A SURVEY OF TRANSCENDENTAL METHODS IN THE STUDY OF CHOW GROUPS OF ZERO-CYCLES

MATT KERR

Abstract. We review an assortment of candidate Bloch-Beilinson filtrations on $CH_0(X)$ (for X smooth projective), together with Hodge-theoretic invariants defined on their graded pieces. A large array of applications is given, especially to detecting nontrivial 0-cycles and the behavior of said invariants with respect to families. Ample background is provided, including many of the theorems and conjectures that have shaped the passage from classical to modern invariants.

Contents

1991 *Mathematics Subject Classification.* 14D07, 14C15, 14C25, 14C30, 14J32.
Key words and phrases. algebraic cycle, Chow group, Bloch-Beilinson filtration, Hodge theory, higher Abel-Jacobi map, Mumford invariants, spreads.

1. Introduction

A driving force behind modern algebraic geometry has been the notion that certain topological, analytic or arithmetic invariants on a projective *algebraic* variety X should be represented or explained by *algebraically* defined objects "on X." So for example the HC (Hodge Conjecture) says that the rational (p, p) classes of X should be generated (modulo torsion) by the fundamental classes of its codimension-p subvarieties. Or for X defined over a number field, the Beilinson conjectures predict that special values of its L-functions should be computable by (determinants of) regulator images of elements in algebraic K-groups of X. These images ultimately boil down to integrals of algebraic differential forms (on X) over subsets of X bounded or cut out by algebraic equations.

In both instances, one has to get a handle on invariants of certain algebraic cycles — whether algebraic K-theory elements or linear combinations of subvarieties of X — and the study of such cycles modulo a suitable equivalence relation becomes a natural pursuit.

Zero-cycles — $\mathbb{Q}$-linear combinations of (zero-dimensional) points on X — are the simplest kind of algebraic cycles, and their study modulo rational equivalence is a rich area with many beautiful results (and unresolved conjectures). Historically, the first example is Abel's theorem, which links an algebraic question (when is a degree-0 divisor on an algebraic curve the divisor of a rational function?) with a transcendental invariant (the image of the divisor in the Jacobian, which involves integration along paths connecting the points).

For 0-cycles on varieties of dimension > 1, on the other hand, recent work has concentrated on the influence of the field of definition on their rational equivalence-classes. This paper is about new techniques and Hodge-theoretic invariants — e.g. higher cycle-classes and higher Abel-Jacobi classes — that reflect this influence, and about interesting examples of cycles they detect. It draws on work of Asakura, Bloch, Green, Griffiths, Lewis, M. Saito, S. Saito, Voevodsky, Voisin and others (and some of our own).

We offer some theorems (with proofs) about 0-cycles on products of curves and complete intersections, and a discussion of infinitesimal/topological invariants for "higher normal functions" arising from the invariants. A brisk review of background material is included, as this is rapidly becoming basic algebraic geometry and we believe it should be accessible.

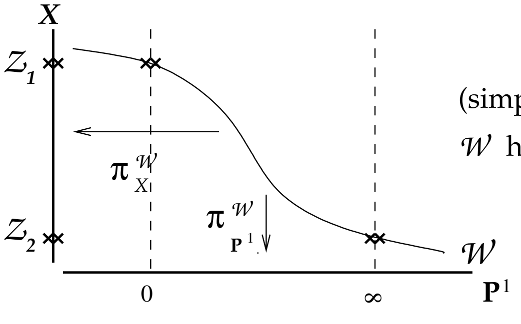
We warmly thank D. Arapura, S. Bloch, P. Brosnan, M. Green, R. de Jeu, J. Lewis, and C. Voisin for helpful conversations and correspondence, and again thank P. Griffiths for introducing us to the subject. We thank the referee for helpful comments which have led to improvements in the exposition. This paper was written at the University of Chicago and MPI-Bonn; we are grateful to both institutions for their support.

2. PRELIMINARIES

2.1. **Cycles and Rational Equivalence.** Even for complex algebraic geometers, a smooth (complex) projective variety X is ultimately cut out by homogeneous polynomials with coefficients belonging to some field $L \subseteq \mathbb{C}$ finitely generated over $\mathbb{Q}$; hence we may consider the underlying variety $/L$ with structure map $X \to Spec(L)$. If $K \supseteq L$ we write $X_K := X_{(L)} \times_L Spec(K)$ for the base change, and throughout the paper $d = \dim(X)$.

All cycle groups and cohomology groups are taken modulo torsion; so the algebraic cycle group $Z^p(X_K) = Z_{d-p}(X_K)$ denotes *rational* linear combinations of irreducible subvarieties of X, defined $/K$ and of codimension p. Hence a 0-cycle $\mathcal{Z} \in Z_0(X_K)$ is (after base-change to $\bar{K}$) of the form $\sum q_i z_i$, with $q_i \in \mathbb{Q}$ and z_i having coordinates in $\bar{K}$, and such that $\sum q_i z_i$ is invariant under the action of $Gal(\bar{K}/K)$ on $X(\bar{K})$. For example, on the affine line $\{\sqrt{3}\} + \{-\sqrt{3}\} - 5\{1\}$ is [the base-change to $\bar{\mathbb{Q}}$ of] a 0-cycle defined $/\mathbb{Q}$, but $2\{\sqrt{3}\} - 2\{-\sqrt{3}\}$ is not.

We say two cycles $\mathcal{Z}_1, \mathcal{Z}_2 \in Z^p(X_K)$ are *rationally equivalent* if (roughly speaking) one can get from $\mathcal{Z}_1$ to $\mathcal{Z}_2$ in a rationally parametrized family $\mathcal{W}$:



(simplest case: $\mathcal{W}$ has one irreducible component.)

More precisely, let $\pi_X (\text{resp. } \pi_{\mathbb{P}^1}) : X \times \mathbb{P}^1 \to X (\text{resp. } \mathbb{P}^1)$ be the projections; then

$$\mathcal{Z}_1 \overset{\text{rat}}{\equiv} \mathcal{Z}_2 \quad (\text{or } \mathcal{Z}_1 - \mathcal{Z}_2 \overset{\text{rat}}{\equiv} 0) \quad \overset{\text{defn}}{\Longleftrightarrow}$$

$$\begin{array}{c} \exists\, \mathcal{W} \in Z^p(X \times \mathbb{P}^1_{/K}) \text{ such that} \\ \pi_{X_*}[\mathcal{W}\cdot(X \times \{0\}) - \mathcal{W}\cdot(X\times\{\infty\})] \,=\, \mathcal{Z}_1 - \mathcal{Z}_2 \end{array} \quad \Longleftrightarrow$$

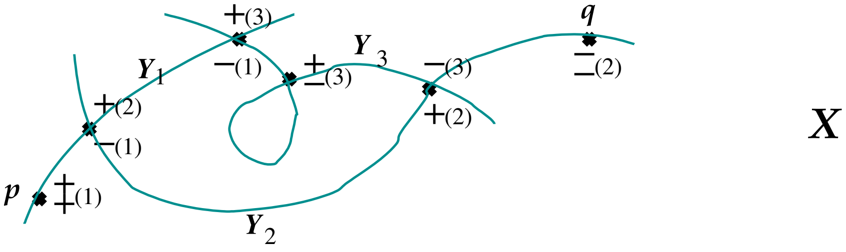
$$\exists \left\{ \begin{array}{l} \widetilde{Y}_i \xrightarrow{\iota_i}\!\!\!\!\!\!\!\!\twoheadrightarrow Y_{i/K} \subseteq X \text{ irreducible codim. } p-1, \\ f_i \in K(\widetilde{Y}_i)^* \text{ rational functions} \end{array} \right\} \begin{array}{c} \text{such that} \\ \sum \iota^i_*(f_i) \,=\, \mathcal{Z}_1 - \mathcal{Z}_2. \end{array}$$

(Here $\widetilde{Y}_i$ denotes a normalization of Y_i.) In the first definition, $\mathcal{W}$ may be reducible; so it changes nothing to replace $\mathbb{P}^1$ by a chain of $\mathbb{P}^1$'s. On the other hand, if one replaces $\mathbb{P}^1$ by a chain of curves of arbitrary genera, one has algebraic equivalence $\mathcal{Z}_1 \overset{\text{alg}}{\equiv} \mathcal{Z}_2$. Finally, a codimension-p cycle is homologous to zero iff on $X_{\mathbb{C}}$ it is the boundary of a topological chain of real dimension $2d - 2p + 1$,

$$\mathcal{Z}_1 \overset{\text{hom}}{\equiv} \mathcal{Z}_2 \quad \overset{\text{defn}}{\Longleftrightarrow} \quad \exists\, \Gamma \in C^{\text{top}}_{2d-2p+1}(X^{\text{an}}_{\mathbb{C}}, \mathbb{Q}) \ s.t.\ \partial\Gamma = \mathcal{Z}_1 - \mathcal{Z}_2,$$

or equivalently if its fundamental class (§2.2) vanishes.

Clearly $\overset{\text{rat}}{\equiv} 0 \implies \overset{\text{alg}}{\equiv} 0 \implies \overset{\text{hom}}{\equiv} 0$. For 0-cycles on a curve ($d = 1$), the second definition gives $\mathcal{Z} \overset{\text{rat}}{\equiv} 0 \iff \mathcal{Z} = (f)$ for $f \in K(X)^*$. If X is a surface ($d = 2$), a rational equivalence between 2 points may look as follows:



where e.g. $\genfrac{}{}{0pt}{}{+}{+}\ (i)$ says f_i has a zero of order 2. (Since we work modulo torsion [i.e. $\otimes\mathbb{Q}$], $2p - 2q \overset{\text{rat}}{\equiv} 0 \implies p \overset{\text{rat}}{\equiv} q$.) Also, for 0-cycles in general, $\overset{\text{hom}}{\equiv}$ and $\overset{\text{alg}}{\equiv}$ are the same.

The object of study will be Chow groups of cycles $/K$ modulo rational equivalences defined $/K$,

$$CH^p(X_K) := \frac{Z^p(X_K)}{(\overset{\text{rat}}{\equiv})_K}.$$

(We also write $CH^p(X_{/K})$, especially when X is replaced by something like $Y \times \mathcal{S}$ so that it is clear that K applies to the whole product.) One consequence of working modulo torsion is that $CH^p(X_K) \hookrightarrow CH^p(X_{\mathbb{C}})$, so one does not have to worry about the field of definition of the $\overset{\text{rat}}{\equiv}$. When clarity is at stake (only), we have written $\langle \mathcal{Z} \rangle \in CH^*(X)$ for the $\overset{\text{rat}}{\equiv}$-class of a cycle $\mathcal{Z} \in Z^*(X)$; classes of cycles $\overset{\text{hom}}{\equiv}$ (resp. $\overset{\text{alg}}{\equiv}$)0 are written CH^*_{hom} (resp. CH^*_{alg}).

Our goal is to detect 0-cycles/ $\overset{\text{rat}}{\equiv}$ in the kernel of the Albanese map; since the invariants make heavy use of Hodge theory we review next what we need.

2.2. **Hodge Structures.** A HS (Hodge Structure) of weight m is a finite-dimensional $\mathbb{Q}$-vector space $\mathcal{H}$, together with a descending filtration $F^\bullet$ on $\mathcal{H}_\mathbb{C} := \mathcal{H} \otimes_\mathbb{Q} \mathbb{C}$ satisfying for each i: $F^i\mathcal{H}_\mathbb{C} \oplus \overline{F^{m-i+1}\mathcal{H}_\mathbb{C}} = \mathcal{H}_\mathbb{C} = F^0\mathcal{H}_\mathbb{C}$. Writing $\mathcal{H}^{i,m-i}_{(\mathbb{C})} := F^i\mathcal{H}_\mathbb{C} \cap \overline{F^{m-i}\mathcal{H}_\mathbb{C}}$, we get a Hodge decomposition $\mathcal{H}_\mathbb{C} = \oplus_{p+q=m}\mathcal{H}^{p,q}$. A $\mathbb{Q}$-subspace $\mathcal{G} \subseteq \mathcal{H}$ is a subHS iff $\mathcal{G}_\mathbb{C} = \oplus_{p+q=m}(\mathcal{G}_\mathbb{C} \cap \mathcal{H}^{p,q})$, and a $\mathbb{Q}$-linear transformation $\mathcal{H} \xrightarrow{\theta} {}'\mathcal{H}$ is a morphism of HS iff over $\mathbb{C}$ it takes the form $\oplus\mathcal{H}^{p,q} \xrightarrow{\oplus\theta^{p,q}} \oplus{}'\mathcal{H}^{p,q}$ (relative to a pair of $\mathbb{C}$-bases subordinate to the resp. Hodge decompositions).

Intersections and sums of subHS are subHS, as are the image and kernel of a morphism of HS; quotients (e.g., $\mathcal{H}/\mathcal{G}$) have a natural HS ($\mathcal{H}_\mathbb{C}/\mathcal{G}_\mathbb{C} \cong \oplus_{p+q=m}\mathcal{H}^{p,q}/\mathcal{G}^{p,q}$), as do tensor products and duals (of HS). We define

$$F^j_h\mathcal{H}_{(\mathbb{Q})} := \text{largest subHS of } H_{(\mathbb{Q})} \text{ contained in } F^j\mathcal{H}_\mathbb{C} \cap \mathcal{H}_{(\mathbb{Q})},$$

and note that for $m > 2j$ equality does *not* in general hold. One also has the Tate HS $\mathbb{Q}(-d)$, of pure type (d,d) and weight $2d$.

From geometry comes a large assortment of examples. For X smooth projective, set $H^m(X) := H^m_{\text{sing}}(X^{(an)}_\mathbb{C}, \mathbb{Q})$, and identify $H^m(X,\mathbb{C}) \cong H^m\{\Gamma(X_\mathbb{C}, \Omega^\bullet_{X^\infty}), d\} \hookleftarrow H^m\{\Gamma(X_\mathbb{C}, F^j\Omega^\bullet_{X^\infty}), d\} =: F^jH^m(X,\mathbb{C})$ using C^∞-forms (de Rham cohomology). By the Hodge theorem $H^j(X)$ is a HS; we also see that $H^{p,q}(X,\mathbb{C})$ is represented by C^∞-forms of type (p,q) (modulo coboundaries). Associated to an algebraic cycle $\mathcal{Z} \in Z^p(X)$ one has the fundamental class

$$[\mathcal{Z}] \in H^{2p}(X) \cap F^pH^{2p}(X,\mathbb{C}) =: Hg^p(X)$$

defined by the functional $\left\{\int_\mathcal{Z}(\cdot)\right\} \in \left\{H^{2d-2p}(X,\mathbb{C})\right\}^\vee$; and $\mathbb{Q}[\mathcal{Z}] \subseteq H^{2p}(X)$ is a (Tate) subHS. Write $H^{2p}_{\text{alg}}(X)$ for the span of all such classes. If $X = Y_1 \times Y_2$ then $H^m(X) \cong \oplus_{r+s=m}H^r(Y_1) \otimes H^s(Y_2)$ as HS, and $[\mathcal{Z}]$ has Künneth components $[\mathcal{Z}]_i \in H^{2p-i}(Y_1) \otimes H^i(Y_2)$ summing to $[\mathcal{Z}]$; again $\mathbb{Q}[\mathcal{Z}]_i$ is a subHS.

A polarization of a HS $\mathcal{H}$ is a choice of bilinear form $\mathcal{Q} : \mathcal{H} \times \mathcal{H} \to \mathbb{Q}$ (symmetric for m even, skew for m odd) which after tensoring with $\mathbb{C}$

obeys the Hodge-Riemann bilinear relations. Polarized HS are *semisimple* in the sense that given $\mathcal{G} \subseteq \mathcal{H}$ subHS, $\mathcal{G}^{\perp}$ is a HS and $\mathcal{H} = \mathcal{G} \oplus \mathcal{G}^{\perp}$; so for example exact sequences (with polarized middle term) split. For X smooth projective, $H^m(X)$ has a natural polarization (using the Lefschetz decomposition and the hyperplane class); hence $H^m(X)$ and its subHS's are semisimple.

From above we have $H^{2p}_{\text{alg}}(X) \subseteq Hg^p(X)$; there is a large bank deposit waiting for whomever proves the equality [HC(p, X)] in general. The Generalized Hodge Conjecture GHC(j, m, X) predicts that $F^j_h H^m(X) = N^j H^m(X)$, where $N^\bullet$ is the descending filtration by coniveau (i.e., codimension of support); clearly HC(p, X) is GHC$(p, 2p, X)$. (Note that $N^j \subseteq F^j_h$ always holds.) Now let $[\mathsf{H}] \in H^2(X)$ be the class of a hyperplane section; a consequence of the Hard Lefschetz theorem is that $\cup[\mathsf{H}]^i : H^{d-i}(X) \to H^{d+i}(X)$ is an $\cong$. This is clearly also induced by an algebraic correspondence (cycle) on $X \times X$; the Lefschetz standard conjecture (of Grothendieck) says the *inverse* is [algebraic-]cycle-induced (for all i). In this paper, we will write HLC(X) (Hard Lefschetz conjecture) for the apparently stronger statement that (for each i) there exists a cycle inducing $(\cup[\mathsf{H}]^i)^{-1} : H^{d+i}(X) \to H^{d-i}(X)$ and also inducing the zero-map in all other degrees. In fact, they are equivalent by [Kl]; and clearly HLC(X) is implied by HC$(d,\, X \times X)$.

2.3. **Jacobians.** Next assume $\mathcal{H}$ is of weight $2p - 1$, and define the Jacobian

$$J^p(\mathcal{H}) := \frac{\mathcal{H}_{\mathbb{C}}}{F^p\mathcal{H}_{\mathbb{C}} + \mathcal{H}_{\mathbb{Q}}}.$$

Injective or surjective morphisms, direct sums, and quotients of HS (as well as commuting diagrams) induce the same behavior in the corresponding Jacobians. We write $J^p(H^{2p-1}(X)) =: J^p(X)$. (In the same way one can define for $'\mathcal{H}$ of weight $2p$ the Hodge group $Hg^p('\mathcal{H}) = {'\mathcal{H}} \cap F^p('\mathcal{H}_{\mathbb{C}})$, so that $Hg^p(H^{2p}(X)) = Hg^p(X)$.) Note that if $\mathcal{H} = \mathcal{K}^\vee \otimes \mathbb{Q}(-d)$ as HS then $J^p(\mathcal{H}) \cong \frac{(F^{d-p+1}\mathcal{K}_{\mathbb{C}})^\vee}{\text{im}(\mathcal{K}_{(\mathbb{Q})})^\vee}$.

Given $\mathcal{Z} \in Z^p_{\text{hom}}(X)$ and $\mathcal{H} \subseteq H^{2p-1}(X)$ (subHS) we may construct an element in $J^p(\mathcal{H})$ as follows. Let $\partial^{-1}\mathcal{Z} \in C^{\text{top}}_{2d-2p+1}(X, \mathbb{Q})$ be a fixed choice of chain bounding on $\mathcal{Z}$, and assume $\mathcal{H}$ and $\mathcal{K} \subseteq H^{2d-2p+1}(X)$ are dual under the perfect pairing $H^{2p-1}(X) \otimes H^{2d-2p+1}(X) \to \mathbb{Q}(-d)$. Since $\mathcal{Z}$ has complex dimension $d - p$, $\int_{\partial^{-1}\mathcal{Z}} d\alpha = \int_{\mathcal{Z}} \alpha = 0$ if $\alpha \in \Gamma(X, F^{d-p+1}\Omega^{2d-2p}_{X^\infty})$. Thus $\left\{\int_{\partial^{-1}\mathcal{Z}}(\cdot)\right\}$ gives a well-defined functional on[1]

$$F^{d-p+1}H^{2d-2p+1}(X,\mathbb{C})\cong\frac{\ker(d)\subseteq\Gamma(X,F^{d-p+1}\Omega^{2d-2p+1}_{X^\infty})}{\operatorname{im}\left\{d:\ \Gamma(X,F^{d-p+1}\Omega^{2d-2p}_{X^\infty})\to\text{num}\right\}},$$

which we may restrict to $F^{d-p+1}\mathcal{K}_{\mathbb{C}}$. Varying now the choice of $\partial^{-1}\mathcal{Z}$ (by topological cycles) changes the functional by rational functionals; hence we still get a well-defined result in $J^p(\mathcal{H})$, the Abel-Jacobi class $AJ_X(\mathcal{Z})$ if $\mathcal{H}=H^{2p-1}(X)$.

2.4. **Deligne cohomology.** For our cycle maps we will use Deligne (co)homology, which for X projective one may define (by [J4]) as cohomology of a complex:

$$H^i_{\mathcal{D}}(X,\mathbb{Q}(j)):=$$

$$H^i\left\{C^{\text{top}}_{2d-\bullet}\left(X,(2\pi\sqrt{-1})^j\mathbb{Q}\right)\oplus\Gamma\left(X,F^{j\prime}\mathcal{D}^\bullet_X\right)\oplus\Gamma\left(X,{}'\mathcal{D}^{\bullet-1}_X\right),\,D\right\}$$

where[2] $D(\Gamma,\Omega,K)=(-\partial_{\text{top}}\Gamma,-d[\Omega],\,d[K]-\Omega+\delta_\Gamma)$. In particular, one has the exact sequence

$$0\to J^p(X)\to H^{2p}_{\mathcal{D}}(X,\mathbb{Q}(p))\to Hg^p(X)\to 0.$$

A mixed Hodge structure (MHS) is a $\mathbb{Q}$-vector space $\mathcal{H}$ with an ascending (weight) filtration $W_\bullet$, plus a descending $F^\bullet$ on $\mathcal{H}_{\mathbb{C}}$, such that $Gr^W_m\mathcal{H}$ together with the induced $F^\bullet$ on $(Gr^W_m\mathcal{H})_{\mathbb{C}}$ is a weight m HS. If Y is smooth quasiprojective, $H^n(Y)$ has a natural MHS with $W_{n-1}H^n(Y)=\{0\}$ and $W_nH^n(Y)=\operatorname{im}\left\{H^n(\bar{Y})\to H^n(Y)\right\}=:\underline{H^n}(Y)$ for *any* smooth compactification $\bar{Y}$ (see [K2, sec. 3]). For $\bar{Y}$ a *good* compactification ($\bar{Y}\setminus Y=D$ a normal-crossings divisor), one may interpret $W_{m+j}H^m(Y)$ in terms of forms with log poles along D locally no worse than $d\log z_{k_1}\wedge\cdots\wedge d\log z_{k_j}$ (for $D=\cup\{z_i=0\}$ locally), see [GS].

We will have extensive use for the following basic computations:

$$Hom_{\text{MHS}}(\mathbb{Q}(-p),\mathcal{H})\ =\ W_{2p}\mathcal{H}\cap F^pW_{2p}\mathcal{H}_{\mathbb{C}},$$

$$Ext^1_{\text{MHS}}(\mathbb{Q}(-p),\mathcal{H})\ =\ \frac{W_{2p}\mathcal{H}_{\mathbb{C}}}{F^pW_{2p}\mathcal{H}_{\mathbb{C}}+W_{2p}\mathcal{H}_{(\mathbb{Q})}}.$$

One notes that if $\mathcal{H}$ is *pure* of weight $2p-1$ (i.e., a HS) then $Ext^1_{\text{MHS}}(\mathbb{Q}(-p),\mathcal{H})=J^p(\mathcal{H})$, and using this (together with the *Ext*

---

[1] here "num" denotes numerator.

[2] ${}'\mathcal{D}^i_X$ denotes the sheaf of i-currents, a local section of which is a bounded linear functional on local sections of C^∞ $(2d-i)$-forms. Integration over topological $(2d-i)$-chains $\Gamma\in C^{\text{top}}_{2d-i}(X,\mathbb{Q})$ includes them as "delta functions" $\delta_\Gamma\in\Gamma(X,{}'\mathcal{D}^i_X)$.

long-exact sequence associated to $0 \to W_{2p-1} \to W_{2p} \to Gr^W_{2p} \to 0$) one shows that

$$\mathrm{im}\left\{ Ext^1_{\mathrm{MHS}}(\mathbb{Q}(-p), H^{2p-1}(\bar{Y})) \to Ext^1_{\mathrm{MHS}}(\mathbb{Q}(-p), H^{2p-1}(Y)) \right\} \cong$$

$$Ext^1_{\mathrm{MHS}}\left(\mathbb{Q}(-p), \underline{H^{2p-1}}(Y)\right) / \,\mathrm{im}\left\{ Hom_{\mathrm{MHS}}\left(\mathbb{Q}(-p), Gr^W_{2p} H^{2p-1}(Y)\right) \right\}.$$

This quotient reflects the fact that in generalizing the construction of classes in Jacobians (§2.3) to the quasiprojective case, one needs to mod out essentially by all classes of cycles supported on $(\bar{Y} \setminus Y)$!

