{\rm II}}))$, from which we obtained the $K$-type decomposition; (b) qualitative results on the action of $\mathfrak{g}$ that follow from the form of that decomposition (and the bracket relations for $\mathfrak{g}$); (c) the definition of $V_{(0,C_{\rm I})}$ and $V_{(0,C_{\rm II})}$ by Zuckerman tensoring; and (d) the consequent degeneration of the Hochschild-Serre spectral sequence leading to the results on their $\mathfrak{n}$-cohomology. (For $V_{(k)}$ and $\tilde{V}_{(k)}$, the results of Schmid on discrete series were sufficient.) This seems promising for the prospects of generalizing the main result to groups of higher rank, or even exceptional groups.

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