To appropriately generalize Deligne cohomology to Y, we need to take the weight filtration into account by using absolute Hodge cohomology $H^*_{\mathcal{H}}(Y, \mathbb{Q}(\cdot))$ (see (§5.5.2), or [L1] for a more conceptual explanation). While this reduces to Deligne if Y is projective, it isn't as obvious how to define it without passing to the derived category (owing to the difficulty of *controlling* poles of currents). Now set

$$\underline{H^{2p}_{\mathcal{D}}}(Y, \mathbb{Q}(p)) := \mathrm{im}\left\{ H^{2p}_{\mathcal{D}}(\bar{Y}, \mathbb{Q}(p)) \to H^{2p}_{\mathcal{H}}(Y, \mathbb{Q}(p)) \right\};$$

once more the image is independent of the choice of smooth compactification $\bar{Y}$ (see [K2, sec. 4]). One gets two exact sequences

$$\begin{tikzcd}
0 \arrow[r] & Ext^1_{\mathrm{MHS}}\left(\mathbb{Q}(-p), H^{2p-1}(Y)\right) \arrow[r] & H^{2p}_{\mathcal{H}}(Y, \mathbb{Q}(p)) \arrow[r] & Hom_{\mathrm{MHS}}\left(\mathbb{Q}(-p), H^{2p}(Y)\right) \arrow[r] & 0 \\
0 \arrow[r] & \frac{Ext^1_{\mathrm{MHS}}\left(\mathbb{Q}(-p), \underline{H^{2p-1}}(Y)\right)}{Hom_{\mathrm{MHS}}\left(\mathbb{Q}(-p), Gr^W_{2p} H^{2p-1}(Y)\right)} \arrow[r] \arrow[u, hook] & \underline{H^{2p}_{\mathcal{D}}}(Y, \mathbb{Q}(p)) \arrow[r] \arrow[u, hook] & Hom_{\mathrm{MHS}}\left(\mathbb{Q}(-p), H^{2p}(Y)\right) \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

where we note that the right hand term is just $Hg^p(\underline{H^{2p}}(Y))$.

3. FROM CLASSICAL INVARIANTS TO THE BLOCH-BEILINSON CONJECTURES

3.1. **Cycle-class and Albanese.** Take $\mathcal{Z} \in Z^p(X_K)$ as above (in this section K plays no role), and assume $\mathcal{Z} \overset{\mathrm{rat}}{\equiv} 0$: then $\exists\, \mathcal{W} \in Z^p(X \times \mathbb{P}^1_{/\mathbb{C}})$ with irreducible components $\{\mathcal{W}_i\}$, such that

$$\mathcal{Z} = \sum_i \pi_{X\,*}^{\widetilde{\mathcal{W}_i}} \left\{ \pi_{\mathbb{P}^1}^{\widetilde{\mathcal{W}_i}\,*} \left(\{0\} - \{\infty\} \right) \right\}.$$

(Here e.g. $\pi_X^{\widetilde{\mathcal{W}_i}} = \pi_X \circ \iota^{\widetilde{\mathcal{W}_i}}$, with $\iota^{\widetilde{\mathcal{W}_i}} : \widetilde{\mathcal{W}_i} \twoheadrightarrow \mathcal{W}_i \subseteq X \times \mathbb{P}^1$ the normalization.) Setting $\Gamma = \sum_i \pi_{X\,*}^{\widetilde{\mathcal{W}_i}} \left\{ \pi_{\mathbb{P}^1}^{\widetilde{\mathcal{W}_i}} \left(\overrightarrow{\infty.0} \right) \right\}$, for $\overrightarrow{\infty.0}$ a path on $\mathbb{P}^1$ (such as $\mathbb{R}^-$), gives $\partial\Gamma = \mathcal{Z}$ and $\mathcal{Z} \overset{\mathrm{hom}}{\equiv} 0$. Thus (using Stokes's theorem) the fundamental (or cycle-) class induces a well-defined map

$$cl_X : CH^p(X_K) \overset{[\cdot]}{\to} Hg^p(X).$$

For 0-cycles this is $\deg : CH_0(X) \to H^{2d}(X) \cong \mathbb{Q}$ sending $\sum q_i z_i \mapsto \sum q_i$.

Above we showed how to obtain a map from $\{\ker([\,\cdot\,]) \subseteq Z^p(X_K)\} = Z^p_{\text{hom}}(X_K) \overset{AJ_X}{\to} J^p(X)$, by integration over an arbitrary chain $\partial^{-1}\mathcal{Z}$ bounding on $\mathcal{Z}$. Choosing now $\partial^{-1}\mathcal{Z} := \Gamma$ (still assuming $\mathcal{Z} \overset{\text{rat}}{\equiv} 0$) and $\omega \in \ker(d) \subseteq \Gamma\left(X,\, F^{d-p+1}\Omega^{2d-2p+1}_{X^\infty}\right)$, we have $\int_\Gamma \omega = \int_\infty^0 \sum_i \pi^{\widetilde{\mathcal{W}_i}}_{\mathbb{P}^1}{}_* \pi^{\widetilde{\mathcal{W}_i}}_X{}^* \omega$. Generically fibers of $\widetilde{\mathcal{W}_i} \to \mathbb{P}^1$ have dimension $d-p$; so intuitively $(\pi^{\widetilde{\mathcal{W}_i}}_{\mathbb{P}^1})_*$ (which integrates fiberwise) eats up $(d-p)$ each of dz's and $d\bar{z}$'s, and the integrand is d-closed of type $(1,0)$ on $\mathbb{P}^1$ (hence $\bar{\partial}$-closed). Now the push-forward *a priori* gives us a current, but a regularity lemma for $\bar{\partial}$ shows it is C^∞. So $\bar{\partial}$-closed + $C^\infty \implies$ holomorphic, which makes the integrand *zero* since $\Omega^1(\mathbb{P}^1) = \{0\}$, and thus $AJ_X(\mathcal{Z}) = 0 \in J^p(X)$. We have shown AJ_X is well-defined modulo $\overset{\text{rat}}{\equiv}$; this is Griffiths's Abel-Jacobi map

$$AJ_X : CH^p_{\text{hom}}(X_K) \to J^p(X)$$

(e.g., see [Gr]). For 0-cycles this is often called the Albanese, $Alb : CH_0^{\text{hom}}(X) \to \frac{\{\Omega^1(X)\}^\vee}{\text{im}H_1(X,\mathbb{Q})}$, sending e.g. $\mathcal{Z} = p - q \longmapsto \left\{\omega \mapsto \int_q^p \omega\right\}$.

The Deligne cycle-class map

$$c_{\mathcal{D}} : CH^p(X_K) \to H^{2p}_{\mathcal{D}}(X^{\text{an}}_{\mathbb{C}}, \mathbb{Q}(p))$$

sends $\mathcal{Z}$ to the D-cohomology class of $\left((2\pi\sqrt{-1})^p \mathcal{Z}_{\text{top}},\, \delta_{\mathcal{Z}},\, 0\right)$ in the complex at the beginning of (§2.4). It allows one to write cl and AJ "in one piece" (in the sense of the short exact sequence). For Y smooth quasiprojective one has a corresponding $c_{\mathcal{H}}$ to $H^{2p}_{\mathcal{H}}$ and a diagram (with $\bar{Y}$ smooth)

$$\begin{tikzcd}
CH^p(\bar{Y}) \arrow[r, "c_{\mathcal{D}}"] \arrow[d] & H^{2p}_{\mathcal{D}}(\bar{Y}_{\mathbb{C}}, \mathbb{Q}(p)) \arrow[d] \\
CH^p(Y) \arrow[r, "c_{\mathcal{H}}"] & H^{2p}_{\mathcal{H}}(Y_{\mathbb{C}}, \mathbb{Q}(p)),
\end{tikzcd}$$

hence a (very useful) map

$$\underline{c_{\mathcal{D}}} : CH^p(Y) \to \underline{H^{2p}_{\mathcal{D}}}(Y_{\mathbb{C}}, \mathbb{Q}(p)).$$

3.2. **Inadequacy of the classical invariants.** Now we want to ask how good a job, in the case of 0-cycles, our two maps cl_X and Alb_X (or just $c_{\mathcal{D}}$) do in detecting $\overset{\text{rat}}{\equiv}$-classes.

Abel's Theorem. $X_{\mathbb{C}}$ a curve $\implies$ Alb_X is an isomorphism.

For $d = 1$, a great job; but for $d > 1$, miserable in general:

Theorem (Voisin, [V1]**).** Let $X_{\mathbb{C}} \subseteq \mathbb{P}^3$ be a smooth projective (hyper)surface, very general[3] of degree $D \geq 7$. Then no 2 points on X are rationally equivalent.

Now $H^1(X) = 0 \implies \Omega^1(X) = 0 \implies Alb_X = 0$, and thus cannot detect $[p - q](\neq 0) \in CH_0^{\mathrm{hom}}(X)$. Fakhruddin (in [Fa]) has generalized this; the following is one case of his result:

Theorem (Fakhruddin). Given integers $d \geq 2$, $n \geq 2$; then for D sufficiently large, a generic degree D hypersurface $X_{\mathbb{C}} \subseteq \mathbb{P}^{d+1}$ has the following property: the classes of any n distinct points in $CH_0(X)$ are linearly independent. (Voisin: $d = 2$, $n = 2$.)

Here is another instance of nonvanishing of $\ker(Alb_X)$:

Theorem (Bloch, [B1]**):** Let $A^d_{\mathbb{C}}$ be a d-dimensional abelian variety. The group law induces a Pontryagin product "$*$" on cycles, and

$$CH_0^{\mathrm{hom}}(A^d)^{*i} \begin{cases} = 0 \ \text{ for } i > d \\ \\ \neq 0 \ \text{ for } i \leq d \end{cases} .$$

But already $CH_0^{\mathrm{hom}}(A^d)^{*2} \subseteq \ker(Alb)$ (in fact $=$) by the "parallelogram law": say

$$\begin{pmatrix} & \bullet + \\ o \bullet - & \end{pmatrix} * \begin{pmatrix} & \\ o \bullet - & \bullet + \end{pmatrix} = \begin{pmatrix} & \bullet - & \dashrightarrow & \bullet + \\ o \bullet + & \dashleftarrow & \bullet - & \end{pmatrix} =: \mathcal{Z}.$$

Then choosing $\partial^{-1}\mathcal{Z}$ as indicated (dotted arrows) and $\omega \in \Omega^1(A^d)$, $\int_{\partial^{-1}\mathcal{Z}} \omega = 0$ by cancellation.

The following result offers a partial "explanation" of the Albanese kernel: it's not just the 1-forms that control $\overset{\mathrm{rat}}{\equiv}$ of cycles.

Mumford's Theorem ([Mu]**).** Let $X_{\mathbb{C}}$ be a smooth projective surface ($d = 2$) with $\Omega^2(X) \neq 0$. Then $\ker(Alb) \subseteq CH_0^{\mathrm{hom}}(X)$ is "infinite-dimensional".

Remark 3.1. (i) Roitman ([R1]) generalized this to $d \geq 2$: one has the same result if $\Omega^i(X) \neq 0$ for any $i > 1$. Also see [L2] for proofs of both.

(ii) "Infinite-dimensionality" means $\ker(Alb)$ "cannot be parametrized by a finite-dimensional algebraic variety" — concretely, this comes from

---

[3] i.e., of maximal transcendence degree in the parameter space

increasing without bound the number of points z_i involved in 0-cycles $\mathcal{Z} = \sum q_i z_i$. This gives us as many parameters as we like, and the *key* is that rational equivalence (i.e., the dimension of the subspace of relations) doesn't keep up.

(iii) This theorem and Griffiths's result that $\overset{\text{hom}}{\equiv}$ and $\overset{\text{alg}}{\equiv}$ are different for higher-dimensional (not 0-)cycles have motivated vast quantities of research aimed at understanding structure / producing examples for well over 3 decades.

Together with Mumford, the following "converse" suggests that $\Omega^2(X)$ (or more accurately, $H^2(X)/F_h^1 H^2(X)$) "controls" the Albanese kernel.

Bloch Conjecture ([B2]). Let $X_{\mathbb{C}}$ be a smooth projective surface with $\Omega^2(X)$ $= 0$. Then $\ker(Alb) = 0$.

More generally, for $d(= dim(X)) \geq 2$ this should hold if $\Omega^i(X) = 0$ for $1 < i \leq d$.

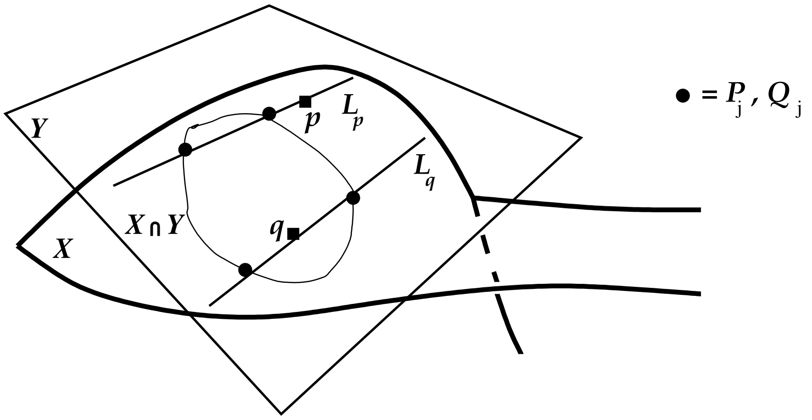
Example 3.2. Bloch-Kas-Lieberman ([BKL]) proved BC for surfaces of Kodaira dimension $\kappa < 2$.

Example 3.3. Consider a smooth hypersurface $X_{(\mathbb{C})} \subseteq \mathbb{P}^{d+1}$ of degree $D \leq d+1$. Then $\deg(K_X) = D-d-2 < 0 \implies \Omega^d(X) = H^0(K_X) = 0$; also, $\Omega^i(X) = 0$ for $1 < i < d$ by the Lefschetz hyperplane theorem.

Theorem (Roitman, [R2]). For X as just described, $CH_0^{\text{hom}}(X) = 0$ (any two points are $\overset{\text{rat}}{\equiv}$).

That is, the BC holds for projective hypersurfaces (in fact, complete intersections) of low degree. We refer to [L3] (or [R2], [L2]) for a full proof, but explain things for the case $1 < D \leq d$ (for simplicity leaving out $D = d+1$). For any point $p \in X(\mathbb{C})$, the set of lines on X through p is described (in terms of local affine coordinates about p) as a complete intersection of multidegree $(1, 2, \ldots, D)$ in $\mathbb{P}^d$; since $D \leq d$ this is nonempty. So take individual lines L_p, $L_q \subseteq X$ through any given p, q; since $H^2(\mathbb{P}^{d+1}) \cong \mathbb{Q}$ and $CH_1^{\text{hom}}(\mathbb{P}^{d+1}) = \{0\}$, $L_p \overset{\text{rat}}{\equiv} L_q$ on $\mathbb{P}^{d+1}$. That is, $\exists$ surfaces $Y_i \subseteq \mathbb{P}^{d+1}$ and $F_i \in \mathbb{C}(\widetilde{Y}_i)^*$ with $\sum_i \iota_*^i(F_i) = L_p - L_q$. (We may also arrange that the Y_i and $Y_i \cap Y_j$ intersect X properly; e.g. a chain of 3 $\mathbb{P}^2$'s will suffice.) Restricting this rational equivalence to X we get curves $\mathcal{C}_i := X \cap Y_i \subseteq X$ and $f_i \in \mathbb{C}(\widetilde{\mathcal{C}}_i)^*$ with $[0 \overset{\text{rat}}{\equiv}] \sum{}' \iota_*^i(f_i) = \sum_{j=1}^{D} P_j - \sum_{j=1}^{D} Q_j$ where $P_j \in L_p$, $Q_j \in L_q$. Obviously $P_j \overset{\text{rat}}{\equiv} p$ and

$Q_j \overset{\text{rat}}{\equiv} q$, so $D \cdot p \overset{\text{rat}}{\equiv} D \cdot q$. In fact, $CH_0^{\text{hom}}(X)[= CH_0^{\text{alg}}(X)]$ is divisible and so Roitman's result holds *without* going modulo torsion. Here is an oversimplified picture with only one Y_i:



3.3. **Albanese kernel and transcendence degree (Part I).** It is instructive to consider the form of the cycles (in $\ker(Alb)$) guaranteed by (Roitman's generalization of) Mumford in case $X_{\mathbb{C}}$ has a model $/\bar{\mathbb{Q}}$. The first clue is given by:

Bloch-Beilinson Conjecture: Let X (smooth projective) be defined $/\bar{\mathbb{Q}}$. Then

$$c_{\mathcal{D}} : CH^p(X_{/\bar{\mathbb{Q}}}) \to H^{2p}_{\mathcal{D}}(X^{\text{an}}_{\mathbb{C}}, \mathbb{Q}(p))$$

(or equivalently, $AJ : CH^p_{\text{hom}}(X_{/\bar{\mathbb{Q}}}) \to J^p(X)$) is injective.

Remark 3.4. Here one can replace $\bar{\mathbb{Q}}$ by a number field, bearing in mind that in this paper CH is taken modulo torsion. Otherwise the statement is $false$: for example, Murre and Ramakrishnan ([MR]) have shown that for any prime p there exists an elliptic curve E over a number field k and points P, $Q \in E(k)$ such that $(P, Q) - (0, Q) - (P, 0) + (0, 0) \in \ker(Alb) \subseteq CH_0^{\text{hom}}(E \times E_{/k})$ is p-torsion.

For 0-cycles, $CH_0^{\text{hom}}(X_{/\bar{\mathbb{Q}}})$ consists of $\sum q_i z_i$ with coordinates of all z_i in $\bar{\mathbb{Q}}$. If BBC holds, any such cycle in $\ker(Alb)$ is $\overset{\text{rat}}{\equiv} 0$. So if $H_1(X) = 0$ (e.g. for $X_{(\mathbb{C})}$ simply connected), then $Alb_X = 0$ and BBC $\Longrightarrow$ $CH_0^{\text{hom}}(X_{/\bar{\mathbb{Q}}}) = 0$. Hence BBC implies that on a projective complete intersection (or K3, or CY) defined $/\bar{\mathbb{Q}}$, any 2 points $\in X(\bar{\mathbb{Q}})$ are rationally equivalent. This does not contradict Voisin's result because there X is not defined $/\bar{\mathbb{Q}}$ (hence doesn't even have $\bar{\mathbb{Q}}$-points).

The flip side of this is that according to BBC, $\ker(Alb) \subseteq CH_0^{\mathrm{hom}}(X_{\mathbb{C}})$ (for X defined $/\bar{\mathbb{Q}}$) is generated by cycles whose "minimal field of definition" $K \subseteq \mathbb{C}$ has $\mathrm{trdeg}(K/\bar{\mathbb{Q}}) \geq 1$. (We simply call this the transcendence degree of the cycle; here we are only considering X with a model $/\bar{\mathbb{Q}}$ because otherwise *all* cycles on X have $\mathrm{trdeg}_{/\bar{\mathbb{Q}}} \geq 1$.) To write down such cycles, we first describe the "very general" points we require in a broader context that will be used in our discussion of spreads below.

3.4. **Interlude on points.** Let $\mathcal{S}/\bar{\mathbb{Q}}$ be smooth projective, and choose some affine Zariski open subset $\mathcal{S}_0/\bar{\mathbb{Q}}$. The embedding $\bar{\mathbb{Q}}[\mathcal{S}_0] \hookrightarrow \bar{\mathbb{Q}}(\mathcal{S}_0) \cong \bar{\mathbb{Q}}(\mathcal{S})$ of the (affine) coordinate ring into its function field induces a morphism $\operatorname{Spec} \bar{\mathbb{Q}}(\mathcal{S}) \to \operatorname{Spec} \bar{\mathbb{Q}}[\mathcal{S}_0]$. Composing this with the inclusion $\mathcal{S}_0 \subseteq \mathcal{S}$ yields a morphism $\operatorname{Spec} \bar{\mathbb{Q}}(\mathcal{S}) \to \mathcal{S}$ which we call p_g, the "generic point" of $\mathcal{S}$. There are two totally different ways to get "points" on $\mathcal{S}_{\mathbb{C}}$ out of this.

(1). Set $p_g^{\mathbb{C}} := p_g \times_{\bar{\mathbb{Q}}} \operatorname{Spec} \mathbb{C}$ (i.e. $\operatorname{Spec} \mathbb{C}(\mathcal{S}) \to \mathcal{S}_{\mathbb{C}}$), the generic point of $\mathcal{S}_{\mathbb{C}}$. We approximate both generic points by shrinking $\mathcal{S}$ as follows: formally write $\eta_{\mathcal{S}} := \varprojlim \mathcal{U}$ (limit over $\mathcal{U}/\bar{\mathbb{Q}} \subseteq \mathcal{S}$ affine Zariski open) and $\eta_{\mathcal{S}}^{\mathbb{C}} := \varprojlim \mathcal{U}_{\mathbb{C}}$ (limit *still* over $\mathcal{U}$ defined $/\bar{\mathbb{Q}}$; $\mathcal{U}_{\mathbb{C}}$ is just $\mathcal{U} \times_{\bar{\mathbb{Q}}} \mathbb{C}$). The limit has practical meaning only under some functor $\mathfrak{F}$, usually contravariant (like H^*, $H^*_{\mathcal{H}}$, CH^*) so that $\mathfrak{F}(\eta_{\mathcal{S}}^{[\mathbb{C}]}) := \varinjlim \mathfrak{F}(\mathcal{U}_{[\mathbb{C}]})$. (We note in particular that direct limits are exact in the category of abelian groups.) The reason for introducing the η's is two-fold. On the one hand, CH^* cannot distinguish between p_g (i.e., pullback to $\operatorname{Spec} \bar{\mathbb{Q}}(\mathcal{S})$) and $\eta_{\mathcal{S}}$; e.g. for $X/\bar{\mathbb{Q}}$, $CH^*(X \times \eta_{\mathcal{S}/\bar{\mathbb{Q}}}) := \varinjlim CH^*(X \times \mathcal{U}_{/\bar{\mathbb{Q}}}) \cong CH^*(X \times \operatorname{Spec} \bar{\mathbb{Q}}(\mathcal{S})_{/\bar{\mathbb{Q}}}) \cong CH^*(X_{/\bar{\mathbb{Q}}(\mathcal{S})})$. But unlike $p_g^{\mathbb{C}}$, $\eta_{\mathcal{S}}^{\mathbb{C}}$ is a limit of analytic spaces (on which one can do Hodge theory); so one has singular and absolute Hodge cohomology groups, e.g. $H^*_{\mathcal{H}}(\eta_{\mathcal{S}}^{\mathbb{C}}, \mathbb{Q}(\cdot)) := \varinjlim H^*_{\mathcal{H}}(\mathcal{U}_{\mathbb{C}}^{\mathrm{an}}, \mathbb{Q}(\cdot))$. One can take the limit of the MHS's on the $H^m(\mathcal{U})$ to produce $W_{\bullet}$ and $F^{\bullet}$ on $H^m(\eta_{\mathcal{S}})$, and in particular

$$\underline{H^m}(\eta_{\mathcal{S}}) := W_m H^m(\eta_{\mathcal{S}}) = \operatorname{im}\left\{ H^m(\mathcal{S}) \to \varinjlim H^m(\mathcal{U}) \right\} \cong \frac{H^m(\mathcal{S})}{N^1 H^m(\mathcal{S})}$$

is a very important (finite-dimensional) HS.

(2). Given an embedding $\bar{\mathbb{Q}}(\mathcal{S}) \xrightarrow[\cong]{\mathrm{ev}} K \subseteq \mathbb{C}$ which restricts to the "identity" on $\bar{\mathbb{Q}}$, set $p := p_g \times_{\mathrm{ev}} \operatorname{Spec} \mathbb{C}$. What we mean by this slight abuse of notation is illustrated by the following pair of diagrams:

$$\begin{tikzcd}
\bar{\mathbb{Q}}\,(\mathcal{S}) \arrow[dd, hook, "\mathrm{ev}"] & \bar{\mathbb{Q}}\,[\mathcal{S}_0] \arrow[l, hook'] \arrow[dd, hook] & & \operatorname{Spec}\bar{\mathbb{Q}}\,(\mathcal{S}) \arrow[r, "p_g"] & \mathcal{S}_0 & \subseteq \mathcal{S} \\
 & & {} \arrow[r, "\text{induces}"] & {} & & \\
\mathbb{C} & \mathbb{C}\,[\mathcal{S}_0] \arrow[l, "\widetilde{\mathrm{ev}}"'] & & \operatorname{Spec}\mathbb{C} \arrow[uu] \arrow[r, "p"] & (\mathcal{S}_0)_{\mathbb{C}} \arrow[uu] & \subseteq \mathcal{S}_{\mathbb{C}}
\end{tikzcd}$$

where $\widetilde{\mathrm{ev}}$ is (the unique lift of ev) chosen to make the first square commute. This is quite different from **(1)** in that p is 0-dimensional (over $\mathbb{C}$) and a geometric point. Now the restriction of $\widetilde{\mathrm{ev}}$ to $\bar{\mathbb{Q}}[\mathcal{S}_0] \subseteq \mathbb{C}[\mathcal{S}_0]$ factors through $\bar{\mathbb{Q}}(\mathcal{S})$, hence through $\bar{\mathbb{Q}}[\mathcal{U}]$ for any affine open $\mathcal{U} \subseteq \mathcal{S}_0$ defined $/\bar{\mathbb{Q}}$. It follows that for all such $\mathcal{U}$, p factors through all $\mathcal{U}_{\mathbb{C}}(= \mathcal{U} \times_{\bar{\mathbb{Q}}} \mathbb{C}) \subseteq \mathcal{S}_{\mathbb{C}}$. That is, p is in the complement of all (hence countably many) proper subvarieties defined $/\bar{\mathbb{Q}}$. We express this by saying $p \in \eta_{\mathcal{S}}(\mathbb{C})$, or that $p \in \mathcal{S}(\mathbb{C})$ is *very general*,[4] or of maximal transcendence degree (as its affine coordinates [in $\mathcal{S}_0$] generate over $\bar{\mathbb{Q}}$ a field of transcendence degree $\dim(\mathcal{S})$). Of course, in the above construction, p is really defined $/K$.

3.5. **Albanese kernel and transcendence degree (Part II).** Here now are some "canonical" examples of 0-cycles in $\ker(Alb_X)$ for X defined $/\bar{\mathbb{Q}}$. Let $\mathcal{C}_j/\bar{\mathbb{Q}}$ $(j = 1, \ldots, d)$ be curves of positive genera, and select base points $o_j \in \mathcal{C}_j(\bar{\mathbb{Q}})$; set $X = \mathcal{C}_1 \times \cdots \times \mathcal{C}_d$. Take $p_j \in \mathcal{C}_j(\mathbb{C})$ very general and "algebraically independent" in the sense that $p_1 \times \cdots \times p_d \in X(\mathbb{C})$ is very general. Finally, let $\mathcal{W}_j \in CH_0^{\mathrm{hom}}(\mathcal{C}_{j/\bar{\mathbb{Q}}})$ (for each j) be such that $AJ_{\mathcal{C}_j}(\mathcal{W}_j) \neq 0$ in $J^1(\mathcal{C}_j)$. (We emphasize once more that for us torsion is zero in cycles and classes; in particular J is the rational Jacobian.)

Theorem 3.5. *For each $\ell \geq 2$, the 0-cycles*
$(A):\quad (p_1 - o_1) \times \cdots \times (p_\ell - o_\ell) \times o_{\ell+1} \times \cdots \times o_d,$
$(B):\quad (p_1 - o_1) \times \cdots \times (p_{\ell-1} - o_{\ell-1}) \times \mathcal{W}_\ell \times o_{\ell+1} \times \cdots \times o_d$
give nonzero classes in $\ker(Alb) \subseteq CH_0^{hom}(X_{\mathbb{C}})$.

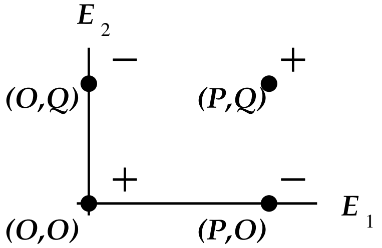
Remark 3.6. (i) For obvious reasons we call (A) "ℓ-box" (e.g., for $\ell = d = 2$ it is $(p_1, p_2) - (p_1, o_2) - (o_1, p_2) + (o_1, o_2)$); the respective transcendence degrees of (A) and (B) are clearly ℓ and $\ell - 1$. (A) is well-known, while (B) follows for $n = \ell = 2$ from [RS] and in general from [K1] (see Theorem 6.1 below).

---

[4] "general" is *not* the term we want here, as this denotes a point in the complement of only a *finite* union of Zariski closed subsets.

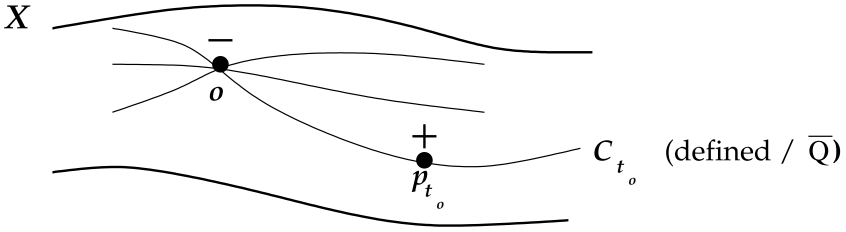
(ii) Any abelian variety A^d is covered by such a product of curves, and cycles of the above two types (A) and (B) project to nontrivial elements of $CH_0^{\mathrm{hom}}(A^d)^{*\ell}$ from the Bloch theorem.

(iii) A simple example of (B) with $d = \ell = 2$ is with the $\mathcal{C}_j$ both elliptic curves $E_i/\bar{\mathbb{Q}}$, $P \in E_1(\mathbb{C})$ very general, and $Q \in E_2(\bar{\mathbb{Q}})$ of infinite rank. Taking $\mathcal{W}_2 = \langle Q\rangle - \langle O\rangle$, we get $(\langle P\rangle - \langle O\rangle) \times (\langle Q\rangle - \langle O\rangle) \overset{\mathrm{rat}}{\not\equiv} 0$ (in the sense of this paper, i.e. modulo torsion).



We will investigate cycles of the form e.g. $(p_1 - o_1) \times (p_1 - o_1)$ on $\mathcal{C}_1 \times \mathcal{C}_1$ in §6.2 below. On the other hand, $\mathcal{W}_1 \times \mathcal{W}_2$ "should" be $\overset{\mathrm{rat}}{\equiv} 0$ since it is in $\ker(Alb) \cap CH_0^{\mathrm{hom}}(\mathcal{C}_1 \times \mathcal{C}_{2/\bar{\mathbb{Q}}})$, though this is only conjectural — even if $\mathcal{C}_1$ and $\mathcal{C}_2$ are elliptic curves.

Finally, Griffiths-Green-Paranjape [GGP] found an interesting example for any smooth projective surface $X/\bar{\mathbb{Q}}$ with $\Omega^2(X) \neq 0$. If $H_1(X) = 0$, here is how this works: take a Lefschetz pencil of X, which we write $\mathbb{P}^1 \xleftarrow{h} \mathfrak{H} \xrightarrow{\mathcal{P}} X$ ($\mathcal{P}$ is a blow-up) with $\mathcal{C}_t := \mathcal{P}(h^{-1}(t)) \subseteq X$ for $t \in \mathbb{P}^1$. For each t let $p_t \in X$ denote the inclusion of a very general point $\in \mathcal{C}_t(\mathbb{C})$, and $o \in X(\bar{\mathbb{Q}})$ lie in the base locus. Then for some $t_0 \in \mathbb{P}^1(\bar{\mathbb{Q}})$, $0 \overset{\mathrm{rat}}{\not\equiv} p_{t_0} - o$ is a nontrivial cycle of transcendence degree 1 in the Albanese kernel. We will generalize this example in §7.3 below.



3.6. **Bloch-Beilinson filtration and Chow-Künneth decomposition.** For 0-cycles on a surface X (whether defined $/\bar{\mathbb{Q}}$ or not), cl (or deg) maps to $H^4(X)$, and AJ (or Alb) to $J^2(X)$ — which involves

$H^3(X)$. Since $\ker(Alb)$ (which is conjecturally generated by cycles of $\mathrm{trdeg}_{/\bar{\mathbb{Q}}} \geq 1$) seems to depend on part of $H^2(X)$, we expect any invariant for it should involve $H^2(X)$.

More generally, for X smooth projective (defined over a subfield of $\mathbb{C}$), Bloch and Beilinson independently predicted a filtration $F^\bullet_{\mathrm{BB}}$ on $CH^p(X_{\mathbb{C}})$ with graded pieces $Gr^i_{F_{\mathrm{BB}}} CH^p(X_{\mathbb{C}})$ (a) generated by cycles of $\mathrm{trdeg}_{/\bar{\mathbb{Q}}} \geq i-1$ and (b) completely described by invariants involving $H^{2p-i}(X)$ (and no other $H^j(X)$). To be precise:

Definition 3.7. A *Bloch-Beilinson filtration* (BBF) is a (conjectural) system of descending filtrations on all Chow groups of smooth projective varieties $/\mathbb{C}$. It must respect the intersection product (in the sense $F^i_{\mathrm{BB}} CH^p \cdot F^j_{\mathrm{BB}} CH^q \subseteq F^{i+j}_{\mathrm{BB}} CH^{p+q}$) and morphisms induced by correspondences ($\langle \Gamma \rangle \in CH^{p-q+d_X}(Y \times X) \implies \langle \Gamma \rangle_* F^i_{\mathrm{BB}} CH^q(Y) \subseteq F^i_{\mathrm{BB}} CH^p(X)$). One also demands $F^0_{\mathrm{BB}} CH^p = CH^p$, $F^\ell CH^p = \{0\}$ for ℓ sufficiently large, $F^1_{\mathrm{BB}} CH^p = CH^p_{\mathrm{hom}}$, and $F^2_{\mathrm{BB}} CH^p \subseteq \ker(AJ)$ (with $F^2_{\mathrm{BB}} CH^p \cap CH^p_{\mathrm{alg}} = \ker(AJ) \cap CH^p_{\mathrm{alg}}$).

But these last two (at least, the $\subseteq$'s) are really part of a larger demand corresponding to (b) above. For its statement, $[\Delta_X] \in H^{2d_X}(X \times X)$ must have "algebraic Künneth components" — that is, there must exist algebraic cycles $\tilde{\Delta}_{X,j}$ (or $\Delta(2d_X - j,\, j)$) on $X \times X$ with $[\tilde{\Delta}_{X,j}] = [\Delta_X]_j \in H^{2d_X - j}(X) \otimes H^j(X)$. The last requirement for $F^\bullet_{\mathrm{BB}}$ (essentially $\forall\, i, j \in \mathbb{Z}$) is then that the action of $\left\langle \tilde{\Delta}_{X,j} \right\rangle_*$ on $Gr^i_{F_{\mathrm{BB}}} CH^p(X)$ must be $\delta_{2p-i,\, j}$ times the identity, independent of the choice of $\{\tilde{\Delta}_{X,j}\}$. If we assume the HLC ($\Leftarrow$HC), not only do such $\{\tilde{\Delta}_{X,j}\}$ exist; they also may be chosen, for $j < p$, to be supported on $W \times X$ with $\mathrm{codim}_X(W) \geq d_X - p - 1$. For $i > p$, this yields $\left\langle \tilde{\Delta}_{X,2p-i} \right\rangle_* \Big|_{Gr^i_{F_{\mathrm{BB}}} CH^p(X)} = 0$, hence $Gr^i_{F_{\mathrm{BB}}} CH^p = 0$; and so $F^{p+1}_{\mathrm{BB}} CH^p(X) = 0$.

Remark. The definition comes from [J4].

Now, one can put multiplicative (hence ring) structures on $CH^{d_X}(X \times X)$, $H^{2d_X}_{\mathrm{alg}}(X \times X)$ by $A \circ B := \pi_{13_*}(\pi_{12}^*(A) \cdot \pi_{23}^*(B))$ (using the 3 obvious projections $X \times X \times X \to X \times X$); this corresponds to composing endomorphisms of $CH^*(X)$, $H^*(X)$ induced by correspondences. Under $\circ$, the Künneth components $\left\{ [\tilde{\Delta}_{X,j}] \right\}$ are orthogonal idempotents in (still assuming HLC) $H^{2d_X}_{\mathrm{alg}}(X \times X)$ summing to $[\Delta_X]$; but the corresponding statements on the $CH^{d_X}(X \times X)$-level are much stronger, and false for the $\left\{ \left\langle \tilde{\Delta}_{X,j} \right\rangle \right\}$ as we have described them. However, *if* $F^\bullet_{\mathrm{BB}}$ *exists* *as described above, then $CH^{d_X}_{hom}(X \times X)$ becomes a nilpotent ideal*[5] in $CH^{d_X}(X \times X)$. It follows (cf. [J1], [L2]) that lifts $\{\Delta_{X,j}\}$ from the quotient $\frac{CH^{d_X}(X\times X)}{CH^{d_X}_{\text{hom}}(X\times X)} \cong H^{2d_X}_{\text{alg}}(X \times X)$ to $CH^{d_X}(X \times X)$ can be chosen so as to preserve the desired relations: (i) $\left\{\langle \Delta_{X,j}\rangle_*\right\}$ are orthogonal idempotents[6] under $\circ$, (ii) $\Delta_X \overset{\text{rat}}{\equiv} \sum_j \Delta_{X,j}$, and (iii) $[\Delta_{X,j}] = [\Delta_X]_j$ as above.

Definition 3.8. A collection of cycles $\{\Delta_{X,j}\}_{j=0}^{2d_X} \subseteq Z^{d_X}(X \times X)$ with properties (i)-(iii), is called a Chow-Künneth decomposition (of Δ_X), terminology due to Murre. We shall call it also *good* (for our purposes with 0-cycles) if $\left\{\langle \Delta_{X,j}\rangle_*\right\}_{j<d_X}$ act as 0 on $CH_0(X)$ and $\left\{\ker \langle \Delta_{X,2d_X}\rangle_* \subseteq CH_0(X)\right\} = CH_0^{\text{hom}}(X)$.

With a little more work the above discussion yields:

Proposition 3.9. *HLC* $+ \,\exists$ *of* $F^\bullet_{BB} \implies \exists\ (\forall\ X)$ *of good C-K decomposition satisfying moreover* $\langle \Delta_{X,j}\rangle_*\big|_{Gr^i_{F_{BB}}CH^p(X)} = \delta_{2p-i,\,j} \cdot Id.$

However, one can construct good C-K decompositions *directly* (with no assumptions) for X a curve, surface, abelian variety, complete intersection, or arbitrary product of these.

4. Higher cycle- and AJ-classes

4.1. **Spreads.** We are about to study several actual filtrations on Chow groups, each of which gives a Bloch-Beilinson filtration under some conjectural conditions. For the construction of many of these, the following idea is crucial.

Let $K \subseteq \mathbb{C}$ be finitely generated $/\bar{\mathbb{Q}}$. Then using a transcendence basis, one constructs (e.g., see [K1] Lemma 1(a)) $\mathcal{S}/\bar{\mathbb{Q}}$ smooth projective and a very general point $s_0 \in \mathcal{S}(\mathbb{C})$ such that $\text{ev}_{s_0} : \bar{\mathbb{Q}}(\mathcal{S}) \overset{\cong}{\to} K$ (in fact, $s_0 \in \mathcal{S}(K)$).

Assume first that X is defined $/\bar{\mathbb{Q}}$ and $\mathcal{Z} \in Z^p(X_K)$. Changing coefficients of $\mathcal{Z}$'s defining (homogeneous) equations from K to $\bar{\mathbb{Q}}(\mathcal{S})$, we have $\mathcal{Z}_g := \mathcal{Z} \times_{\text{ev}_{s_0}^{-1}} \text{Spec}\,\bar{\mathbb{Q}}(\mathcal{S}) \in Z^p(X_{/\bar{\mathbb{Q}}(\mathcal{S})})$. Now rational functions (in $\bar{\mathbb{Q}}(\mathcal{S})$) have denominators; we may clear these (nonuniquely; this introduces ambiguities) to yield bihomogeneous equations cutting out $\bar{\mathfrak{Z}} \in Z^p(X \times \mathcal{S}_{/\bar{\mathbb{Q}}})$. Owing to the ambiguities the $\overset{\text{rat}}{\equiv}$-class of $\bar{\mathfrak{Z}}$ is not well-defined, in contrast to that of its restriction $\mathfrak{Z} \in Z^p(X \times \eta_{\mathcal{S}/\bar{\mathbb{Q}}})$.

---

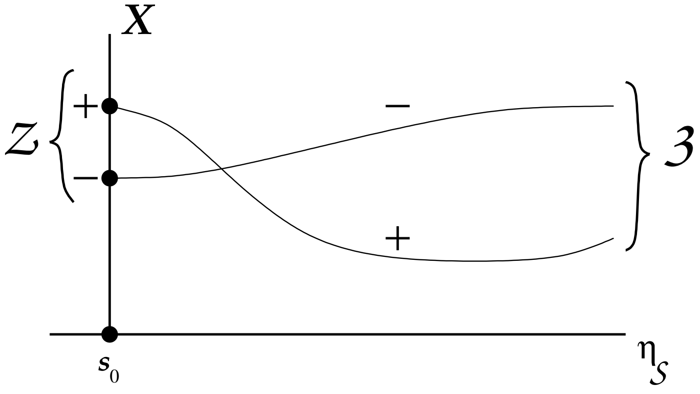
[5] It is known that $CH^{d_X}_{\text{alg}}(X \times X)$ is nilpotent; this clearly follows from §6.2 below.

[6] usually written $\{\pi_j\}$ in the literature, but we avoid this notation.

(In particular, one may freely modify $\bar{\mathfrak{Z}}$ by cycles supported on $X \times D$, for $D \subseteq \mathcal{S}$ an arbitrary divisor defined $/\bar{\mathbb{Q}}$.) We refer to $\mathfrak{Z}$ and $\bar{\mathfrak{Z}}$ as the $\bar{\mathbb{Q}}$-spread and *complete* $\bar{\mathbb{Q}}$-spread of $\mathcal{Z}$; clever *choices* of the latter play a role in proofs.

Once one has spread out, going back is easy: tensor everything with K and pull back (restrict) along the inclusion $X = X \times \{s_0\} \hookrightarrow X \times \eta_{\mathcal{S}}$. As we can spread and restrict rational equivalences $\{(Y_i,\, f_i)\}$ too, $\mathcal{Z} \overset{\text{rat}}{\equiv} 0 \iff \mathfrak{Z} \overset{\text{rat}}{\equiv} 0$.

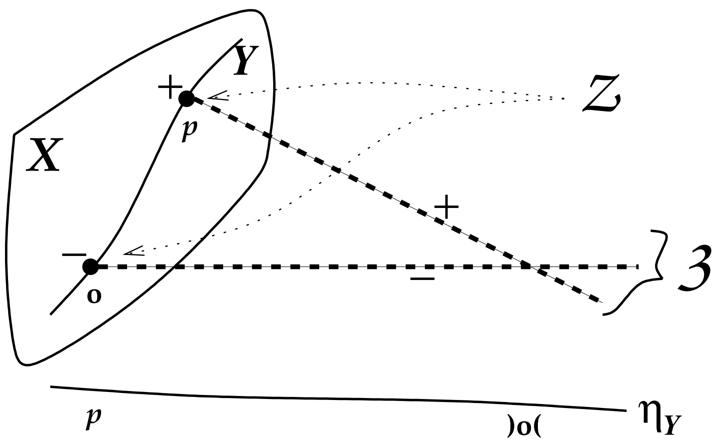
Now, different very general points $s \in \mathcal{S}(\mathbb{C})$ correspond to different embeddings $\text{ev}_s : \bar{\mathbb{Q}}(\mathcal{S}) \hookrightarrow \mathbb{C}$ respecting $\bar{\mathbb{Q}}$. So if we tensor everything with $\mathbb{C}$ and set $\mathcal{Z}_s := \mathfrak{Z}|_{X \times \{s\}}$, letting s vary shows how the original cycle deforms in X as its defining equations' coefficients change along with $\text{ev}_s(\bar{\mathbb{Q}}(\mathcal{S})) =: K_s$.



Clearly, spreading out exchanges field-theoretic information for additional geometric structure. That extra structure means that $c_{\mathcal{H}}(\mathfrak{Z})$ is going to capture more information than $c_{\mathcal{D}}(\mathcal{Z})$, and this is what the Lewis, M. Saito, and Griffiths-Green filtrations all capitalize on.

Example 4.1. $X/\bar{\mathbb{Q}}$, $\mathcal{Z} = p \in X(\mathbb{C})$ very general $\implies$ $\mathcal{S} = X$ and (we can choose) $\bar{\mathfrak{Z}} = \Delta_X \in Z^d(X \times X_{/\bar{\mathbb{Q}}})$.

Example 4.2. Consider a smooth curve $Y/\bar{\mathbb{Q}}$ on a surface $X/\bar{\mathbb{Q}}$, $o \in Y(\bar{\mathbb{Q}})$ and $p \in Y(\mathbb{C})$ very general; of course, p has a model $/K$ $(\subseteq \mathbb{C})$ s.t. $K \cong \bar{\mathbb{Q}}(Y)$. We write $\iota : Y \hookrightarrow X$ and define $\mathcal{Z} := \iota_*(p - o) \in Z_0^{\text{hom}}(X_K)$. Taking $\mathcal{S} = Y$, we can choose $\bar{\mathfrak{Z}} = \mathsf{I}_*(\Delta_Y - Y \times \{o\}) \in Z^2(X \times Y)$ where $\mathsf{I} : Y \times Y \hookrightarrow X \times Y$.



Now, so far we have only considered the case where "X does not spread". If X does *not* have a model $/\bar{\mathbb{Q}}$, it sits in something that does; and one constructs the $\bar{\mathbb{Q}}$-spread $\mathfrak{X}(\overset{\pi}{\to} \eta_{\mathcal{S}})$ of X_K just as one did $\mathfrak{Z}$. One can always choose $\bar{\mathfrak{X}}$ to be smooth, but not $\bar{\pi} : \bar{\mathfrak{X}} \to \mathcal{S}$ (which we note is of relative dimension d). So in general $\mathcal{Z}$ has $\bar{\mathbb{Q}}$-spreads $\mathfrak{Z} \in Z^p(\mathfrak{X}/\bar{\mathbb{Q}}) \leftarrow Z^p(\bar{\mathfrak{X}}/\bar{\mathbb{Q}}) \ni \bar{\mathfrak{Z}}$. (As above $\bar{\mathfrak{Z}}$ is defined modulo the addition of cycles supported on $\pi^{-1}(D)$, for any codim-1 $D_{/\bar{\mathbb{Q}}} \subseteq \mathcal{S}$.) We always refer to $\pi^{-1}(\mathcal{U} \subseteq \mathcal{S})$ as $\mathfrak{X}_{\mathcal{U}}$, so $\mathfrak{X}_{\mathcal{S}} = \bar{\mathfrak{X}}$ and $\mathfrak{X}_{\eta_{\mathcal{S}}} = \mathfrak{X}$; we write X_s for $\pi^{-1}(s)$. Of course, if on the other hand X is defined $/\bar{\mathbb{Q}}$, the ($\bar{\mathbb{Q}}$-)spread of X_K is $\mathfrak{X} = X \times \eta_{\mathcal{S}}$.

4.2. **Lewis filtration.** We will work with this for the remainder of the paper.

By identifying formally $\mathfrak{X}$ and $\varprojlim \pi^{-1}(\mathcal{U})$ (as we did above for $\eta_{\mathcal{S}}$ and $\varprojlim \mathcal{U}$), we obtain a composition

$$CH^p(X_{/K}) \xrightarrow[\text{spread}]{\cong} CH^p(\mathfrak{X}_{/\bar{\mathbb{Q}}}) \xrightarrow[\underline{c_{\mathcal{D}}}]{} \underline{H^{2p}_{\mathcal{D}}}(\mathfrak{X}, \mathbb{Q}(p))$$

which we denote by ψ. Lewis [L1] puts a Leray filtration $\mathcal{L}^\bullet$ (induced by π) on $\underline{H^{2p}_{\mathcal{D}}}(\mathfrak{X}_{\mathbb{C}}, \mathbb{Q}(p))$ with graded pieces

$$(4.1) \quad 0 \to Gr_{\mathcal{L}}^{i-1}\underline{J^p}(\mathfrak{X}) \overset{\beta_{i-1}}{\to} Gr^i_{\mathcal{L}}\underline{H^{2p}_{\mathcal{D}}}(\mathfrak{X}_{\mathbb{C}}, \mathbb{Q}(p)) \overset{\alpha_i}{\to} Gr^i_{\mathcal{L}}\underline{Hg^p}(\mathfrak{X}) \to 0$$

where

$$Gr^i_{\mathcal{L}}\underline{Hg^p}(\mathfrak{X}) := Hom_{\text{MHS}}\left(\mathbb{Q}(-p),\, [W_{2p}]H^i(\eta_{\mathcal{S}}, R^{2p-i}\pi_*\mathbb{Q})\right)$$

and

$$Gr_{\mathcal{L}}^{i-1}\underline{J^p}(\mathfrak{X}) := \frac{Ext^1_{\text{MHS}}\left(\mathbb{Q}(-p),\, W_{2p-1}H^{i-1}(\eta_{\mathcal{S}}, R^{2p-i}\pi_*\mathbb{Q}\right)}{\text{im}\, Hom_{\text{MHS}}\left(\mathbb{Q}(-p),\, Gr^W_{2p}H^{2p-1}(\mathfrak{X}, \mathbb{Q})\right)}.$$

The notational abuse taking place here (e.g., $Gr^W_{2p}H^{2p-1}(\mathfrak{X}, \mathbb{Q})$ on its own is not finite-dimensional) refers as usual to the obvious limits. We define the Lewis filtration

$$\mathcal{L}^i CH^p(X_{/K}) := \psi^{-1}\left(\mathcal{L}^i \underline{H^{2p}_{\mathcal{D}}}(\mathfrak{X}_{\mathbb{C}}, \mathbb{Q}(p))\right),$$

and for $\mathcal{Z} \in \mathcal{L}^i CH^p$ the i^{th} higher cycle-class

$$cl^i_X(Z) := \alpha_i\left(Gr^i_{\mathcal{L}}\psi(\mathcal{Z})\right) = \alpha_i\left(Gr^i_{\mathcal{L}}\underline{c_{\mathcal{D}}}(\mathfrak{Z})\right) =: [\mathfrak{Z}]_i.$$

If the latter vanishes, we can define an $(i-1)^{\text{st}}$ higher AJ class

$$AJ_X^{i-1}(\mathcal{Z}) = [AJ(\mathfrak{Z})]_{i-1}$$

by demanding that β_{i-1} of it gives $Gr^i_{\mathcal{L}}\psi(\mathcal{Z}) = Gr^i_{\mathcal{L}}\underline{c_{\mathcal{D}}}(\mathfrak{Z})$.[7]

We may summarize our invariants in a chart

$$\begin{array}{ccccccccc} \mathcal{L}^0 & \supseteq & \mathcal{L}^1 & \supseteq & \mathcal{L}^2 & \supseteq & \mathcal{L}^3 & \supseteq \cdots\cdots & \end{array}$$

$$\begin{array}{c|c|c|c|c} [\mathfrak{Z}]_0 & [\mathfrak{Z}]_1 & [\mathfrak{Z}]_2 & [\mathfrak{Z}]_3 & \\ & & & & \cdots\cdots \\ \text{---} & [AJ(\mathfrak{Z})]_0 & [AJ(\mathfrak{Z})]_1 & [AJ(\mathfrak{Z})]_2 & \end{array}$$

(which mimics that in [GG1] although the invariants there are only verifiably the same if HC holds). Here

$$[\mathfrak{Z}]_0 \in Hom_{\text{MHS}}\left(\mathbb{Q}(-p),\, H^0(\eta_{\mathcal{S}}, R^{2p}\pi_*\mathbb{Q})\right) \cong Hg^p(X)$$

identifies with $[\mathcal{Z}]$, while if $[\mathfrak{Z}]_0 = [\mathfrak{Z}]_1 = 0$ then $[AJ(\mathfrak{Z})]_0$ essentially identifies with $AJ(\mathcal{Z})$ (or more precisely, with a flat normal function); certainly $\mathcal{L}^2 \subseteq \ker(AJ)$. Writing $t = \text{trdeg}(K/\bar{\mathbb{Q}}) = \dim(\mathcal{S})$, we have (mostly from [L1]) these facts:

Proposition 4.3. *(a)* $Gr^k_{\mathcal{L}}CH^p(X_K) = 0$ *if* $k > p$ *or* $k > t+1$. *One may replace* $Gr^k_{\mathcal{L}}$ *by* $\mathcal{L}^k$ *provided* $\underline{c_{\mathcal{D}}}$ *is injective.*

(b) If X *has algebraic Künneth components, then* $\mathcal{L}^{\bullet}$ *satisfies the second paragraph of Defn.* 3.7.

---

[7] Note that by writing

$$\psi_i : \mathcal{L}^i CH^p(X_K) \to Gr^i_{\mathcal{L}}\underline{H^{2p}_{\mathcal{D}}}(\mathfrak{X}, \mathbb{Q}(p))$$

for the map $\mathcal{Z} \mapsto Gr^i_{\mathcal{L}}\psi(\mathcal{Z})$, one trivially expresses $\mathcal{L}^{\bullet}$ [$= \ker(\psi_{\bullet -1})$] as a filtration by kernels of successive maps.

Here (b) comes from the fact that in (4.1) only $R^{2p-i}\pi_*\mathbb{Q}$ appears, while (a) follows from the hard and weak Lefschetz theorems (respectively). Weak Lefschetz also implies $[\mathfrak{Z}]_{t+1} = 0$; however $[AJ(\mathfrak{Z})]_t$ (hence $Gr_{\mathcal{L}}^{t+1}CH^p$) need not vanish. In connection with (a) one makes the following Bloch-Beilinson-type conjecture for quasiprojective varieties:

Conjecture (BBC^q). For $Y/\bar{\mathbb{Q}}$ smooth quasiprojective, $\underline{c_{\mathcal{D}}} : CH^p(Y_{/\bar{\mathbb{Q}}}) \to \underline{H^{2p}_{\mathcal{D}}}(Y_{\mathbb{C}}, \mathbb{Q}(p))$ is injective.

Lewis ([L1]) deduces BBC^q from a conjecture of Jannsen ([J2]); one can also prove (with some work) that it follows from BBC+HC.

Now taking a limit over $K \subseteq \mathbb{C}$ f.g. $/\bar{\mathbb{Q}}$ (and using $CH^p(X_K) \hookrightarrow CH^p(X_{\mathbb{C}})$ and the corresponding limit of $\mathcal{S}$'s) we obtain filtrations and maps

$$\psi_i : \mathcal{L}^i CH^p(X_{\mathbb{C}}) \to \varinjlim_{\mathcal{S}} \; Gr^i_{\mathcal{L}} \underline{H^{2p}_{\mathcal{D}}}(\mathfrak{X}_{(\eta_S)}, \mathbb{Q}(p)).$$

While we prefer to work $/K$ (and without this extra limit) in what follows, this gives a candidate Bloch-Beilinson filtration.

Theorem (Lewis, [L1]**).** HLC+BBC^q $\implies \mathcal{L}^\bullet$ is a BBF.

If BBC^q holds then the above invariants (see chart) completely describe $CH^p(X_K)$, and in particular $[\mathfrak{Z}]_2$ and $[AJ(\mathfrak{Z})]_1$ describe $\ker(Alb)$ for 0-cycles on a surface$/\bar{\mathbb{Q}}$.[8] For computing these, we can use $\underline{c_{\mathcal{D}}}(\mathfrak{Z}) = \mathrm{im}(c_{\mathcal{D}}(\bar{\mathfrak{Z}}))$ for a convenient choice of $\bar{\mathfrak{Z}}$.

Proposition 4.4. *If* $\mathcal{Z} \in \mathcal{L}^i CH^p(X_K)$ *then* $[\mathfrak{Z}]_i$ *is the image of* $[\bar{\mathfrak{Z}}]$ *under the projection*

$$Hg^p\left(H^{2p}(\bar{\mathfrak{X}})\right) \twoheadrightarrow Hg^p\left(Gr^i_{\mathcal{L}}\underline{H^{2p}}(\mathfrak{X})\right) = Gr^i_{\mathcal{L}}\underline{Hg^p}(\mathfrak{X}).$$

Here $Gr^i_{\mathcal{L}}\underline{H^{2p}}(\mathfrak{X}) = W_{2p}H^i(\eta_{\mathcal{S}}, R^{2p-i}\pi_*\mathbb{Q})$ is justifiable by [Ar].

According to Propositions 4.4 and 4.3(a), if $[\mathfrak{Z}]_0 = [\mathfrak{Z}]_1 = \cdots = [\mathfrak{Z}]_p = 0$, then $[\bar{\mathfrak{Z}}]$ projects to 0 in $Hg^p(\underline{H^{2p}}(\mathfrak{X}))$ (one says $[\mathfrak{Z}] = 0$). It follows (using [De, Cor. 8.2.8]) that for some divisor $D_{/\bar{\mathbb{Q}}} \subseteq \mathcal{S}$, $[\bar{\mathfrak{Z}}]$ comes from a class in $Hg^{p-1}(H^{2p-2}(\widetilde{\mathfrak{X}_D}))$. Assuming HC, this is the class of a cycle, by whose image (in $\bar{\mathfrak{X}}$, supported on $\mathfrak{X}_D$) we may modify our choice of $\bar{\mathfrak{Z}}$ to get $[\bar{\mathfrak{Z}}] = 0$. In specific cases (like those in §6), one may be able to explicitly construct (*without* HC) a complete spread $\bar{\mathfrak{Z}}$ which is $\overset{\text{hom}}{\equiv} 0$. Here is the reason we want this property:

---

[8] It is worth noting that $\mathcal{L}^2 = \ker(Alb)$ holds for 0-cycles if X (of any dimension) is defined $/\bar{\mathbb{Q}}$, see [K2].

Proposition 4.5. *Suppose $\mathcal{Z} \in \mathcal{L}^i CH^p(X_K)$ has a homologically trivial complete $\bar{\mathbb{Q}}$-spread $\bar{\mathfrak{Z}}$. Then $[AJ(\mathfrak{Z})]_i$ is the image of $AJ_{\bar{\mathfrak{X}}}(\bar{\mathfrak{Z}})$ under the projection*

$$J^p\left(H^{2p-1}(\bar{\mathfrak{X}})\right) \twoheadrightarrow J^p\left(Gr_{\mathcal{L}}^{i-1}\underline{H^{2p-1}}(\mathfrak{X})\right) \twoheadrightarrow Gr_{\mathcal{L}}^{i-1}\underline{J^p}(\mathfrak{X}).$$

If X is defined $/\bar{\mathbb{Q}}$ then we get simplifications: $\mathfrak{X} = X \times \eta_{\mathcal{S}}$ and (taking $p = d (= \dim(X))$ for 0-cycles)

$$Gr_{\mathcal{L}}^{i}\underline{Hg^d}(X \times \eta_{\mathcal{S}}) = Hom_{\text{MHS}}\left(\mathbb{Q}(-d),\, \underline{H^i}(\eta_{\mathcal{S}}) \otimes H^{2d-i}(X)\right)$$

$$\cong Hom_{\text{MHS}}\left(H^i(X),\, \underline{H^i}(\eta_{\mathcal{S}})\right) \twoheadrightarrow Hom_{\mathbb{C}}\left(\Omega^i(X),\, \Omega^i(\mathcal{S})\right);$$

we write $[\mathfrak{Z}]_i^*$ for $[\mathfrak{Z}]_i$'s induced Hom's. Moreover,

$$Gr_{\mathcal{L}}^{i-1}\underline{J^d}(X \times \eta_{\mathcal{S}}) = \frac{Ext^1_{\text{MHS}}\left(\mathbb{Q}(-d),\, \underline{H^{i-1}}(\eta_{\mathcal{S}}) \otimes H^{2d-i}(X)\right)}{\operatorname{im} Hom_{\text{MHS}}\left(\mathbb{Q}(-d),\, Gr_i^W H^{i-1}(\eta_{\mathcal{S}}) \otimes H^{2d-i}(X)\right)}$$

$$\overset{\xi}{\twoheadrightarrow} J^d\left(\underline{H^{i-1}}(\eta_{\mathcal{S}}) \otimes \frac{H^{2d-i}(X)}{F_h^{d-i+1}H^{2d-i}(X)}\right),$$

where ξ is worked out in [K2, sec. 12]; we define $AJ_X^{i-1}(\mathcal{Z})^{tr} =$

$$[AJ(\mathfrak{Z})]_{i-1}^{tr} := \xi\left([AJ(\mathfrak{Z})]_{i-1}\right).$$

Here then is how to compute the invariants for 0-cycles:

Corollary 4.6. *Let $X/\bar{\mathbb{Q}}$ and $\mathcal{Z} \in \mathcal{L}^i CH_0(X_K)$; then*

(a) $[\mathfrak{Z}]_i$ is the image of the Künneth component $[\bar{\mathfrak{Z}}]_i \in H^{2d-i}(X) \otimes H^i(\mathcal{S})$, hence $[\mathfrak{Z}]_i^*$ is the composition of $[\bar{\mathfrak{Z}}]_i^*$ with e.g. $H^i(\mathcal{S}) \twoheadrightarrow \underline{H^i}(\eta_{\mathcal{S}})$.

(b) If in addition $\bar{\mathfrak{Z}} \overset{hom}{\equiv} 0$, then $[AJ(\mathfrak{Z})]_{i-1}^{tr}$ is the projected image of

$$\left\{\int_{\partial^{-1}\bar{\mathfrak{Z}}}(\cdot)\right\} \in \left(F^{t+1}\left\{H^{2t-i+1}(\mathcal{S},\mathbb{C}) \otimes H^i(X,\mathbb{C})\right\}\right)^{\vee}\Big/\, \textit{periods}$$

$\cong J^d\left(H^{i-1}(\mathcal{S}) \otimes H^{2d-i}(X)\right)$, *under the obvious projection of HS.*

Example 4.7. Recall the case $d = \ell = 2$ of Thm. 3.5(A), the cycle $\mathcal{Z} = (p_1 - o_1) \times (p_2 - o_2) \in \ker(Alb) \subseteq CH_0^{\text{hom}}(\mathcal{C}_1 \times \mathcal{C}_{2/K})$ where $\mathcal{C}_1, \mathcal{C}_2/\bar{\mathbb{Q}}$ and $K \cong \bar{\mathbb{Q}}(\mathcal{C}_1 \times \mathcal{C}_2)$. In fact this lives in $\mathcal{L}^2$, see §5.1. Noting that $X = \mathcal{C}_1 \times \mathcal{C}_2 = \mathcal{S}$, we need only check that $[\bar{\mathfrak{Z}}]_2^* : \Omega^2(X) \to \Omega^2(\mathcal{S})$ is nontrivial. This follows from the fact that the spread of (p_1, p_2) is $\Delta_{\mathcal{C}_1 \times \mathcal{C}_2}$ (which induces the identity on Ω^2), while the spreads of the other points of $\mathcal{Z}$ induce the zero map. Here the connection between $\Omega^2(X)$ and the Albanese kernel is quite plain! The other cases of Thm. 3.5(A) are dealt with similarly.

4.3. **Definition of reduced higher AJ maps.** Continue to assume X is defined $/\bar{\mathbb{Q}}$. At times it is too difficult to compute even $[AJ(\mathfrak{Z})]^{tr}_{i-1}$ fully; instead one can make use of the following quotient. Writing D for divisors$/\bar{\mathbb{Q}} \subseteq \mathcal{S}$, we can define

$$\overline{H^*}(\mathcal{S}_{\text{rel}}) := (\text{co})\text{im}\left\{ \varprojlim_{D} \; H^*(\mathcal{S}, D) \to H^*(\mathcal{S}) \right\},$$

$$\underline{H_*}(\eta_{\mathcal{S}}) := (\text{co})\text{im}\left\{ \varprojlim_{D} \; H_*(\mathcal{S} \setminus D) \to H_*(\mathcal{S}) \right\};$$

the latter are homology classes on $\mathcal{S}$ which can be moved to avoid any divisor. The former is a (pure weight) HS.

By restricting functionals $\left\{F^{t+1}\left(\overline{H^{2t-i+1}}(\mathcal{S}_{\text{rel}}, \mathbb{C}) \otimes \frac{H^i(X,\mathbb{C})}{F^1_h}\right)\right\}^\vee \twoheadrightarrow$ $\left\{\overline{H^{2t-i+1}}(\mathcal{S}_{\text{rel}}, \mathbb{C}) \otimes F^i H^i(X, \mathbb{C})\right\}^\vee \cong \left\{\underline{H_{i-1}}(\eta_{\mathcal{S}}, \mathbb{Q}) \otimes F^i H^i(X, \mathbb{C})\right\}^\vee$, we induce a map

$$J^d\left(\underline{H^{i-1}}(\eta_{\mathcal{S}}) \otimes \frac{H^{2d-i}(X)}{F_h^{d-i+1}}\right) \twoheadrightarrow Hom_{\mathbb{Q}}\left(\underline{H_{i-1}}(\eta_{\mathcal{S}}, \mathbb{Q}), \frac{\Omega^i(X)^\vee}{\text{im}\{H_i(X, \mathbb{Q})\}}\right)$$

and the image of $[AJ(\mathfrak{Z})]^{tr}_{i-1}$ is denoted $\overline{[AJ(\mathfrak{Z})]_{i-1}}$. Assuming $\bar{\mathfrak{Z}} \overset{\text{hom}}{\equiv} 0$, recall that $[AJ(\mathfrak{Z})]^{tr}_{i-1}$ (as a functional) is induced by $\int_{\partial^{-1}\bar{\mathfrak{Z}}}$. Applying this to a class $[\hat{\gamma}] \otimes \omega$ for $\omega \in \Omega^i(X)$, $[\gamma] \in \underline{H_{i-1}}(\eta_{\mathcal{S}}, \mathbb{Q})$ [with $[\hat{\gamma}] \in \overline{H^{2t-i+1}}(\mathcal{S}_{\text{rel}}, \mathbb{C})$ its Poincaré dual], gives

$$\int_{\partial^{-1}\bar{\mathfrak{Z}}} \hat{\gamma} \wedge \omega = \int_{\pi_X\left\{(\partial^{-1}\bar{\mathfrak{Z}}) \cap (\gamma \times X)\right\}} \omega =: \int_{\Gamma} \omega\,,$$

where Γ is a topological i-chain bounding on $\bar{\mathfrak{Z}}_*(\gamma) := \pi_X\left\{\bar{\mathfrak{Z}} \cap (\gamma \times X)\right\}$.[9]

Now let $\{\omega\} = \{\omega^k\}_{k \in \mathsf{K}} \subseteq \Omega^i(X)$ denote a basis; evaluating functionals on it gives a map $\Omega^i(X)^\vee \underset{\text{ev}_{\{\omega\}}}{\overset{\cong}{\longrightarrow}} \mathbb{C}^{|\mathsf{K}|}$, and we write $\Lambda_{\{\omega\}}$ for the $\text{ev}_{\{\omega\}}$-image of $\text{im}\{H_i(X, \mathbb{Q})\}$. Then the vector $\left(\int_\Gamma \omega^k\right)_{k \in \mathsf{K}}$ does not change modulo

---

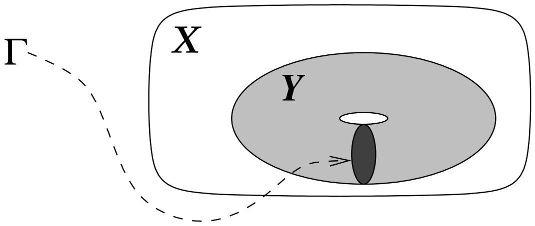
[9] One can ensure this intersection is proper by moving γ in its homology class on $\mathcal{S}$. The easiest way to do this is to choose a divisor $D \subseteq \mathcal{S}$ sufficiently "large" that the restriction of $\bar{\mathfrak{Z}}$ to $X \times (\mathcal{S} \setminus D)$ has relative dimension 0 over $\mathcal{S} \setminus D$, and move γ to avoid D. (This is possible because $[\gamma] \in \underline{H_{i-1}}(\eta_{\mathcal{S}}, \mathbb{Q})$; of course, sometimes D can be chosen empty, as in example 4.2 above.)

$\Lambda_{\{\omega\}}$(= periods) if we choose a different Γ bounding on $\bar{\mathfrak{Z}}_*(\gamma)$ (provided we use the same Γ for all ω^k). We have proved the following:

Proposition 4.8. *Given* $\mathcal{Z} \in \mathcal{L}^i CH_0(X)$ *with nullhomologous complete spread* $\bar{\mathfrak{Z}}$*,* $\gamma \in \underline{H_{i-1}}(\eta_{\mathcal{S}}, \mathbb{Q})$ *and* $\Gamma \in C_i^{top}(X, \mathbb{Q})$ *with* $\partial\Gamma = \bar{\mathfrak{Z}}_*(\gamma)$*,* $\overline{[AJ(\mathfrak{Z})]_{i-1}}(\gamma)$ *evaluates to* $\overline{\left(\int_\Gamma \omega^k\right)_{k\in\mathsf{K}}} \in \mathbb{C}^{|\mathsf{K}|}/\,\Lambda_{\{\omega\}}$ *on a basis* $\{\omega\}$*.*

The $\int_\Gamma \omega^k$ are called membrane integrals; for $i = 2$ this once again makes the connection between $\overset{\text{rat}}{\not\equiv}$ of cycles and holomorphic 2-forms quite clear. We also note that $\Lambda_{\{\omega\}} \subseteq \mathbb{C}^{|\mathsf{K}|}$ is a rational lattice of dimension *at least* $2|\mathsf{K}|$.

Example 4.9. In the situation of Ex. 4.2 (assume $H^1(X) = 0$), the membrane integrals are simply over chains $\subseteq X$ bounding loops $\subseteq Y$



These $\left\{\int_\Gamma \omega_X^{(2,0)}\right\}$ are related to extensions of MHS.

Remark 4.10. (a) We can interpret $[\mathfrak{Z}]_{i-1}$ as corresponding to an element of $Hom_{\mathbb{Q}}\left(\underline{H_{i-1}}(\eta_{\mathcal{S}}, \mathbb{Q}),\, H_{i-1}(X, \mathbb{Q})\right)$, and thus deduce existence of the membrane Γ (hence the membrane integrals) from vanishing of $[\mathfrak{Z}]_{i-1}$ (true for $\mathcal{Z} \in \mathcal{L}^i$) rather than $[\bar{\mathfrak{Z}}]$.

(b) With some work (see [K2]), one may furthermore relax the extra condition in Proposition 4.8 from $[\bar{\mathfrak{Z}}] = 0$ to only $[\mathfrak{Z}]_i = 0$. This is a vast improvement, because it says the reduced invariants $\overline{[AJ(\mathfrak{Z})]_{i-1}}$ are computable by membrane integrals whenever they are defined — hence under weaker conditions than allow $[AJ(\mathfrak{Z})]_{i-1}^{(tr)}$ to be computed by Griffiths's prescription $\int_{\partial^{-1}\bar{\mathfrak{Z}}}(\cdot)$. This is needed to deal with the n-box cycles of §6.3 for $n > 2$.

5. ADDITIONAL FILTRATIONS AND INVARIANTS

In this section we will study a few other "candidates" for $F^\bullet_{\text{BB}}$. We are more interested in what one can do with them as is, than with the fact that under reasonable conjectural conditions they all coincide.

5.1. **Murre filtration**[10]**.** Assume X (defined / any subfield of $\mathbb{C}$) is such that Δ_X has a good Chow-Künneth decomposition. Writing $\Delta^X_{\leq a} := \sum_{(0\leq)j\leq a} \Delta_{X,j}$ and $\Delta^X_{>b} := \sum_{b<j(\leq 2d)} \Delta_{X,j}$, we set

$$F^i_M CH_0(X) := \operatorname{im} \left\langle \Delta^X_{\leq 2d-i} \right\rangle_* = \ker \left\langle \Delta^X_{>2d-i} \right\rangle_* .$$

Note that $F^1_M CH_0 = CH_0^{\text{hom}}$, $F^2_M CH_0 \subseteq \ker(Alb)$, $F^{d+1}_M CH_0 = \{0\}$, and $\langle \Delta_{X,j} \rangle_* \big|_{Gr^i_{F_M} CH_0(X)} = \delta_{2d-i,j} \cdot Id$. This latter property does not necessarily hold for arbitrary algebraic Künneth components $\tilde{\Delta}_{X,j}$ (required of a BBF), but if $F^\bullet_{\text{BB}}$ exists then one easily shows $F^\bullet_M$ must coincide with it. Under the assumptions of HLC+BBC^q, $\mathcal{L}^\bullet$ gives such a filtration, so then $\mathcal{L}^\bullet = F^\bullet_M$. Without such assumptions, always $F^\bullet_M \subseteq \mathcal{L}^\bullet$, and conveniently $\left\langle \Delta^X_{\leq 2d-i} \right\rangle_*$ gives a projection to F^i_M.

Example: products of curves. For one curve $\mathcal{C}$ with a base point o, set $\Delta_{\mathcal{C},2} = \mathcal{C} \times \{o\}$, $\Delta_{\mathcal{C},0} = \{o\} \times \mathcal{C}$, $\Delta_{\mathcal{C},1} = \Delta_{\mathcal{C}} - \Delta_{\mathcal{C},0} - \Delta_{\mathcal{C},2}$. If $X = \mathcal{C}_1 \times \cdots \times \mathcal{C}_n$ (so $d = n$), write $\sigma : (\mathcal{C}_1 \times \mathcal{C}_1) \times \cdots \times (\mathcal{C}_n \times \mathcal{C}_n) \to X \times X$ for the obvious permutation of factors; then

$$\sigma^* \Delta_{X,j} := \sum_{j_1 + \cdots + j_n = j} \Delta_{\mathcal{C}_1, j_1} \times \cdots \times \Delta_{\mathcal{C}_n, j_n}$$

yields a good C-K decomposition of Δ_X. We can obtain an alternate description of the resulting $F^\bullet_M$ as follows. Let Ξ^n_i denote the set of strictly increasing functions $\varepsilon : \{1, \ldots, i\} \hookrightarrow \{1, \ldots, n\}$ and (for each $\varepsilon \in \Xi^n_i$) write $\pi_\varepsilon : X \twoheadrightarrow X_\varepsilon := \mathcal{C}_{\varepsilon(1)} \times \cdots \times \mathcal{C}_{\varepsilon(i)}$. Set $F^0_\times CH_0(X) = CH_0(X)$ and for $1 \leq i \leq n+1$ define

$$F^i_\times CH_0(X) := \bigcap_{\varepsilon \in \Xi^n_{i-1}} \ker(\pi_{\varepsilon_*}) \subseteq CH_0(X).$$

Clearly $F^{n+1}_\times CH_0 = \{0\}$, and in fact $F^\bullet_\times = F^\bullet_M (\subseteq \mathcal{L}^\bullet)$.

Choosing base points $o_j \in \mathcal{C}_j$ (implicit already in the above), we may also define inclusions $\iota_\varepsilon : X_\varepsilon \hookrightarrow X$. [For $\varepsilon \in \Xi^n_0$ the trivial map $\emptyset \hookrightarrow \{1, \ldots, n\}$, π_ε is the structure map and ι_ε incudes $\operatorname{Spec} \mathbb{C} \hookrightarrow X$ as $(o_1, \ldots, o_n)$.] Then we can write the projections $\left\langle \Delta^X_{\leq 2d-i} \right\rangle_*$ to the $F^i_\times$ as follows:

$$P^{[i]}_\times (\mathcal{Z}) := \mathcal{Z} + \sum_{j=0}^{i-1} (-1)^{i-j} \left(\sum_{\varepsilon \in \Xi^n_j} (\iota_\varepsilon \circ \pi_\varepsilon)_* \mathcal{Z} \right)$$

for $\mathcal{Z} \in Z_0(X)$. To write it in a more attractive way, define $Box^n(\mathcal{Z}) := P^{[n]}_\times(\mathcal{Z})$ so that e.g. for $\mathcal{Z}$ one point we have $Box^n\{(p_1, \ldots, p_n)\} =$

---

[10] We caution that this is not quite Murre's conjectural filtration (unless conjectures or further hypotheses on the $\{\Delta_{X,j}\}$ are assumed), see [M].

$(p_1 - o_1) \times \cdots \times (p_n - o_n)$. (For $n = 2$ this is just $Box^2(p_1, p_2) = (p_1, p_2) - (p_1, o_2) - (o_1, p_2) + (o_1, o_2)$.) Then

$$P_{\times}^{[i]}(\mathcal{Z}) = \mathcal{Z} - \sum_{j=0}^{i-1} \sum_{\varepsilon \in \Xi_j^n} \iota_*^{\varepsilon} Box^j(\pi_*^{\varepsilon} \mathcal{Z}).$$

To be clear: we shall only take "Box^j" of a cycle defined on a product of j curves.

5.2. **S. Saito filtration.** The form of this filtration (we use the version in [LS]) precludes discussion exclusively for 0-cycles. Let Sm =category of smooth projective varieties $/\mathbb{C}$, and define inductively $\forall\, X \in Sm$, $p \in \mathbb{N}$: $F_S^0 CH^p(X) := CH^p(X)$,

$$F_S^{i+1} CH^p(X) := \sum_{\substack{Y \in Sm \\ q \in \mathbb{N}}} \sum_{\Gamma} \operatorname{im}\left\{ \langle \Gamma \rangle_* : F_S^i CH^q(Y) \to CH^p(X) \right\},$$

where the sum is over all $\Gamma \in Z^{p-q+d_X}(Y \times X)$ for which $[\Gamma]_* : H^{2q-i}(Y) \to H^{2p-i}(X)$ is the zero map.

Proposition 5.1. *(i)* $F_S^\bullet \subseteq \mathcal{L}^\bullet$.

(ii) If there exists a C-K decomposition of Δ_X *then im* $\langle \Delta_{X,j} \rangle_* \subseteq F_S^i CH^p(X)\ \forall\, j \leq 2p - i$.

(iii) In addition to the decomposition, if one assumes BBC^q *then* $F_S^\bullet = \mathcal{L}^\bullet$.

Proof. [arbitrary codimension]

(i) Inductively assume $F_S^i \subseteq \mathcal{L}^i$ for all X, p; then fix X, p and let $\langle \mathcal{Z} \rangle \in F_S^{i+1} CH^p(X)$. That is, $\mathcal{Z} = \Gamma_* \mathcal{W}$ for $\Gamma \in Z^{p-q+d_X}(Y \times X)$, $\mathcal{W} \in Z^q(Y)$ all defined / some K f.g. $/\bar{\mathbb{Q}}$, and $\langle \mathcal{W} \rangle \in F_S^i CH^q(Y)$. Spreading out X, Y, $\mathcal{W}$, Γ, $\mathcal{Z}$ over the same $\mathcal{S}$, the spread $\tilde{\Gamma} \in Z^{p-q+d_X}(\mathfrak{Y} \times_{\mathcal{S}} \mathfrak{X})$ induces[11] a map $[\tilde{\Gamma}]_*^{\mathcal{D}} : \underline{H_{\mathcal{D}}^{2q}}(\mathfrak{Y}, \mathbb{Q}(q)) \to \underline{H_{\mathcal{D}}^{2p}}(\mathfrak{X}, \mathbb{Q}(p))$ respecting $\mathcal{L}^\bullet$. So $\psi(\mathcal{W}) \in \mathcal{L}^i \underline{H_{\mathcal{D}}^{2q}}(\mathfrak{Y}, \mathbb{Q}(q)) \implies \psi(\mathcal{Z}) = [\tilde{\Gamma}]_*^{\mathcal{D}} \psi(\mathcal{W}) \in \mathcal{L}^i$ and (using Lewis's description of the $Gr_{\mathcal{L}}^i \underline{H_{\mathcal{D}}^*}$) $Gr_{\mathcal{L}}^i \psi(\mathcal{Z}) = \left(Gr_{\mathcal{L}}^i [\tilde{\Gamma}]_*^{\mathcal{D}} \right) \psi(\mathcal{W}) = 0$.

(ii) For $\mathcal{Z} \in CH^p(X)$, one inductively assumes for $i < 2p - j$ that $\langle \Delta_{X,j} \rangle_* \mathcal{Z} \in F_S^i CH^p(X)$ and uses the definition of $F_S^\bullet$ to show $\langle \Delta_{X,j} \rangle_* \mathcal{Z} \in F_S^{i+1}$. Namely, taking $Y = X$, $\Gamma = \Delta_{X,j}$ we have $0 = [\Delta_{X,j}]_{2p-i} : H^{2p-i}(X) \to H^{2p-i}(X)$ (since $j \neq 2p - i$) and $\langle \Delta_{X,j} \rangle_* \langle \Delta_{X,j} \rangle_* \mathcal{Z} = \langle \Delta_{X,j} \rangle_* \mathcal{Z}$, done.

---

[11] using cup product with $\underline{c_{\mathcal{D}}}(\tilde{\Gamma})$ together with the (essentially fiberwise) pullbacks and pushforwards along $\mathfrak{X} \times_{\mathcal{S}} \mathfrak{Y} \twoheadrightarrow \mathfrak{X}, \mathfrak{Y}$ resp. In particular, pushforward is OK because fibers are compact.

(iii) If $\mathcal{Z} \in \mathcal{L}^i CH^p(X)$ then by part (ii), $\sum_{p\leq j\leq 2p-i} \langle \Delta_{X,j}\rangle_* \mathcal{Z} \in F_S^i\,[\subseteq \mathcal{L}^i]$; but restricting $\mathcal{Z} - \sum_{p\leq j\leq 2p-i} \langle \Delta_{X,j}\rangle_* \mathcal{Z}$ (iteratively for $k = i,\, i+1,\, \ldots,\, p$) to $Gr^k_{\mathcal{L}}$ yields $Gr^k_{\mathcal{L}}\left(\mathcal{Z} - \langle \Delta_{X,2p-k}\rangle_* \mathcal{Z}\right)$ which is zero (by property of $\mathcal{L}^\bullet$). Hence, $\mathcal{Z} \equiv \sum_{p\leq j\leq 2p-i} \langle \Delta_{X,j}\rangle_* \mathcal{Z} \mod \mathcal{L}^{p+1}$, which by BBC^q is zero; and so $\mathcal{Z} \in F_S^i$. □

Having set the conjectural merry-go-round into motion, we give it one more push:

Corollary 5.2. *(i) If a good C-K decomposition (hence $F_M^\bullet$) exists for Δ_X, then $F_M^\bullet \subseteq F_S^\bullet$ (on $CH_0(X)$).*

(ii) BBC^q+HLC $\implies$ ($\exists\ F_M^\bullet$ and) $F_M^\bullet = F_S^\bullet = \mathcal{L}^\bullet$.

5.3. **Mumford-Griffiths invariants.** We now show how to define invariants on the $Gr^i_{F_S}$ which *together* carry the same information as the cl^i on the $Gr^i_{\mathcal{L}}$. This will involve putting a Leray filtration on $\mathbb{H}^{2d}(\mathfrak{X}, \Omega^{\bullet\geq d}_{\mathfrak{X}})$. (These invariants are really the only established way for computing when X is not defined $/\bar{\mathbb{Q}}$ and hence spreads along with $\mathcal{Z}$.)

As in §4 we let $\mathfrak{X}$, $\mathcal{S}$ be smooth projective $/\bar{\mathbb{Q}}$ with $\pi : \mathfrak{X}_{\mathcal{S}} \to \mathcal{S}$ proper, and let $\mathcal{U} = (\mathcal{S} \setminus D) \subseteq \mathcal{S}$ (affine Zariski open) be such that $\pi|_{\mathfrak{X}_{\mathcal{U}}}$ is smooth (the singular fibers are contained in $\mathfrak{X}_D$). Define a Leray filtration for forms on $\mathfrak{X}_{\mathcal{U}}$

$$\mathcal{L}^k \Omega^{\bullet\geq d}_{\mathfrak{X}_{\mathcal{U}}} := \mathrm{im}\left\{\pi^*\Omega^k_{\mathcal{U}} \otimes \Omega^{\bullet\geq d-k}_{\mathfrak{X}_{\mathcal{U}}}[-k] \to \Omega^{\bullet\geq d}_{\mathfrak{X}_{\mathcal{U}}}\right\}$$

so that $Gr^k_{\mathcal{L}}\Omega^{\bullet\geq d}_{\mathfrak{X}_{\mathcal{U}}} = \pi^*\Omega^k_{\mathcal{U}} \otimes \Omega^{\bullet\geq d-k}_{\mathfrak{X}_{\mathcal{U}}/\mathcal{U}}[-k]$; also set $\mathcal{L}^k\mathbb{H}^*(\mathfrak{X}_{\mathcal{U}}, \Omega^{\bullet\geq d}_{\mathfrak{X}_{\mathcal{U}}}) := \mathrm{im}\left\{\mathbb{H}^*(\mathfrak{X}, \mathcal{L}^k\Omega^{\bullet\geq d}_{\mathfrak{X}_{\mathcal{U}}}) \to \mathbb{H}^*(\mathfrak{X}_{\mathcal{U}}, \Omega^{\bullet\geq d}_{\mathfrak{X}_{\mathcal{U}}})\right\}$. Standard formal nonsense using the $\mathbb{H}^*$-long-exact sequences associated to $0 \to \mathcal{L}^{k+1} \to \mathcal{L}^k \to Gr^k_{\mathcal{L}} \to 0$ (for each k) leads to a spectral sequence computing $Gr^k_{\mathcal{L}}\mathbb{H}^{j+k}(\mathfrak{X}_{\mathcal{U}}, \Omega^{\bullet\geq d}_{\mathfrak{X}_{\mathcal{U}}}) = E^{j,k}_\infty$. This is

$$E_1^{j,k} := \mathbb{H}^{j+k}\left(\mathfrak{X}_{\mathcal{U}},\, Gr^k_{\mathcal{L}}\Omega^{\bullet\geq d}_{\mathfrak{X}_{\mathcal{U}}}\right) = \mathbb{H}^j\left(\mathfrak{X}_{\mathcal{U}},\, \pi^*\Omega^k \otimes \Omega^{\bullet\geq d-k}_{\mathfrak{X}_{\mathcal{U}}/\mathcal{U}}\right)$$

$$= H^0\left(\mathcal{U},\, \Omega^k_{\mathcal{U}} \otimes \mathbb{R}^j\pi_*\Omega^{\bullet\geq d-k}_{\mathfrak{X}_{\mathcal{U}}/\mathcal{U}}\right)$$

where $d_1 : E^{j,k} \to E^{j,k+1}$ is induced by ∇ (the Gauss-Manin connection). (The last equality comes from the Leray s.s. for $\mathfrak{X}_{\mathcal{U}} \xrightarrow{\pi} \mathcal{U}$ and observing that since $\mathcal{U}$ is affine and $\Omega^k_{\mathcal{U}} \otimes \mathbb{R}^j\pi_*\Omega^{\bullet\geq d-k}_{\mathfrak{X}_{\mathcal{U}}/\mathcal{U}}$ are coherent, all $H^{i>0}$'s vanish.) Writing[12] $\mathcal{F}^a\mathcal{H}^b_{\mathfrak{X}_{\mathcal{U}}/\mathcal{U}}$ for $\mathbb{R}^b\pi_*\Omega^{\bullet\geq a}_{\mathfrak{X}_{\mathcal{U}}/\mathcal{U}}$, we conclude that

---

[12] More generally, one writes $\mathcal{H}^b_{\mathfrak{X}/\mathcal{U}}$ for $\mathbb{R}^b\pi_*\Omega^\bullet_{\mathfrak{X}/\mathcal{U}} \cong (R^b\pi_*\mathbb{C}) \otimes \Omega_{\mathcal{U}}$, $\mathcal{H}^b_{\mathfrak{X}/\mathcal{U},\mathbb{Q}}$ for $\mathbb{R}^b\pi_*\mathbb{Q}$. (Also note: $\mathcal{H}^b_{X_s}$ means the same thing as $\mathcal{H}^b_{\mathfrak{X}/\mathcal{U}}$.)

$$(5.1)\qquad Gr^k_{\mathcal{L}}\mathbb{H}^{2d}\left(\mathfrak{X}_{\mathcal{U}},\, \Omega^{\bullet\geq d}_{\mathfrak{X}_{\mathcal{U}}}\right) \cong H^k_{\nabla}\left\{H^0\left(\mathcal{U},\, \Omega^*_{\mathcal{U}} \otimes \mathcal{F}^{d-*}\mathcal{H}^{2d-k}_{\mathfrak{X}_{\mathcal{U}}/\mathcal{U}}\right)\right\}.$$

Next, consider the composition $\phi_{\mathcal{U}}$:

$$Hom_{\text{MHS}}\left(\mathbb{Q}(-d),\, H^{2d}(\mathfrak{X}_{\mathcal{U}}, \mathbb{Q})\right) \hookrightarrow F^d H^{2d}(\mathfrak{X}_{\mathcal{U}}, \mathbb{C}) \cong \mathbb{H}^{2d}\left(\bar{\mathfrak{X}},\, \Omega^{\bullet\geq d}_{\bar{\mathfrak{X}}}\langle\log\, \mathfrak{X}_D\rangle\right)$$

$$\to \mathbb{H}^{2d}\left(\mathfrak{X}_{\mathcal{U}},\, \Omega^{\bullet\geq d}_{\mathfrak{X}_{\mathcal{U}}}\right).$$

One can show that this is independent of the choice of good compactification $\bar{\mathfrak{X}}$ of $\mathfrak{X}_{\mathcal{U}}$, and functorial for restrictions from $\mathcal{U}$ to $\mathcal{U}_0 \subseteq \mathcal{U}$. Hence it makes sense in the limit over $\mathcal{U} \subseteq \mathcal{S}$, and we may now consider Φ:

$$CH_0(X_K) \overset{\cong}{\to} CH^d(\mathfrak{X}_{(\bar{\mathbb{Q}})}) \overset{[\cdot]}{\to} Hom_{\text{MHS}}\left(\mathbb{Q}(-d),\, H^{2d}(\mathfrak{X}, \mathbb{Q})\right) \overset{\phi_\eta}{\to} \mathbb{H}^{2d}(\mathfrak{X}, \Omega^{\bullet\geq d}_{\mathfrak{X}})$$

(where as before $\mathfrak{X} = \varprojlim_{\mathcal{U}} \pi^{-1}(\mathcal{U}) = \mathfrak{X}_\eta$). An argument exactly like that for Prop. 5.1(ii) above, shows

Theorem 5.3. $\Xi\left(F^i_S CH_0\right) \subseteq \mathcal{L}^i\mathbb{H}^{2d}$, *hence* $\exists$ *a map*

$$\Phi_i :\, F^i_S CH_0(X_K) \to H^i_{\nabla}\left\{\Gamma\left(\eta_{\mathcal{S}},\, \Omega^*_{\mathcal{S}} \otimes \mathcal{F}^{d-*}\mathcal{H}^{2d-i}_{X_s}\right)\right\} =: \nabla J^{d,i}(\mathfrak{X}/\eta_{\mathcal{S}}).$$

Remark 5.4. The notation $\nabla J^{\cdots}$ (due to S. Saito) suggests that we think of these Φ_i as infinitesimal invariants of higher normal functions of some kind (an analogous point of view will be considered below in §7). Accordingly, $\Phi_i(\mathscr{Z})$ will be denoted $\delta_i\mathfrak{Z}$.

The question arises as to whether $\delta_i\mathfrak{Z}$ and $[\mathfrak{Z}]_i$ record the same information *for each* i, or equivalently (see [LS]) whether Φ_i may be defined on $\mathcal{L}^i CH_0$. The problem is that it is not clear that ϕ_η maps $\mathcal{L}^i$ on $F^d H^{2d}(\mathfrak{X}_\eta, \mathbb{C})$ to $\mathcal{L}^i$ on $\mathbb{H}^{2d}(\mathfrak{X}_{\mathcal{U}},\, \Omega^{\bullet\geq d}_{\mathfrak{X}_{\mathcal{U}}})$, owing to the lack of a de Rham-theoretic description[13] of the first $\mathcal{L}^i$. However, from the work of Lewis-Saito (*op. cit.*) [(i)] and the author (unpublished) [(ii)] one has the following partial results:

---

[13] Without something of this sort, $\mathcal{L}^i F^d H^{2d}(\mathfrak{X}_{\mathcal{U}}, \mathbb{C})$ (the group to which $[\cdot]$ takes the spread of a cycle in $\mathcal{L}^i CH_0$) is just $\mathcal{L}^i\mathbb{H}^{2d}(\mathfrak{X}_{\mathcal{U}}, \Omega^\bullet_{\mathfrak{X}_{\mathcal{U}}}) \cap \text{im}\left\{\mathbb{H}^{2d}(\bar{\mathfrak{X}}, \Omega^{\bullet\geq d}_{\bar{\mathfrak{X}}}\langle\log\, \mathfrak{X}_D\rangle) \to \mathbb{H}^{2d}(\mathfrak{X}_{\mathcal{U}}, \Omega^\bullet_{\mathfrak{X}_{\mathcal{U}}})\right\}$. Assuming $\bar{\mathfrak{X}} \overset{\pi}{\to} \mathcal{S}$ is log-D-regular, one can prove (using techniques from [P]) that the $E_2^{j,k}$-term of a spectral sequence computing $\mathbb{H}^*\left(\bar{\mathfrak{X}},\, \Omega^{\bullet\geq d}_{\bar{\mathfrak{X}}}\langle\log\, \mathfrak{X}_D\rangle\right)$ $\left[\cong F^d H^{2d}(\mathfrak{X}_{\mathcal{U}}, \mathbb{C}) \text{ for } * = 2d\right]$ is $\mathbb{H}^k\left(\mathcal{S},\, \Omega^*_{\mathcal{S}}\langle\log\, D\rangle \otimes \mathbb{R}^j\pi_*\Omega^{\bullet\geq d-*}_{\bar{\mathfrak{X}}/\mathcal{S}}\langle\log(\mathfrak{X}_D/D)\rangle\right)$ but proving $E_2 = E_\infty$ or that E_∞ yields $Gr^\bullet_{\mathcal{L}}$ seems difficult.

Proposition 5.5. $\Phi\left(\mathcal{L}^i CH_0(X)\right) \subseteq \mathcal{L}^i \mathbb{H}^{2d}\left(\mathfrak{X},\, \Omega^{\bullet \geq d}_{\mathfrak{X}}\right)$*, hence one can define* Φ_i *on* $\mathcal{L}^i$*, in case* either

(i) $\bar{\mathfrak{X}} \xrightarrow{\pi} \mathcal{S}$ *can be chosen to be smooth (no singular fibers in the compactified family),* or

(ii) $H^0(\Omega^i_X) = 0$ *for* $0 < i < d$*.*

Remark 5.6. Lewis has reduced the general case to HC, a conjecture of Zucker-Brylinski, and a conjecture about morphisms of MHS involving intersection cohomology; this is still a vast improvement over HC+BBC^q.

Example 5.7. *for* (i): $X/\bar{\mathbb{Q}}$ so that $\bar{\mathfrak{X}} = X \times \mathcal{S}$, though applicability is not limited to this.

for (ii): X a smooth complete intersection, $K3$ surface, CY 3-fold, etc. (defined / any field).

It is instructive to work out the target of Φ_i for $\bar{\mathfrak{X}} = X \times \mathcal{S}$: the connection ∇ is just differentiation along $\mathcal{S}$ and since $H^i_{d_{\mathcal{S}}}\left\{\Gamma(\eta_{\mathcal{S}}, \Omega^*_{\mathcal{S}})\right\} = H^i(\eta_{\mathcal{S}}, \mathbb{C})$,
$\nabla J^{d,i}(X \times \eta_{\mathcal{S}}/\eta_{\mathcal{S}}) = H^i(\eta_{\mathcal{S}}) \otimes F^{d-i}H^{2d-i}(X) = H^i(\eta_{\mathcal{S}}) \otimes H^{2d-i}(X)$. So in this case $\delta_i \mathfrak{Z}$ and $[\mathfrak{Z}]_i$ are precisely the same thing.

Now we refine the maps Φ_i in a couple of different directions (the second in the next section).

The first "refinement" requires a momentary switching of gears, since up to this point all sheaf (hyper)cohomologies have been computed with respect to the analytic topology. Working in the $\bar{\mathbb{Q}}$-Zariski topology, one has the "dlog" (or El Zein) cycle-class map on $CH^d(\mathfrak{X}_{/\bar{\mathbb{Q}}})$ which we compose with the spread to get $\Phi^{\text{Zar}/\bar{\mathbb{Q}}} : CH_0(X_K) \to \mathbb{H}^{2d}_{\text{Zar}/\bar{\mathbb{Q}}}(\mathfrak{X},\, \Omega^{\bullet \geq d}_{\mathfrak{X}/\bar{\mathbb{Q}}})$. (See [LS], [G1].) An argument practically identical to the one above, yields a Leray filtration on $\mathbb{H}^{2d}_{\text{Zar}/\bar{\mathbb{Q}}}$ and maps from $F^k_S CH_0(X_K)$ to $Gr^k_{\mathcal{L}}\mathbb{H}^{2d}_{\text{Zar}/\bar{\mathbb{Q}}}$, which is

$$H^k_{\nabla}\left\{H^0_{\text{Zar}/\bar{\mathbb{Q}}}\left(\eta_{\mathcal{S}},\, \Omega^*_{\eta_{\mathcal{S}}/\bar{\mathbb{Q}}} \otimes \mathbb{R}^{2d-k}\pi_*\Omega^{\bullet \geq d-*}_{\mathfrak{X}/\eta_{\mathcal{S}}}\right)\right\} \tag{5.2}$$

$$\cong H^0\left(\eta_{\mathcal{S}},\, \mathcal{H}^k_{\nabla,\text{Zar}/\bar{\mathbb{Q}}}\left\{\Omega^*_{\eta_{\mathcal{S}}/\bar{\mathbb{Q}}} \otimes \mathcal{F}^{p-*}\mathcal{H}^{2d-k}_{\text{DR}}(\mathfrak{X}_{(\eta)}/\eta_{\mathcal{S}})\right\}\right)$$

Here the first $\cong$ comes from Γ-acyclicity of $\eta_{\mathcal{S}}$ in the $\bar{\mathbb{Q}}$-Zariski topology. Now $H^0(\eta_{\mathcal{S}}, \Omega^*_{\eta_{\mathcal{S}}/\bar{\mathbb{Q}}}) \cong \Omega^*_{\bar{\mathbb{Q}}(\eta_{\mathcal{S}})/\bar{\mathbb{Q}}} \cong \Omega^*_{K/\bar{\mathbb{Q}}}$ (Kähler differentials where $\bar{\mathbb{Q}}(\eta_{\mathcal{S}}) \underset{\cong}{\overset{\text{ev}}{\to}} K$), while in the algebraic context $\mathfrak{X}_\eta = X_K \times_{\text{ev}^{-1}} \text{Spec}\, \bar{\mathbb{Q}}(\eta_{\mathcal{S}})$,

$\eta_{\mathcal{S}} = \operatorname{Spec} \bar{\mathbb{Q}}(\eta_{\mathcal{S}}) \cong \operatorname{Spec} K$. Hence we have maps

$$\begin{aligned}\Phi_i^{\mathrm{Zar}} : F_{\mathcal{S}}^i CH_0(X_K) \to H_{\nabla}^i \left(\Omega^{\bullet}_{K/\bar{\mathbb{Q}}} \otimes F^{d-\bullet} H_{\mathrm{DR}}^{2d-i}(X_K/K) \right) \\ =: \nabla J^{d,i}(X_K/K)\end{aligned}$$

(here $\nabla =$ "arithmetic" Gauss-Manin connection), which factor the Φ_i by mapping (5.2) $\to \varinjlim_{\mathcal{U}}$(5.1). This is just the restriction from the Zariski to the analytic site; one conjectures it is injective. The Φ_i^{Zar} are called *Mumford-Griffiths invariants*, since (in the limit over $K \subseteq \mathbb{C}$) they pick up the information used in the proof of Mumford/Roitman, while also being a form of generalization of Griffiths's infinitesimal invariant.

5.4. **Green-Voisin invariants and applications.** On the other hand, the Γ-acyclic objects in the analytic topology are balls. To make a ball avoid any $D/\bar{\mathbb{Q}} \subseteq \mathcal{S}$ (hence live "in" $\eta_{\mathcal{S}}$), we take a limit over balls containing a very general point $s_0 \in \mathcal{S}(\mathbb{C})$, and obtain local invariants[14]

$$\Phi_i^{\mathrm{an}} : F_{\mathcal{S}}^i CH_0(X_K) \to \varinjlim_{B \ni s_0} \Gamma \left(B, \mathcal{H}_{\nabla}^i \left\{ \Omega^*_{\mathcal{S}} \otimes \mathcal{F}^{d-\bullet} \mathcal{H}_{X_s}^{2d-i} \right\} \right).$$

The important point is that the cohomology sheaves $\mathcal{H}_{\nabla}^i$ may be filtered and graded by using the fiberwise Hodge filtration (on $\mathcal{H}_{X_s}^{2d-i}$). Specifically, one has a spectral sequence (*defined* to be 0 for $m < 0$) with

$$\mathcal{E}_1^{\ell,m}(i) := \mathcal{H}_{\bar{\nabla}}^{\ell+m} \left\{ \Omega^{\bullet}_{\mathcal{S}} \otimes \mathcal{H}_{X_s}^{d+m-\bullet, d-m-i+\bullet} \right\}$$

and $\mathcal{E}_{\infty}^{\ell,m}(i) =: Gr_{\mathsf{f}}^m \mathcal{H}_{\nabla}^{\ell+m} \left\{ \Omega^{\bullet}_{\mathcal{S}} \otimes \mathcal{F}^{d-\bullet} \mathcal{H}_{X_s}^{2d-i} \right\}$; but it does *not* always (for $\dim(X) > 1$, X not defined $/\bar{\mathbb{Q}}$) degenerate at $\mathcal{E}_1$ or even $\mathcal{E}_2$. (However, noting that d_j sends $\mathcal{E}_j^{\ell,m} \to \mathcal{E}_j^{\ell-j+1,m+j}$, we always have $\mathcal{E}_{\infty}^{\ell,0}(i) \subseteq \mathcal{E}_1^{\ell,0}(i)$.) Likewise, we warn that while $\bar{\nabla}(= d_0)$ may be computed fiberwise (by cupping with a Kodaira-Spencer class), the higher $d_{j \geq 1}$'s cannot.

Since each B is acyclic and the (co)limit is exact, we denote the limit by $\hat{B}$ and obtain iteratively (for $m = 0, 1, \ldots, i$)

$$\left\{ \varinjlim_{B \ni s_0} \Gamma \left(B, \mathcal{H}_{\nabla}^i \left\{ \Omega^*_{\mathcal{S}} \otimes \mathcal{F}^{d-\bullet} \mathcal{H}_{X_s}^{2d-i} \right\} \right) \right\} \supset \ker \left(\Xi_i^{(m-1)} \right) \xrightarrow{\Xi_i^{(m)}}$$

---

[14] one can actually replace the r.h.s. by $\Gamma(\eta_{\mathcal{S}}, \cdots)$ but this will no longer be true for the $\Phi_i^{(m)}$ that follows.

$$\varinjlim_{B\ni s_0} \Gamma\left(B,\, \mathcal{E}_{\infty}^{i-m,m}(i)\right) =: Gr_{\mathsf{f}}^{m}\nabla J^{d,i}\left(\mathfrak{X}_{\hat{B}}/\hat{B}\right),$$

and define $\Phi_i^{(m)}$ to be the composition

$$(\Phi_i^{\mathrm{an}})^{-1}\left(\ker\, \Xi_i^{(m-1)}\right) =: \mathsf{f}^m F_S^i CH_0(X_K) \overset{\Xi_i^{(m)}\circ\Phi_i^{\mathrm{an}}}{\longrightarrow} Gr_{\mathsf{f}}^{m}\nabla J^{d,i}\left(\mathfrak{X}_{\hat{B}}/\hat{B}\right);$$

write $\Phi_i^{(m)}(\mathcal{Z}) = \delta_i^{(m)}\mathfrak{Z}$ for the corresponding *Green-Voisin invariants.* (These are, for example, essentially defined in [G2] from the VHS point of view.)

The reason for further "chopping up" the $\delta_i\mathfrak{Z}$ like this is for greater computability, esp. of $\delta_i^{(0)}\mathfrak{Z}$, by e.g. polynomial algebra. For example, if $X = \{\mathsf{F}(\{z_i\}) = 0\}$ is a very general degree-D hypersuurface in $\mathbb{P}^{d+1}$ and $\mathcal{Z} \in \mathcal{L}^i = F_S^i CH_0(X)$, $\delta_d^{(0)}\mathfrak{Z} \in \Gamma(\hat{B},\, \mathcal{E}_1^{d,0}(d))$ where $\mathcal{E}_1^{d,0}(d) = \frac{\Omega_{\mathcal{S}}^d \otimes \mathcal{H}_{X_s}^{0,d}}{\bar{\nabla}\left(\Omega_{\mathcal{S}}^{d-1}\otimes\mathcal{H}_{X_s}^{1,\,d-1}\right)}$ dualizes to $\ker\left\{\bigwedge^d \theta_{\mathcal{S}}^1 \otimes \mathcal{H}_{X_s}^{d,0} \to \bigwedge^{d-1}\theta_{\mathcal{S}}^1 \otimes \mathcal{H}_{X_s,\mathrm{prim}}^{d-1,\,1}\right\} \cong \ker\left\{\bigwedge^d S^D \otimes R_{\mathsf{F}}^{D-d-2} \xrightarrow{\mu} \bigwedge^{d-1} S^D \otimes R_{\mathsf{F}}^{2D-d-2}\right\}$, where $S^\bullet$ and $R_{\mathsf{F}}^\bullet = S^\bullet \Big/ \left(\left\{\frac{\partial\mathsf{F}}{\partial z_i}\right\}\right)$ are polynomial and Jacobian rings (in the $\{z_i\}$) graded by homogeneous degree, and μ involves polynomial multiplication.

Voisin's result (§3.2) on rational inequivalence of points p, q on a very general surface $X_{(s_0)} \subseteq \mathbb{P}^3$ $(d = 2)$ comes from showing $\delta_2^{(0)}\mathfrak{Z} \neq 0$ for $\mathcal{Z} = p - q$. Roughly speaking, the image of $\mathfrak{Z}$ in $\left.\frac{\Omega_{\mathcal{S}}^2\otimes\mathcal{H}_{X_s}^{0,2}}{\bar{\nabla}\left(\Omega_{\mathcal{S}}^1\otimes\mathcal{H}_{X_s}^{1,1}\right)}\right|_{s_0}$ is identified with the difference of the functionals $\tilde{p}_*,\, \tilde{q}_* : H^0\left(X_{s_0},\, \left.\Omega_{\mathfrak{X}}^{\dim(\mathcal{S})}\right|_{X_{s_0}}\right) \to K_{\mathcal{S}}|_{s_0} \cong \mathbb{C}$ induced by the spreads of p and q. As a vector bundle, $\left.\Omega_{\mathfrak{X}}^{\dim(\mathcal{S})}\right|_{X_{s_0}}$ $(= V \to X_{s_0})$ is very ample for $D \geq 7$ (this is the technical part and involves the sort of polynomial algebra just described); hence the natural map $\mathbb{P}(V^\vee) \to \mathbb{P}\left(H^0(X_{s_0},\, \Omega_{\mathfrak{X}}^{\dim(\mathcal{S})}|_{X_{s_0}})^\vee\right)$ is injective. Since $\tilde{p}$, $\tilde{q}$ plainly yield elements of $V^\vee$ lying over different points of X_{s_0} (hence different in $\mathbb{P}V^\vee$) and mapping resp. to $\tilde{p}_*$, $\tilde{q}_*$, these latter are independent as desired. Fakhruddin elegantly generalized this approach to $d > 2$ and essentially shows $\delta_d^{(0)}\mathfrak{Z} \neq 0$ (for $\mathcal{Z}$ any $\mathbb{Q}$-linear degree-0 combination of the n points in question).

Below in §6.2 we will study "general" cycles in $\ker(Alb) \subseteq CH_0(\mathcal{C} \times \mathcal{C}_{/\mathbb{C}})$ where $\mathcal{C}$ is defined $/\bar{\mathbb{Q}}$ (i.e., "special"), but there is the following result for a "special" cycle on the self-product of a general curve. Write $K_{\mathcal{C}}$ for the canonical divisor and $\iota^\Delta : \mathcal{C} = \Delta_{\mathcal{C}} \hookrightarrow \mathcal{C} \times \mathcal{C}$.

Theorem (Griffiths-Green, [GG2]**).** Let $\mathcal{C}$ be general of genus $g \geq 4$; then

$$\mathcal{Z}_{K_{\mathcal{C}}} := K_{\mathcal{C}} \times K_{\mathcal{C}} - (2g-2)\iota_*^{\Delta} K_{\mathcal{C}} \overset{\text{rat}}{\not\equiv} 0$$

in $CH^2(\mathcal{C} \times \mathcal{C})$.

Once again the proof shows $\delta_2^{(0)}\mathfrak{Z} \neq 0$. Since Griffiths and Green use Schiffer variations to spread out $\mathcal{C} \times \mathcal{C}$, an explicit local computation replaces polynomial algebra.

We conclude with an example to demonstrate the entry of the higher differentials in the spectral sequence $\mathcal{E}_r^{\ell,m}(i)$. As a reference point, first consider curves $\mathcal{C}_1$, $\mathcal{C}_2 \,/\bar{\mathbb{Q}}$ with $o_i \in \mathcal{C}_i(\bar{\mathbb{Q}})$, $p_i \in \mathcal{C}_i(\mathbb{C})$ very general (and such that $p_1 \times p_2 \in \mathcal{C}_1 \times \mathcal{C}_2(\mathbb{C})$ is too). Then $p_i - o_i =: \mathcal{Z}_i$ each have $0 \neq \delta_1^{(0)}\mathfrak{Z}_i \in \Gamma(\hat{B},\, \Omega^1_{\mathcal{C}_i} \otimes H^{0,1}(\mathcal{C}_i))$, while $(p_1 - o_1) \times (p_2 - o_2) =: \mathcal{Z}$ has spread $\mathfrak{Z}_1 \times \mathfrak{Z}_2 =: \mathfrak{Z}$ with essentially $[0 \neq]\, \delta_1^{(0)}\mathfrak{Z}_1 \otimes \delta_1^{(0)}\mathfrak{Z}_2 = \delta_2^{(0)}\mathfrak{Z} \in \Gamma(\hat{B},\, \Omega^2_{\mathcal{C}_1 \times \mathcal{C}_2} \otimes H^{0,2}(\mathcal{C}_1 \times \mathcal{C}_2))$. (Note that the $\bar{\nabla}$'s are 0 as the $\mathcal{C}_i$ do not spread. The d_1's actually are not zero — $\mathcal{E}_2^{1,1}(2)$ and $\mathcal{E}_2^{0,2}(2)$ are 0 — but this does not affect $\mathcal{E}_2^{2,0}(2)$.)

Now we replace the $\mathcal{C}_i$ by two general elliptic curves. Write $E_\lambda = \overline{\{y^2 = x(x-1)(x-\lambda)\}} \overset{\mathsf{X}_\lambda}{\to} \mathbb{P}^1_{[x]}$ for the fibers of the Legendre family $\mathcal{E} \overset{\pi}{\to} \mathcal{U} \subseteq \mathbb{A}^1_{[\lambda]}$ (omit singular fibers). Fix $x_0 \in \mathbb{P}^1(\bar{\mathbb{Q}}) \setminus \{0, 1, \infty\}$ and choose continuously in λ a lift $q_\lambda \in \mathsf{X}_\lambda^{-1}(x_0)$; this will force us to pass to a double cover of Legendre, $\widetilde{\mathcal{E}} \to \widetilde{\mathcal{U}}$. Write $\mathcal{Q} = \bigcup_{\lambda \in \widetilde{\mathcal{U}}} \{q_\lambda\}$ and $o_\lambda = (0,0) \in E_\lambda$, $\mathcal{O} = \bigcup_{\lambda \in \widetilde{\mathcal{U}}} \{o_\lambda\}$. Take λ_1, λ_2 to be algebraically independent transcendental numbers.

Then $\mathcal{Z}_{\lambda_i} = q_{\lambda_i} - o_{\lambda_i} \in CH_0(E_{\lambda_i/\mathbb{C}})$ $(i = 1, 2)$ have the same spread $\mathfrak{Z}_1 = \mathfrak{Z}_2 = \mathcal{Q} - \mathcal{O}$ on $\widetilde{\mathcal{E}}$, which yields a normal function $\nu \in \Gamma(\widetilde{\mathcal{U}},\, \mathcal{J}^1_{\widetilde{\mathcal{E}}/\widetilde{\mathcal{U}}})$ defined by $\nu(\lambda) = AJ_{E_\lambda}(q_\lambda - o_\lambda)$. As this is 2-torsion for $\lambda = x_0$ but not elsewhere, it is nontrivial; by a monodromy argument with Picard-Lefschetz, it cannot be flat. Hence its infinitesimal invariant, which is $\delta_1\mathfrak{Z}_i \in \Gamma\left(\widetilde{\mathcal{U}},\, \frac{\Omega^1_{\widetilde{\mathcal{U}}} \otimes \mathcal{H}^1_{\widetilde{\mathcal{E}}/\widetilde{\mathcal{U}}}}{\nabla\left(\mathcal{O}_{\widetilde{\mathcal{U}}} \otimes \mathcal{F}^1\mathcal{H}^1_{\widetilde{\mathcal{E}}/\widetilde{\mathcal{U}}}\right)}\right)$, must be nonzero. However $\delta_1^{(0)}\mathfrak{Z}_i = 0$ because $\bar{\nabla} :\, \mathcal{H}^{1,0}_{\widetilde{\mathcal{E}}/\widetilde{\mathcal{U}}} \to \Omega^1_{\widetilde{\mathcal{U}}} \otimes \mathcal{H}^{0,1}_{\widetilde{\mathcal{E}}/\widetilde{\mathcal{U}}}$ is surjective, hence $0 \neq \delta_1^{(1)}\mathfrak{Z}_i \in \Gamma\left(B,\, \Omega^1_{\widetilde{\mathcal{U}}} \otimes \mathcal{H}^{1,0}_{\widetilde{\mathcal{E}}/\widetilde{\mathcal{U}}} = \mathcal{E}_\infty^{0,1}(1)\right)$.

Now the spread of $\mathcal{Z} = \mathcal{Z}_{\lambda_1} \times \mathcal{Z}_{\lambda_2}$ is $\mathfrak{Z} = \mathfrak{Z}_1 \times \mathfrak{Z}_2$ (self-product) on $\widetilde{\mathcal{U}} \times \widetilde{\mathcal{U}} \times \widetilde{\mathcal{E}} \times \widetilde{\mathcal{E}}$, and one expects $0 = \delta_1^{(0)}\mathfrak{Z} = \delta_1^{(1)}\mathfrak{Z}$, $0 \neq \delta_1^{(1)}\mathfrak{Z}_1 \otimes \delta_1^{(1)}\mathfrak{Z}_2 = \delta_2^{(2)}\mathfrak{Z} \in \Gamma(B, \mathcal{E}_\infty^{0,2}(2))$. However, $\mathcal{E}_\infty^{0,2}(2)$ is not $\Omega^2_{\widetilde{\mathcal{U}} \times \widetilde{\mathcal{U}}} \otimes \mathcal{H}^{2,0}_{\widetilde{\mathcal{E}} \times \widetilde{\mathcal{E}}/\widetilde{\mathcal{U}} \times \widetilde{\mathcal{U}}}$ but a quotient; moreover, $\mathcal{Z} \overset{\text{rat}}{\equiv} 0$, hence $\delta_2^{(2)}\mathfrak{Z} = 0$. The rational equivalence is direct: let B (the bielliptic curve) denote the normalization of $E_{\lambda_1} \times_{\mathbb{P}^1_{[x]}} E_{\lambda_2} \subseteq E_{\lambda_1} \times E_{\lambda_2}$ and $\iota : B \to E_{\lambda_1} \times E_{\lambda_2}$ be the composition. Then clearly $\iota(B) \ni (q_1, q_2)$ and (o_1, o_2); let $\iota(o) = (o_1, o_2)$ and $\iota(q) = (q_1, q_2)$. Define $\Pi : B \times B \to E_{\lambda_1} \times E_{\lambda_2}$ by $\Pi(b, b') = (\pi_1(\iota(b)), \pi_2(\iota(b')))$, and note that $\Pi_* \{(q, q) - (0, q) - (q, 0) + (0, 0)\} = \mathcal{Z}$. But B is genus 2, hence hyperelliptic, and so by Beauville-Voisin (see §6.2 below) the cycle in brackets is $\overset{\text{rat}}{\equiv} 0$.

Write $\mathcal{X} = \widetilde{\mathcal{E}} \times \widetilde{\mathcal{E}}$, $S = \widetilde{\mathcal{U}} \times \widetilde{\mathcal{U}}$. We conclude that the higher differentials — maps from $\ker \left\{ \Omega^1_S \otimes \mathcal{H}^{2,0}_{\mathcal{X}/S} \overset{\bar{\nabla}}{\to} \Omega^2_S \otimes \mathcal{H}^{1,1}_{\mathcal{X}/S} \right\}$ and

$$\ker \left\{ \ker \left\{ \Omega^1_S \otimes \mathcal{H}^{1,1}_{\mathcal{X}/S} \to \Omega^2_S \otimes \mathcal{H}^{0,2}_{\mathcal{X}/S} \right\} \to \Omega^2_S \otimes \mathcal{H}^{1,1}_{\mathcal{X}/S} \right\}$$

to $\Omega^2_S \otimes \mathcal{H}^{2,0}_{\mathcal{X}/S}$ (or a quotient of this in the latter case) — must hit the class of $\delta_2^{(2)} \mathfrak{Z}$ locally, though this looks like a hard computation to do explicitly.

5.5. **Other filtrations.** We finish with a brief description of 3 additional filtrations obtained by taking the inverse image of a filtration under a cycle map of some kind.

(i) Raskind ([Ra]). This served as the original motivation for [L1]. Assume X/K, K finitely generated $/\mathbb{Q}$ (not $\bar{\mathbb{Q}}$). By work of Jannsen [J3], the Hochschild-Serre spectral sequence for continuous étale cohomology

$$E_2^{p,q}(d) = H^p_{\text{cont}} \left(Gal(\bar{K}/K),\, H^q_{et}(X_{\bar{K}}, \mathbb{Q}_\ell(d)) \right) \implies H^{p+q}_{\text{cont}}(X, \mathbb{Q}_\ell(d))$$

degenerates at E_2. Pulling back the resulting filtration by his cycle map $CH_0(X) \to H^{2d}_{\text{cont}}(X, \mathbb{Q}_\ell(d))$ gives $F^\bullet_R$ and one has "higher ℓ-adic AJ" maps from $Gr^i_{F_R}$ to $E_2^{i,\,2d-i}(d)$. Clearly $F^\bullet_R \supseteq F^\bullet_S$; this is equality if HLC holds and Jannsen's cycle map is injective.

For the remaining filtrations, assume for simplicity that X is defined over a number field k, with cycles $/K$ f.g. $/k$.

(ii) M. Saito ([mS]). By means of various categories of realizations $\mathcal{M}$, one can put a variety of [hopefully not!] different filtrations $F^\bullet_{\mathcal{M}}$ on $CH^*(X_K)$. The idea is once again to pull back a suitable Leray filtration along the composition of spreading out $(/k)$ and taking some kind of Deligne (or absolute Hodge) class. For example, $F^\bullet_{\text{MHS}}$ would just be $\mathcal{L}^\bullet$. One always has $F^\bullet_S \subseteq F^\bullet_{\mathcal{M}}$.

We are interested (for §7.3 below) in $\mathcal{M} = \mathcal{M}_\ell$, whose objects consist roughly of MHS (w./ polarizable $Gr^W_\bullet$'s) and $\mathbb{Q}_\ell$-vector spaces with continuous $Gal(\bar{k}/k)$-action, together with comparison isomorphisms (under which polarizations and Galois action must be compatible). Cohomology $H^*(X/k, \mathbb{Q}) \in \mathcal{M}_\ell$ is then defined using $H^*(X_{\mathbb{C}})$ for the MHS and $H^*_{et}(X_{\bar{k}}, \mathbb{Q}_\ell)$ for the $\mathbb{Q}_\ell$-v.s.

Saito defines Deligne cohomology[15] for $\mathcal{M}$, e.g. for our purposes[16]

$$H^{2d}_{\mathcal{D},\mathcal{M}}(X \times \mathcal{U}/k, \mathbb{Q}(d)) := Ext^{2d}_{D^b\mathcal{M}}(\mathbb{Q}(-d),\, \mathcal{K}^\bullet_{\mathcal{H}}(X \times \mathcal{U}/k))$$

where $D^b\mathcal{M}$ is the bounded derived category and $\mathcal{K}^\bullet_{\mathcal{H}}$ is a complex of $\mathcal{M}$-objects with $H^i(\mathcal{K}^\bullet_{\mathcal{H}}) \cong H^i(X \times \mathcal{U}/k, \mathbb{Q})$. This is computed by a spectral sequence $E_2^{p,q} = Ext^p_{\mathcal{M}}(\mathbb{Q}(-d),\, H^q(X \times \mathcal{U}/k))$, in the sense that there is an increasing filtration $\mathcal{E}^\bullet$ on $H^{2d}_{\mathcal{D},\mathcal{M}}$ with $Gr^j_{\mathcal{E}} \cong E_\infty^{j,2d-j}$. (For $j \geq 2$ this is known to be zero for $\mathcal{M} =$ MHS but not $\mathcal{M}_\ell$.)

Moreover, one has a Leray filtration $\mathcal{L}^\bullet_{\mathcal{M}}$ (with respect to $X \times \mathcal{U} \xrightarrow{\pi} \mathcal{U}$) on whose graded pieces one can put a similar filtration $\mathcal{E}^\bullet$ so that [writing $H^*(\cdot)$ for $H^*(\cdot/k,\, \mathbb{Q})$]

$$Gr^0_{\mathcal{E}} Gr^i_{\mathcal{L}_{\mathcal{M}}} H^{2d}_{\mathcal{D},\mathcal{M}}\,(X \times \mathcal{U}/k,\, \mathbb{Q}(d)) \hookrightarrow Hom_{\mathcal{M}}\left(\mathbb{Q}(-d),\, \underline{H^i}(\mathcal{U}) \otimes H^{2d-i}(X)\right)$$

$$Gr^1_{\mathcal{E}} Gr^i_{\mathcal{L}_{\mathcal{M}}} H^{2d}_{\mathcal{D},\mathcal{M}}\,(X \times \mathcal{U}/k,\, \mathbb{Q}(d)) \hookrightarrow Ext^1_{\mathcal{M}}\left(\mathbb{Q}(-d),\, [W_i]H^{i-1}(\mathcal{U}) \otimes H^{2d-i}(X)\right)$$

$$\to Ext^1_{\mathcal{M}}\left(\mathbb{Q}(-d),\, H^{i-1}(\mathcal{U}) \otimes \frac{H^{2d-i}(X)}{F_h^{d-i+1}}\right)$$

and for $j \geq 2$, $Gr^j_{\mathcal{E}} Gr^i_{\mathcal{L}_{\mathcal{M}}}$ is a subquotient of $Ext^j_{\mathcal{M}}\left(\mathbb{Q}(-d),\, H^{i-j}(\mathcal{U}) \otimes H^{2d-i}(X)\right)$ (again, zero for $\mathcal{M} =$ MHS but not $\mathcal{M}_\ell$).

Finally, we take the limit over $\mathcal{U} \subseteq \mathcal{S}$ and pull $\mathcal{L}^\bullet_{\mathcal{M}}$ back by the cycle map $CH_0(X_K) \xrightarrow{\cong} CH^d(X \times \eta_{\mathcal{S}/k}) \to \varinjlim_{\mathcal{U}} H^{2d}_{\mathcal{D},\mathcal{M}}(X \times \mathcal{U}/k, \mathbb{Q}(d))$ to get $F^\bullet_{\mathcal{M}}$. As there are "forgetful" maps[17] from the graded pieces of $H^{2d}_{\mathcal{D},\mathcal{M}}$ to those of $H^{2d}_{\mathcal{D}(,\mathrm{MHS})}$, $F^\bullet_{\mathcal{M}} \subseteq \mathcal{L}^\bullet$ on CH_0; if HC+BBC^q hold then they are equal.

(iii) Griffiths-Green ([GG1]). This filtration is closest in spirit to $\mathcal{L}^\bullet$. (Indeed, our presentation of the latter, including some notation, is influenced by [GG1].) In the construction of §4.2, instead of filtering $\underline{H^{2d}_{\mathcal{D}}}(X \times \eta_{\mathcal{S}}, \mathbb{Q}(d))$, filter $H^{2d}_{\mathcal{D}}(X \times \mathcal{S}, \mathbb{Q}(d))$ by $\mathcal{L}^\bullet_{\mathcal{S}}$ (Leray for $X \times \mathcal{S} \to \mathcal{S}$), and pull this back to $CH^d(X \times \mathcal{S}_{/\bar{\mathbb{Q}}})$. We say $\langle \mathcal{Z} \rangle \in CH^d(X_K)$

---

[15] for $\mathcal{M} = MHS$ this is really absolute Hodge cohomology.

[16] as usual $\mathcal{U}[\subseteq \mathcal{S}]$ affine Zar. op.

[17] One caveat here is that $F_h^j(H^*(X))$ in the category of $\mathcal{M}_\ell$-objects (F_h^j always means the maximal subobject contained in F^j) may not map (under the forgetful functor $\mathcal{M}_\ell \to$ MHS) to $F_h^j(H^*(X))$ in the category of MHS, but to something between this and $N^j(H^*(X))$ (which is the same for both categories). Of course, this is no problem if you assume GHC, but we don't. However, this issue can be safely ignored in §7.3.

belings to F^i_{GG} iff it has a choice of complete spread $\bar{\mathfrak{Z}}$ with $\langle\bar{\mathfrak{Z}}\rangle \in \mathcal{L}^i_{\mathcal{S}}CH^d(X \times \mathcal{S}_{/\bar{\mathbb{Q}}})$ [$\Leftrightarrow$ $c_{\mathcal{D}}(\bar{\mathfrak{Z}}) \in \mathcal{L}^i_{\mathcal{S}}H^{2d}_{\mathcal{D}}$].

A priori this is a stronger condition than $\mathcal{L}^i$ on the cycle $\mathcal{Z}$. Under

$$H^{2d}_{\mathcal{D}}(X \times \mathcal{S}, \mathbb{Q}(d)) \to \underline{H^{2d}_{\mathcal{D}}}(X \times \eta_{\mathcal{S}}, \mathbb{Q}(d)),$$

one easily shows $\mathcal{L}^\bullet$ (on $\underline{H^{2d}_{\mathcal{D}}}$) is the *image* of $\mathcal{L}^\bullet_{\mathcal{S}}$ (on $H^{2d}_{\mathcal{D}}$), but $\mathcal{L}^\bullet_{\mathcal{S}}$ is *not* the preimage of $\mathcal{L}^\bullet$. Hence (for cycles) $\bar{\mathfrak{Z}} \in \mathcal{L}^i_{\mathcal{S}} \implies \mathfrak{Z} \in \mathcal{L}^i$ but not the other way around. However, if we assume the HC (only!), $F^\bullet_{GG} = \mathcal{L}^\bullet$ (on cycles $Z^d(X_K)$) by the argument in [mS, sec. 1.6].

6. Zero-cycles on products of varieties

We want to investigate the behavior of our invariants with respect to *products* of 0-cycles, considered on the *product* of the varieties on which they lie individually. The resulting cycles are called *exterior products*; this is clearly a well-defined operation on the level of Chow groups.

In the three subsections that follow, we consider the following cases:

(1) cycles $\mathcal{Z} = \mathcal{Z}_1 \times \mathcal{Z}_2$ such that $\mathfrak{Z} = \mathfrak{Z}_1 \times \mathfrak{Z}_2$ — the spread respects the product structure. In this case we can essentially take "products" of higher invariants of $\mathcal{Z}_1$ and $\mathcal{Z}_2$ to get higher invariants of $\mathcal{Z}$, and we have some quite general nonvanishing results.

(2) self-products $\mathcal{Z} \times \mathcal{Z}$ (and higher powers) of 0-cycles, especially for $\mathcal{Z}$ a divisor on a curve.

(3) 0-cycles on a product of curves whose spreads wildly disrespect the product structure.

The first section contains no proofs, as they are like more complicated versions of the proof of Asakura's result in §6.2 and appear in [K1]; stating the results is complicated enough anyway. A fourth subsection explains a situation similar to (3) for certain relative cycles.

6.1. Products of algebraically independent cycles.

Let Y_1, Y_2 be smooth projective $/\bar{\mathbb{Q}}$ of respective dimensions d_1, d_2 and let $K \subseteq \mathbb{C}$ be finitely generated $/\bar{\mathbb{Q}}$ of trdeg. j. Take

(a) $\mathcal{V} \in \mathcal{L}^j CH_0(Y_{1/K})$ such that a complete spread $\overline{\mathfrak{V}} \in Z^{d_1}(Y_1 \times \mathcal{S})$ induces a nonzero map $\Omega^j(Y_1) \to \Omega^j(\mathcal{S})$,

(b) $\mathcal{W} \in CH_0^{\text{hom}}(Y_{2/\bar{\mathbb{Q}}})$ such that $0 \neq AJ_{Y_2}(\mathcal{W}) \in J^{d_2}(Y_2)$.

Then from [K1] one has the following.

Theorem 6.1. *$\mathcal{Z} := \mathcal{V} \times \mathcal{W} \overset{rat}{\not\equiv} 0$ in $CH_0(Y_1 \times Y_{2/K})$; in particular, $[AJ(\mathfrak{Z})]_j^{(tr)} \neq 0$.*

Example 6.2. This immediately justifies Thm. 3.5(B), by taking

$$\begin{aligned} \mathcal{V} &= (p_1 - o_1) \times \cdots \times (p_{\ell-1} - o_{\ell-1}) \qquad && [d_1 = j = \ell - 1], \\ \text{and} \quad \mathcal{W} &= \mathcal{W}_\ell \times o_{\ell+1} \times \cdots \times o_d && [d_2 = d - \ell + 1]. \end{aligned}$$

Remark 6.3. (i) For 0-cycles, the Theorem generalizes a result of Rosenschon and M. Saito ([RS]) which gives the special case $d_1 = 1 (= j)$. (This includes the $(\langle P \rangle - \langle O \rangle) \times (\langle Q \rangle - \langle 0 \rangle)$ example of Remark 3.6(iii).) Note that in this case, one can replace (a) by the simpler statement that $\langle \mathcal{V} \rangle$ belongs to $CH_0^{\text{hom}}(Y_{1/\mathbb{C}})$ and is *not* defined $/\bar{\mathbb{Q}}$. (That is, no cycle rationally equivalent to $\mathcal{V}$ is defined $/\bar{\mathbb{Q}}$.)

(ii) The $(\langle P \rangle - \langle O \rangle) \times (\langle Q \rangle - \langle O \rangle)$ cycle can be used ([K2]) to construct explicit 0-cycles of trdeg. 1 in $\ker(Alb)$ on any special Kummer (K3) surface. In §8.2 we will show how to use the case $d_1 = 2$, $d_2 = 1$ to construct 0-cycles in $\mathcal{L}^3 CH_0$ on CY 3-folds arising from the Borcea-Voisin construction.

To generalize this even further, we define (for all $1 \leq j_2 \leq d_2$)

$$SF_h^{(1,\, d_2 - j_2 + 1)} \left\{ H^{j_2 - 1}(\mathcal{S}_2) \otimes H^{2d_2 - j_2}(Y_2) \right\} :=$$

$$\text{largest subHS of } H^{j_2-1}(\mathcal{S}_2) \otimes H^{2d_2 - j_2}(Y_2) \text{ lying in}$$

$$\ker \left(H^{j_2 - 1}(\mathcal{S}_2, \mathbb{C}) \otimes H^{2d_2 - j_2}(Y_2, \mathbb{C}) \to H^{j_2 - 1, 0}(\mathcal{S}_2, \mathbb{C}) \otimes H^{d_2 - j_2,\, d_2}(Y_2, \mathcal{C}) \right).$$

This contains

$$N^1 H^{j_2 - 1}(\mathcal{S}_2) \otimes H^{2d_2 - j_2}(Y_2) \; + \; H^{j_2 - 1}(\mathcal{S}_2) \otimes F_h^{d_2 - j_2 + 1} H^{2d_2 - j_2}(Y_2),$$

so one has a projection

$$J^{d_2} \left(\underline{H^{j_2 - 1}}(\eta_{\mathcal{S}_2}) \otimes \frac{H^{2d_2 - j_2}(Y_2)}{F_h^{d_2 - j_2 + 1}} \right) \overset{\Xi}{\twoheadrightarrow} J^{d_2} \left(\frac{H^{j_2 - 1}(\mathcal{S}_2) \otimes H^{2d_2 - j_2}(Y_2)}{SF_h^{(1,\, d_2 - j_2 + 1)}} \right)$$

and we define $\overline{[AJ(\mathfrak{Z})]_{j_2 - 1}^{tr}} := \Xi \left([AJ(\mathfrak{Z})]_{j_2 - 1}^{tr} \right)$. One can show that the cycles constructed by Theorem 6.1 have nontrivial $\overline{[AJ(\mathfrak{Z})]_j^{tr}}$.

Now take (subfields of $\mathbb{C}$) K_1, K_2 f.g. $/\bar{\mathbb{Q}}$ and

(a) $\mathcal{V} \in \mathcal{L}^{j_1} CH_0(Y_{1/K_1})$ with $cl_{Y_1}^{j_1}(\mathcal{V}) \neq 0$,

(b) $\mathcal{W} \in \mathcal{L}^{j_2} CH_0(Y_{2/K_2})$ with either
  (i) $cl_{Y_2}^{j_2}(\mathcal{W}) \neq 0$ OR
  (ii) $\overline{AJ_{Y_2}^{j_2 - 1}(\mathcal{W})^{tr}} \neq 0$ and $cl_{Y_2}^{j_2}(\mathcal{W}) = \cdots = cl_{Y_2}^{d_2}(\mathcal{W}) = 0$.

Set $K = \bar{\mathbb{Q}}(K_1,\, K_2)$, the field generated by K_1 and K_2 over $\bar{\mathbb{Q}}$ — this is the natural field of definition of the product $\mathcal{V}\times\mathcal{W}$. For the "algebraic independence" requirement (to force the spread of $\mathcal{V}\times\mathcal{W}$ to be $\mathfrak{V}\times\mathfrak{W}$), we demand $\mathrm{trdeg}_{/\bar{\mathbb{Q}}}(K) = \mathrm{trdeg}_{/\bar{\mathbb{Q}}}(K_1) + \mathrm{trdeg}_{/\bar{\mathbb{Q}}}(K_2)$. Then from [K1] we have:

Theorem 6.4. *Assume the GHC; then* $0 \overset{rat}{\not\equiv} \mathcal{V}\times\mathcal{W} \in \mathcal{L}^{j_1+j_2}CH_0(Y_1\times Y_{2/K})$. *In particular, when (i) holds,* $cl^{j_1+j_2}_{Y_1\times Y_2}(\mathcal{V}\times\mathcal{W}) \neq 0$*; while if (ii) holds,* $AJ^{j_1+j_2-1}_{Y_1\times Y_2}(\mathcal{V}\times\mathcal{W})^{(tr)} \neq 0$.

This, then, is the sense in which invariants "multiply" nontrivially. The theorem is obviously false without the independence requirement, e.g. for self-products ($\mathcal{V}=\mathcal{W}$, $K=K_1=K_2$) — see the Beauville-Voisin theorem below.

6.2. **Self-products of cycles.** In this subsection we prove two results (approximate converses of one another) on the "symmetric square" of a divisor $p-o$ on a curve, concluding with a brief treatment of its higher "powers". As above, denote by $K_\mathcal{C}$ the canonical divisor of $\mathcal{C}$.

Theorem (Beauville-Voisin, [BV]**).** Let o, p be two points on a smooth hyperelliptic curve $\mathcal{C}/\mathbb{C}$ of genus $g\geq 1$, with $(2g-2)o \overset{\text{rat}}{\equiv} K_\mathcal{C}$ in $CH_0(\mathcal{C})$. Then $\mathcal{Z} := (p-o)\times(p-o) \overset{\text{rat}}{\equiv} 0$ in $CH_0(\mathcal{C}\times\mathcal{C})$.

Proof. The assumption implies $o\in\mathcal{C}(\mathbb{C})$ is Weierstrass, hence a ramification point of the (intrinsic) canonical map $\varphi : \mathcal{C} \overset{\deg 2}{\to} \mathbb{P}^1[\subseteq \mathbb{P}^{g-1}]$. Writing $\sigma : \mathcal{C}\to\mathcal{C}$ for the corresponding involution, we set $f(z) := \frac{z-\varphi(p)}{z-\varphi(o)}$ and $F := f\circ\varphi$, and note $(F) = p+\sigma(p)-2\cdot o$. Hence (and this is worth drawing for oneself)

$$(\iota_{\Delta_\mathcal{C}})_*(F) + \left(\iota_{\{o\}\times\mathcal{C}}\right)_*\left(\frac{1}{F^2}\right) + \left(\iota_{\{p\}\times\mathcal{C}}\right)_*(F) + \left(\iota_{\mathcal{C}\times\{\sigma(p)\}}\right)_*\left(\frac{1}{F}\right) =$$

$$2\cdot\{(p,p)-(o,p)-(p,o)+(o,o)\} \;=\; 2\cdot\mathcal{Z}$$

shows that (always modulo torsion) $\mathcal{Z} \overset{\text{rat}}{\equiv} 0$. □

In addition, this result holds *vacuously* if $\mathcal{C}$ is a general curve of genus $g\geq 3$; that is, there cannot exist $\omega\in\Omega^1(\mathcal{C})$ with $(\omega) = (2g-2)\cdot o$. Indeed, for such a curve all Weierstrass points are normal, hence cannot have $2g-1$ as a gap value. (This is false for arbitrary nonhyperelliptic curves.)

Before giving the "converse", a beautiful theorem due to Asakura, we digress briefly on AJ maps. Consider Y_1, Y_2 smooth projective of dimensions d_1, d_2, $d_1+d_2 =: d$, and let $\Pi_i : Y_1\times Y_2 \to Y_i$ be the natural projections ($i = 1, 2$). Take $\mathcal{V} \in CH^{d-p}(Y_1)$ and $\bar{\mathfrak{Z}} \in CH^p_{\text{hom}}(Y_1 \times Y_2)$. The restriction of functionals

$$\left\{F^{d-p+1}H^{2d-2p+1}(Y_1 \times Y_2, \mathbb{C})\right\}^\vee \twoheadrightarrow \left\{F^{d-p+1}\left(\mathbb{Q}[\mathcal{V}] \otimes H^1(Y_2, \mathbb{C})\right)\right\}^\vee$$

$$\xrightarrow{\cong} \left\{F^1H^1(Y_2, \mathbb{C})\right\}^\vee$$

is well-defined modulo periods, hence induces

$$\text{pr}_{\mathcal{V}} : \ J^p(Y_1 \times Y_2) \to J^{d_2}(Y_2).$$

To describe this map on $\bar{\mathfrak{Z}}$, let $\partial^{-1}\bar{\mathfrak{Z}}$ be a bounding chain and restrict the functional $\int_{\partial^{-1}\bar{\mathfrak{Z}}}$ to $\mathbb{Q}[\mathcal{V}] \otimes F^1H^1(Y_2, \mathbb{C})$. If $\xi_{\mathcal{V}} \in \Gamma(\Omega^{d-p,\, d-p}_{Y_1^\infty})$ represents $[\mathcal{V}]$ and $\omega \in \Omega^1(Y_2)$, then

$$\int_{\partial^{-1}\bar{\mathfrak{Z}}} \Pi_1^*\xi_{\mathcal{V}} \wedge \Pi_2^*\omega \; = \int_{(\partial^{-1}\bar{\mathfrak{Z}}) \cap (\mathcal{V} \times Y_2)} \Pi_2^*\omega \; = \int_{\Pi_2^*\left\{(\partial^{-1}\bar{\mathfrak{Z}}) \cap (\mathcal{V} \times Y_2)\right\}} \omega.$$

Noting that $\partial\left(\Pi_2\left\{(\partial^{-1}\bar{\mathfrak{Z}}) \cap (\mathcal{V} \times Y_2)\right\}\right) = \Pi_2\left\{\bar{\mathfrak{Z}} \cap (\mathcal{V} \times Y_2)\right\} =: \bar{\mathfrak{Z}}_*\mathcal{V}$, we may write this $\int_{\partial^{-1}(\bar{\mathfrak{Z}}_*\mathcal{V})} \omega$; it follows that

$$\text{pr}_{\mathcal{V}}\left(AJ_{Y_1 \times Y_2}(\bar{\mathfrak{Z}})\right) = Alb_{Y_2}\left(\bar{\mathfrak{Z}}_*\mathcal{V}\right).$$

Using this we now prove:

Theorem (Asakura, [As]**).** Let $\mathcal{C}$ be a smooth projective curve $/\bar{\mathbb{Q}}$, with $p \in \mathcal{C}(\mathbb{C})$ very general and $o \in \mathcal{C}(\bar{\mathbb{Q}})$. Then $(2g-2)o \overset{\text{rat}}{\not\equiv} K_{\mathcal{C}}$ in $CH_0(\mathcal{C}_{/\bar{\mathbb{Q}}}) \implies \mathcal{Z} := (p-o) \times (p-o) \overset{\text{rat}}{\not\equiv} 0$ in $CH_0(\mathcal{C} \times \mathcal{C}_{/\mathbb{C}})$; more precisely, $[AJ(\mathfrak{Z})]_1^{tr} \neq 0$.

Proof. We first construct a choice of complete $\bar{\mathbb{Q}}$-spread $\bar{\mathfrak{Z}} \in Z^2(\mathcal{C} \times \mathcal{C} \times \mathcal{C}_{/\bar{\mathbb{Q}}})$ for $\mathcal{Z}$. Write $\mathcal{P}_i$, $\mathcal{P}_{ij}$ and ι_i, ι_{ij} for projections and inclusions, e.g. $\iota_{13} : \mathcal{C} \times \mathcal{C} \hookrightarrow \mathcal{C} \times \mathcal{C} \times \mathcal{C}$ sends $(q_1, q_2) \mapsto (q_1, o, q_2)$ and $\mathcal{P}_{23} : \mathcal{C} \times \mathcal{C} \times \mathcal{C} \twoheadrightarrow \mathcal{C} \times \mathcal{C}$ sends $(q_1, q_2, q_3) \mapsto (q_2, q_3)$. Set $\Delta_{\mathcal{C}}^{[3]} := \{(Q, Q, Q) \,|\, Q \in \mathcal{C}\}$ and

$$\bar{\mathfrak{Z}} := \Delta_{\mathcal{C}}^{[3]} - \iota_*^{12}\Delta_{\mathcal{C}} - \iota_*^{13}\Delta_{\mathcal{C}} - \iota_*^{23}\Delta_{\mathcal{C}} + \iota_*^1\mathcal{C} + \iota_*^2\mathcal{C} + \iota_*^3\mathcal{C}\,;$$

and note that $\bar{\mathfrak{Z}} \cap (\{p\} \times \mathcal{C} \times \mathcal{C}) = (p, p) - (o, p) - (p, o) + (o, o) = \mathcal{Z}$. Moreover $(\mathcal{P}_{23})_* : H^4(\mathcal{C} \times \mathcal{C} \times \mathcal{C})^{\mathfrak{S}_3} \to H^2(\mathcal{C} \times \mathcal{C})^{\mathfrak{S}_2}$ is an isomorphism[18] while $(\mathcal{P}_{23})_*\bar{\mathfrak{Z}} = 0$, so $\bar{\mathfrak{Z}} \overset{\text{hom}}{\equiv} 0$.

Consider the commuting diagram of Jacobians

---

[18] $\mathfrak{S}_n$ denotes the n^{th}symmetric group

$$\begin{tikzcd}
J^2\left(H^3(\mathcal{C}\times\mathcal{C}\times\mathcal{C})\right) \arrow[r, two heads] \arrow[dr, "\mathrm{pr}_{\otimes 3}"] & J^2\left(H^1(\mathcal{C})\otimes H^2(\mathcal{C}\times\mathcal{C})\right) \arrow[r, two heads] \arrow[d, two heads] & J^2\left(H^1(\mathcal{C})\otimes \frac{H^2(\mathcal{C}\times\mathcal{C})}{F_h^1 H^2(\mathcal{C}\times\mathcal{C})}\right) \arrow[d, two heads] \\
J^2\left(\{H^1(\mathcal{C})^{\otimes 3}\}^{\mathfrak{S}_3}\right) \arrow[r, hook, "\mathcal{I}"'] & J^2\left(H^1(\mathcal{C})^{\otimes 3}\right) \arrow[r, two heads, "\mathrm{pr}_F"'] & J^2\left(H^1(\mathcal{C})\otimes \frac{H^1(\mathcal{C})\otimes H^1(\mathcal{C})}{F_h^1\{H^1(\mathcal{C})^{\otimes 2}\}}\right) \arrow[d, equal] \\
 & & \frac{J^2\left(H^1(\mathcal{C})^{\otimes 3}\right)}{J^2\left(H^1(\mathcal{C})\otimes F_h^1\{H^1(\mathcal{C})^{\otimes 2}\}\right)}
\end{tikzcd}$$

where the composition in the top row sends $AJ_{\mathcal{C}\times\mathcal{C}\times\mathcal{C}}(\bar{\mathfrak{Z}})$ to $[AJ(\mathfrak{Z})]_1^{tr}$, which we must show nonzero. By the diagram, this is accomplished if we have $(\mathrm{pr}_F\circ\mathrm{pr}_{\otimes 3})(AJ(\bar{\mathfrak{Z}}))\neq 0$. The remainder of the proof therefore proceeds in 2 steps: we show first (i) that $0\neq\mathrm{pr}_{\otimes 3}(AJ(\bar{\mathfrak{Z}}))\in\mathrm{im}(\mathcal{I})$, then (ii) that $\mathrm{pr}_F\circ\mathcal{I}$ is injective.

We apply the discussion preceding this theorem with $Y_1=\mathcal{C}\times\mathcal{C}$, $Y_2=\mathcal{C}$, and $\mathcal{V}=\Delta_{\mathcal{C}}-\{0\}\times\mathcal{C}-\mathcal{C}\times\{0\}\in CH^1(\mathcal{C}\times\mathcal{C})$. As the Künneth components $[\mathcal{V}]_0\in H^0(\mathcal{C})\otimes H^2(\mathcal{C})$ and $[\mathcal{V}]_2\in H^2(\mathcal{C})\otimes H^0(\mathcal{C})$ are zero, $[\mathcal{V}]=[\mathcal{V}]_1$ and $\mathbb{Q}[\mathcal{V}]\otimes H^1(\mathcal{C})$ is a subHS of $H^1(\mathcal{C})^{\otimes 3}$. Hence $\mathrm{pr}_{\mathcal{V}}:\ J^2(\mathcal{C}\times\mathcal{C}\times\mathcal{C})\to J^1(\mathcal{C})$ factors through $\mathrm{pr}_{\otimes 3}$; so if $Alb_{\mathcal{C}}(\bar{\mathfrak{Z}}^*\mathcal{V})\neq 0$ then $\mathrm{pr}_{\otimes 3}(AJ(\bar{\mathfrak{Z}}))\neq 0$. Indeed, we have[19]

$$\begin{aligned}\bar{\mathfrak{Z}}\cdot\mathcal{P}_{12}^*\mathcal{V} &= \Delta_{\mathcal{C}}^{[3]}\cdot(\Delta_{\mathcal{C}}\times\mathcal{C})-(\Delta_{\mathcal{C}}\times\{0\})\cdot(\Delta_{\mathcal{C}}\times\mathcal{C})\\ &= -(\iota^{\Delta_{\mathcal{C}}^{[3]}})_*K_{\mathcal{C}}+(\iota^{\Delta_{\mathcal{C}}\times\{0\}})_*K_{\mathcal{C}},\end{aligned}$$

and applying $(\mathcal{P}_3)_*$ gives $\bar{\mathfrak{Z}}^*\mathcal{V}=-K_{\mathcal{C}}+(2g-2)o$. Since $K_{\mathcal{C}}\overset{\text{rat}}{\not\equiv}(2g-2)o$, $AJ_{\mathcal{C}}(\bar{\mathfrak{Z}}^*\mathcal{V})\neq 0$.

Moreover, $\mathfrak{S}_3$ operates on $CH^2(\mathcal{C}\times\mathcal{C}\times\mathcal{C})$ and $J^2\left((H^1(\mathcal{C})^{\otimes 3})\right)$; and since $\bar{\mathfrak{Z}}\in\left(CH^2(\mathcal{C}\times\mathcal{C}\times\mathcal{C})\right)^{\mathfrak{S}_3}$, we have $[0\neq]\,\mathrm{pr}_{\otimes 3}(AJ(\bar{\mathfrak{Z}}))\in\left\{J^2\left(H^1(\mathcal{C})^{\otimes 3}\right)\right\}^{\mathfrak{S}_3}=J^2\left(\{H^1(\mathcal{C})^{\otimes 3}\}^{\mathfrak{S}_3}\right)$ and (i) is verified.

Injectivity of the bottom composition will follow from the fact that

$$\left\{H^1(\mathcal{C})^{\otimes 3}\right\}^{\mathfrak{S}_3}\cap H^1(\mathcal{C})\otimes F_h^1\left\{H^1(\mathcal{C})^{\otimes 2}\right\} = \{0\}.$$

These are $\mathbb{Q}$-vector spaces. Given a basis $\{\omega_i\}\subseteq\Omega^1(\mathcal{C})$, an element of $F_h^1\left\{H^1(\mathcal{C})^{\otimes 2}\right\}$ may be written uniquely as $\sum_j\omega_j\otimes a_j+\sum_j\bar{\omega}_j\otimes\alpha_j$

---

[19] This uses the following self-intersection formula adapted from [Fu, pp. 59, 103]: let ι denote the composition $\mathcal{C}\overset{\jmath}{\underset{\sim}{\to}}\Delta_{\mathcal{C}}\overset{\iota^{\Delta}}{\hookrightarrow}\mathcal{C}\times\mathcal{C}$. Then $\jmath^*\left(N_{(\Delta_{\mathcal{C}})/(\mathcal{C}\times\mathcal{C})}\right)\cong(\iota^*T_{\mathcal{C}\times\mathcal{C}})/T_{\mathcal{C}}\cong T_{\mathcal{C}}$, and so $\iota^*\iota_*\langle\mathcal{C}\rangle=c_1(T_{\mathcal{C}})\cap\langle\mathcal{C}\rangle=-c_1(K_{\mathcal{C}})\cap\langle\mathcal{C}\rangle$, which is to say the negative of the canonical divisor (which we write $-K_{\mathcal{C}}$).

for $\alpha_j \in H^{1,0}(\mathcal{C},\mathbb{C})$, $a_j \in H^{0,1}(\mathcal{C},\mathbb{C})$. For coefficients (hence periods) in $\mathbb{Q}$, the element must be fixed under conjugation, hence $a_j = \bar{\alpha}_j$. An element of $H^1(\mathcal{C}) \otimes F_h^1\{H^1(\mathcal{C})^{\otimes 2}\}$ thus takes the form[20]

$$\sum_{i,j} \omega_i \otimes \omega_j \otimes \bar{\alpha}_j + \sum_{i,j} \omega_i \otimes \bar{\omega}_j \otimes \alpha_j + \sum_{i,j} \bar{\omega}_i \otimes \omega_j \otimes \bar{\beta}_j + \sum_{i,j} \bar{\omega}_i \otimes \bar{\omega}_j \otimes \beta_j.$$

If the last line is invariant under $\mathfrak{S}_3$ then, applying (12) and switching i and j, it must equal

$$\sum_{i,j} \omega_i \otimes \omega_j \otimes \bar{\alpha}_i + \sum_{i,j} \omega_i \otimes \bar{\omega}_j \otimes \bar{\beta}_i + \sum_{i,j} \bar{\omega}_i \otimes \omega_j \otimes \alpha_i + \sum_{i,j} \bar{\omega}_i \otimes \bar{\omega}_j \otimes \beta_i.$$

But if $\alpha_j = \bar{\beta}_i$ then they lie in $H^{1,0} \cap H^{0,1} = \{0\}$. □

Remark 6.5. It would be interesting to see whether any similar results can be proved (using intersection theory) if we replace $\mathcal{C}$ by X of higher dimension. The generalization would not be straightforward, however. If X is a smooth projective surface with $H^1(X) = H^3(X) = 0$, the cycle analogous to $\mathcal{Z}$ in Asakura's theorem is $\overset{\text{rat}}{\equiv} 0$ if BBC^q holds. This is because an exactly analogous choice of $\bar{\mathfrak{Z}}$ on $X \times X \times X$ gives $\bar{\mathfrak{Z}} \overset{\text{hom}}{\equiv} 0$, and then necessarily $AJ(\bar{\mathfrak{Z}}) = 0$ since $H^7(X \times X \times X) = 0$. For X a $K3$, Beauville-Voisin ([BV]) have actually *proved* $\mathcal{Z} \overset{\text{rat}}{\equiv} 0$ if $24\, o \overset{\text{rat}}{\equiv} c_2(T_X)$, but it should be true for *any* choice of $o \in X(\bar{\mathbb{Q}})$.

Next we ask what happens if we take higher self-products $(p-o) \times \cdots \times (p-o)$ on $\mathcal{C} \times \cdots \times \mathcal{C}$? Here is a vanishing result (like the B-V theorem above).

Theorem **([Vo][21]).** Let $\mathcal{C}$ be a smooth projective curve $/\mathbb{C}$ with $\mathcal{Z} \in CH_0^{\text{hom}}(\mathcal{C})$. Then the n-fold self-product $\mathcal{Z}^{\times n} \overset{\text{rat}}{\equiv} 0$ in $CH_0(\mathcal{C}^{\times n})$ for $n \geq g+1$.

Remark. One should note immediately that $\mathcal{Z}$ here is *not* limited to cycles of transcendence degree 1 (like $p-o$ above) on curves defined $/\bar{\mathbb{Q}}$. For that situation one expects a much stronger result, namely $(p-o)^{\times n} \overset{\text{rat}}{\equiv} 0$ for $n \geq 3$ (regardless of g). Indeed, if BBC^q holds then cycles of trdeg. 1 in $\mathcal{L}^3 CH_0(\mathcal{C}^{\times n})$ are $\overset{\text{rat}}{\equiv} 0$ (see Prop. 4.3(a)); while from §5.1 $(p-o)^{\times n} \in F_{\times}^n CH_0 \subseteq \mathcal{L}^n CH_0$.

---

[20] in fact (applying complex conjugation again) $\alpha_j = \beta_j$, but we don't need this.

[21] C. Voisin has pointed out that this is really due to Bloch [B1]. Indeed, it follows easily from the Theorem quoted in §3.2 above (take $A = J^1(\mathcal{C})$). Note also that Voevodsky arrives at the weaker lower bound $n \geq 2g$.

Proof. First let n be arbitrary, and write $[p_1, \ldots, p_n]$ for the image of $(p_1, \ldots, p_n)$ under the standard map $\tilde{\alpha} : \mathcal{C}^{\times n} \to S^n\mathcal{C}$ onto the n^{th} symmetric product. Let $o \in \mathcal{C}$ be a choice of base point and define

$$\tilde{\xi}_k : S^k\mathcal{C} \to J^1(\mathcal{C}) \text{ by } [p_1, \ldots, p_k] \mapsto AJ_{\mathcal{C}}\left(\sum p_i - k \cdot o\right),$$

$$\tilde{\gamma} : S^{n-1}\mathcal{C} \to S^n\mathcal{C} \text{ by } [p_1, \ldots, p_{n-1}] \mapsto [p_1, \ldots, p_{n-1}, o];$$

denote the induced maps on CH_0 by α, ξ_k, γ. Next, writing $\tilde{\mathcal{W}}$ for a lift of $\mathcal{W} \in Z_0(S^k\mathcal{C})$ to $Z_0(\mathcal{C}^{\times k})$ and $Sym_k := \frac{1}{k!}\sum_{\sigma \in \mathfrak{S}_k} \sigma_* \in End\left(Z_0(\mathcal{C}^{\times k})\right)$ (and referring to §5.1 for $P_\times^{[k]}$, Box^k) we define

$$\beta : CH_0(S^n\mathcal{C}) \to CH_0(\mathcal{C}^{\times n}) \text{ by } Sym_n\left\{P_\times^{[n-1]}\tilde{\mathcal{W}}\right\},$$

$$b : CH_0(S^{n-1}\mathcal{C}) \to CH_0(\mathcal{C}^{\times n-1}) \text{ by}$$

$$b(\mathcal{W}) := Sym_{n-1}\left\{P_\times^{[n-1]}\tilde{\mathcal{W}}\right\} = Sym_{n-1}\left\{Box^{n-1}(\tilde{\mathcal{W}})\right\}.$$

Finally, let $\iota_j : \mathcal{C}^{\times n-1} \hookrightarrow \mathcal{C}^{\times n}$ be the inclusion putting $\{0\}$ in the j^{th} coordinate and $\pi_j : \mathcal{C}^{\times n} \twoheadrightarrow \mathcal{C}^{\times n-1}$ be the projection forgetting that coordinate; and define

$$\iota_*^{(1)} : CH_0(\mathcal{C}^{\times n-1}) \to CH_0(\mathcal{C}^{\times n}) \text{ by } \iota_*^{(1)}(\mathcal{W}) := \frac{1}{n}\sum_{j=1}^{n}(\iota_j)_*\mathcal{W},$$

$$\pi_*^{(1)} : CH_0(\mathcal{C}^{\times n}) \to CH_0(\mathcal{C}^{\times n-1}) \text{ by } \pi_*^{(1)}(\mathcal{W}) := \sum_{j-1}^{n}(\pi_j)_*\mathcal{W}.$$

We summarize these maps in the diagram

$$\begin{tikzcd}
\left(F_\times^n CH_0(\mathcal{C}^{\times n})\right)^{\mathfrak{S}_n} \arrow[r, hook, "I"'] & CH_0(\mathcal{C}^{\times n}) \arrow[r, "\alpha"] & CH_0(S^n\mathcal{C}) \arrow[r, "\beta"] \arrow[dl, "\xi_n"'] & \left(F_\times^{n-1} CH_0(\mathcal{C}^{\times n})\right)^{\mathfrak{S}_n} \arrow[dd, shift left, "\pi_*^{(1)}"] \\
 & CH_0(J^1(\mathcal{C})) & & \\
 & & CH_0(S^{n-1}\mathcal{C}) \arrow[uu, "\gamma"] \arrow[ul, "\xi_{n-1}"] \arrow[r, "b"] & \left(F_\times^{n-1} CH_0(\mathcal{C}^{\times n-1})\right)^{\mathfrak{S}_{n-1}} . \arrow[uu, shift left, "\iota_*^{(1)}"]
\end{tikzcd}$$

Clearly $\xi_{n-1} = \xi_n \circ \gamma$. While $\iota_*^{(1)} \circ \pi_*^{(1)}$ is not the identity, $(\pi_i)_* \circ (\iota_j)_* = \delta_{ij}$ on $\left(F_\times^{n-1} CH_0(\mathcal{C}^{\times n-1})\right)^{\mathfrak{S}_{n-1}} \implies \pi_*^{(1)} \circ \iota_*^{(1)} =$ identity there. It is also easy to see that $\beta \circ \gamma = \iota_*^{(1)} \circ b$, and applying $\pi_*^{(1)}$ on the left yields $\pi_*^{(1)} \circ \beta \circ \gamma = b$. Moreover, the top composition is just the obvious inclusion, and $\pi_*^{(1)} \circ \beta \circ \alpha \circ I = 0$. Since the $\overset{\text{rat}}{\equiv}$-class of $\mathcal{Z}^{\times n}$ belongs to $\left(F_\times^n CH_0(\mathcal{C}^{\times n})\right)^{\mathfrak{S}_n}$, it is obviously trivial if $\beta \circ \alpha \circ I$ is itself zero.

This is precisely what happens when we take $n \geq g+1$. Indeed, we claim in this case that ξ_n and ξ_{n-1}, hence γ, are isomorphisms.[22] Assuming this, we can apply $\gamma^{-1} \circ \alpha \circ I$ on the right to $\beta \circ \gamma = \iota_*^{(1)} \circ b$, obtaining

$$\beta \circ \alpha \circ I \;=\; \iota_*^{(1)} \circ b \circ \gamma^{-1} \circ \alpha \circ I .$$

On the other hand, applying $\gamma^{-1} \circ \alpha \circ I$ on the right and $\iota_*^{(1)}$ on the left to $b = \pi_*^{(1)} \circ \beta \circ \gamma$ gives

$$\iota_*^{(1)} \circ b \circ \gamma^{-1} \circ \alpha \circ I \;=\; \iota_*^{(1)} \circ \pi_*^{(1)} \circ \beta \circ \alpha \circ I \;=\; \iota_*^{(1)} \circ 0 \;=\; 0\,,$$

and we are done modulo our claim.

Recall that $\tilde{\xi}_k$ is surjective for $k \geq g$, with generic fiber $\mathbb{P}^{k-g}$. Indeed, all its fibers are rational: given $x \in J^1(\mathcal{C})$, pick any $[q_1, \ldots, q_k] \in \tilde{\xi}_k^{-1}(x)$; then $\tilde{\xi}_k^{-1}(x) \cong \mathbb{P}H^0(\mathcal{C}, \mathcal{O}(q_1 + \cdots + q_k)) \cong \mathbb{P}^M$ for some $M \geq k - g$. (These facts follow from Jacobi inversion and Riemann-Roch.) For $k = g$, one obtains that $\tilde{\xi}_g$ is birational; hence ξ_g is an isomorphism.[23] For $k = g+1$, one notes that $\mathrm{im}\{\tilde{\gamma} : S^g\mathcal{C} \to S^{g+1}\mathcal{C}\}$ meets every fiber of $\tilde{\xi}_{g+1}$, so any point of $S^{g+1}\mathcal{C}$ is $\overset{\text{rat}}{\equiv}$ to a point of $\mathrm{im}(\tilde{\gamma})$, hence γ is surjective; and since $\xi_{g+1} \circ \gamma = \xi_g$, γ is injective. Thus ξ_{g+1} is an isomorphism; repeating this argument shows ξ_k is an isomorphism for $k \geq g$ as claimed. $\square$

This theorem has the exciting corollary that all cycles algebraically equivalent to 0 are "nilpotent" modulo $\overset{\text{rat}}{\equiv}$. To see this, let $Y \in CH_m^{\text{alg}}(X)$. Then there exist $\mathcal{Y}_i \in CH_{m+1}(X \times \mathcal{C}_i)$ and $p_i,\ q_i \in \mathcal{C}_i$ [=curve of genus g_i] s.t. $Y = \sum_i (\mathcal{Y}_i(p_i) - \mathcal{Y}_i(q_i))$, where $\mathcal{Y}_i(c) := \pi_X(\mathcal{Y}_i \cap (\{c\} \times X))$ for $c \in \mathcal{C}_i$. For each i one considers the map

$$\phi_n : CH_0(\mathcal{C}^{\times n}) \to CH_{m \cdot n}(X^{\times n})$$

$$c_1 \times \cdots \times c_n \mapsto \mathcal{Y}(c_1) \times \cdots \times \mathcal{Y}(c_n)$$

induced by $\mathcal{Y}^{\times n}$. Since $\phi_n\{(p-q)^{\times n}\} = (\mathcal{Y}(p) - \mathcal{Y}(q))^{\times n}$ and $(p-q)^{\times n} \overset{\text{rat}}{\equiv} 0$ for $n \geq g+1$, the terms of the $\sum_i$ are nilpotent [i.e. $(\mathcal{Y}_i(p_i) - \mathcal{Y}_i(q_i))^{\times(g_i+1)} \overset{\text{rat}}{\equiv} 0$]. It follows easily that $Y^{\times\{\sum_i (g_i+1)\}} \overset{\text{rat}}{\equiv} 0$.

Corollary 6.6. *All degree-*0 *zero-cycles are nilpotent.*

---

[22] Voevodsky uses the classical fact that (essentially by Riemann-Roch) $S^k\mathcal{C} \overset{\tilde{\xi}_k}{\to} J^1(\mathcal{C})$ is a projective bundle for $k \geq 2g-1$, to see this for $n \geq 2g$.

[23] For this implication see the argument in [V4], §11.1.1 (p. 310), which is stated for surfaces but valid also for higher-dimensional varieties (e.g., see also [L2], pf. of 13.3).

Voevodsky [Vo] also outlines an argument to the effect that extending the nilpotence statement from cycles $\overset{\text{alg}}{\equiv} 0$ to cycles $\overset{\text{num}}{\equiv} 0$ would amount to proving the Bloch conjecture. The same argument also appeared in Voisin's paper [V3]. Kimura ([Ki, sec. 7]) picks up this theme and proves nilpotence of cycles $\overset{\text{hom}}{\equiv} 0$ (and BC) for varieties of "finite dimensional" Chow motives. This includes (smooth) varieties covered by a product of curves (and the proof of this utilizes once again the classical fact regarding $S^{n \geq 2g-1}\mathcal{C} \to J^1(\mathcal{C})$ used in [Vo]).

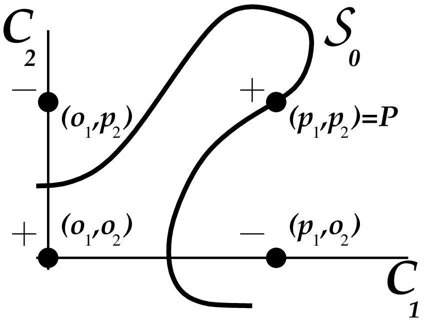
6.3. **Correspondence cycles.** If $X = \mathcal{C}_1 \times \cdots \times \mathcal{C}_n \,/\bar{\mathbb{Q}}$ (here $n = d$) then we may consider 0-cycles of the form $\mathcal{Z} = Box^n(p_1, \ldots, p_n)$ for $p_\ell \in \mathcal{C}_\ell(\mathbb{C})$; we have automatically $\mathcal{Z} \in F^n_\times \subseteq \mathcal{L}^n CH_0(\mathcal{C}_1 \times \cdots \times \mathcal{C}_{n/\mathbb{C}})$. Thus, to be *detectably* nontrivial, $\mathcal{Z}$ must *not* be defined over a field of transcendence degree less than $(n-1)$ over $\bar{\mathbb{Q}}$. The interesting situation is therefore where $\mathcal{S}_{0/\bar{\mathbb{Q}}} \subseteq \mathcal{C}_1 \times \cdots \times \mathcal{C}_n$ is a (possibly singular) $(n-1)$-dimensional subvariety, and $P = (p_1, \ldots, p_n)$ is the inclusion of a very general point $P_0 \in \mathcal{S}_0(\mathbb{C})$, so that the spread $\mathfrak{P}$ of P is the graph of a desingularization $\mathcal{S} \twoheadrightarrow \mathcal{S}_0 \subseteq X$ as a subvariety of $\mathcal{S} \times X$. We write in an obvious notation $\bar{\mathfrak{Z}} := Box^n_{\mathcal{S}}(\mathfrak{P})$.

A technique for detecting such 0-cycles $\mathcal{Z}$ is developed in [K2, sec. 14], by writing an explicit $\Omega^n(X)^\vee$-valued $(n-1)$-current $R_{\mathfrak{Z}} = \sum_{k \in \mathsf{K}} [\omega^k]^\vee \otimes R^k_{\mathfrak{Z}}$ on $\mathcal{S}$. Integrating $R_{\mathfrak{Z}}$ on topological cycles $\gamma \in \text{im}\{\varprojlim_D H_{n-1}(\mathcal{S} \setminus D) \to H_{n-1}(\mathcal{S})\}$ then computes $\overline{[AJ(\mathfrak{Z})]_{n-1}} \in Hom\left(\underline{H_{n-1}}(\eta_{\mathcal{S}}, \mathbb{Q}), \frac{\Omega^n(X)^\vee}{\text{im}\, H_n(X,\mathbb{Q})}\right)$; i.e. as vectors $\in \mathbb{C}^{|\mathsf{K}|}$, $\left(\int_\gamma R^k_{\mathfrak{Z}}\right)_{k \in \mathsf{K}} \equiv \left(\int_\Gamma \omega^k\right)_{k \in \mathsf{K}}$ modulo $\Lambda_{\{\omega\}}$.[24] Remarkably, in case $n = 2$ (only) $\int_\gamma R^k_{\mathfrak{Z}}$ reduces to an iterated integral, so it is in this case that we review the construction.

Henceforth $\mathcal{S}_0 \subseteq \mathcal{C}_1 \times \mathcal{C}_2 = X$ are defined $/\bar{\mathbb{Q}}$ with $o_\ell \in \mathcal{C}_\ell(\bar{\mathbb{Q}})$, $P_0 \in \mathcal{S}_0(\mathbb{C})$ is very general with image $(p_1, p_2) \in X(\mathbb{C})$, and $\mathcal{Z} = (p_1 - o_1) \times (p_2 - o_2) \in \mathcal{L}^2 CH_0(X_K)$ where $K \cong \bar{\mathbb{Q}}(\mathcal{S})$.

---

[24] We note that $\dim(\mathcal{S}) = n-1 \implies [\mathfrak{Z}]_n = 0 \implies \overline{[AJ(\mathfrak{Z})]_{n-1}}$ is defined and computed by the membrane integrals of Proposition 4.8. For $n > 2$ it is not clear (without HC) how to modify $\bar{\mathfrak{Z}}$ over $D \subseteq \mathcal{S}$ to get $\bar{\mathfrak{Z}} \overset{\text{hom}}{\equiv} 0$, so this uses Remark 4.10(b); for $n = 2$, see below.



$\mathcal{Z}$ is called an "$\mathcal{S}_0$-" or "correspondence-" 0-cycle on $\mathcal{C}_1 \times \mathcal{C}_2$; in fact, we have already encountered the diagonal 0-cycle on $\mathcal{C} \times \mathcal{C}$ ($\mathcal{S}_0 = \Delta_{\mathcal{C}}$) and the bielliptic 0-cycle on $E_1 \times E_2$ ($\mathcal{S} = \mathcal{B}$). Note that since $[\mathfrak{Z}]_0 = [\mathfrak{Z}]_1 = [\mathfrak{Z}]_2 = 0$ (i.e. $[\mathfrak{Z}] = 0$ in $\underline{H}^4(\eta_{\mathcal{S}} \times \mathcal{C}_1 \times \mathcal{C}_2, \mathbb{Q})$), $[\bar{\mathfrak{Z}}] \in H^4(\mathcal{S} \times \mathcal{C}_1 \times \mathcal{C}_2)$ comes from a class in $H^2(\{Q\} \times \mathcal{C}_1 \times \mathcal{C}_2)$. As this must also be a rational $(1,1)$ class, Lefschetz $(1,1)$ says that it is the class of a cycle on $\{Q\} \times \mathcal{C}_1 \times \mathcal{C}_2$. If we modify $\bar{\mathfrak{Z}} = Box^2_{\mathcal{S}}(\mathfrak{P})$ by subtracting off this cycle, then the new $\bar{\mathfrak{Z}} \overset{\text{hom}}{\equiv} 0$.

Let $\{\alpha_\ell^j\}_{j=1,\ldots,2g_\ell}$, $\{\hat{\alpha}_\ell^j\}$ be dual bases of $H_1(\mathcal{C}_\ell, \mathbb{Q})$ based at o_ℓ resp. o'_ℓ, while avoiding o'_ℓ resp. o_ℓ; and take $\mathbb{D}_\ell := \overline{\mathcal{C}_\ell \setminus \bigcup_j |\hat{\alpha}_\ell^j|}$ as fundamental domain for $\mathcal{C}_\ell$. Writing $\{\omega_\ell^k\}_{k=1,\ldots,g_\ell} \subseteq \Omega^1(\mathcal{C}_\ell)$ for a basis, we obtain functions $z_\ell^k := \int_{o_\ell}^* \omega_\ell^k$ on $\mathbb{D}_\ell$ (which are 0 at o_ℓ), or equivalently 0-currents on $\mathcal{C}_\ell$ with "branch cuts" at the $\{\alpha_\ell^j\}$. One has in particular $d[z_\ell^k] = \omega_\ell^k - \sum_{j=1}^{2g_\ell} \left(\int_{\alpha_\ell^j} \omega_\ell^k \right) \cdot \delta_{\hat{\alpha}_\ell^j}$.

Write $\underline{\mathrm{k}} = (k_1, k_2)$, $\mathsf{K} = \{1, \ldots, g_1\} \times \{1, \ldots, g_2\}$ so that the basis $\{\omega^{\underline{\mathrm{k}}}\} \subseteq \Omega^1(X)$ takes the form $\{\omega_1^{k_1} \wedge \omega_2^{k_2}\}_{(k_1,k_2)\in\mathsf{K}}$; also denote the pullbacks of z, ω, α, $\hat{\alpha}$ to $\mathcal{S}$ no differently. Then from [K2]

$$R_{\mathfrak{Z}}^{(k_1,k_2)} = z_1^{k_1}\omega_2^{k_2} - \sum_{j=1}^{2g_1} \left(\int_{\alpha_1^j} \omega_1^{k_1} \right) z_2^{k_2} \cdot \delta_{\hat{\alpha}_1^j} \ \in {}'\mathcal{D}^1(\mathcal{S}).$$

Now assume $\mathcal{S}_0 \ni (o_1, o_2) =: o$ and also write o for a point on $\mathcal{S}$ mapping to it. Choose γ to pass through o (and such that $\gamma \cap \hat{\alpha}_1^{j_1} \cap \hat{\alpha}_2^{j_2} = \emptyset \ \forall\, j_1,\, j_2$) and let $\varphi : [0,1] \to \gamma \subseteq \mathcal{S}$ be a parametrization with $\varphi(0) = o$. Define the iterated integral

$$\int_{\gamma;\, o} \alpha \circ \beta \ := \int_{x=0}^{1} \left(\int_{t=0}^{x} \varphi_t^* \alpha \right) \varphi_x^* \beta$$

for any $\alpha, \beta \in \Omega^1(\mathcal{S})$. (*A priori* this is only an invariant of the homotopy class of $\gamma \in \pi_1(\mathcal{S}; o)$.) We claim that $\int_\gamma R_{\mathfrak{Z}}^{(k_1,k_2)} = \int_{\gamma;\, o} \omega_1^{k_1} \circ \omega_2^{k_2}$

on the nose; to check this we drop the k_1,k_2 and write $R = z_1\omega_2 - \sum_j \left(\int_{\alpha_j} \omega_1 \right) z_2 \cdot \delta_{\hat{\alpha}_1^j}$. Pulling this back to γ gives

$$z_1\omega_2 - \sum_j \sum_{q \in \hat{\alpha}_1^j \cap \gamma} \left(\int_{\alpha_1^j} \omega_1 \right) z_2(q) \cdot \delta_{\{q\}}. \tag{6.1}$$

Denote the path along γ from q to o by $\overrightarrow{q.o}$; on this path let $\tilde{z}_2$ be the continuous function with $d\tilde{z}_2 = \omega_2$ and $\tilde{z}_2(o) = 0$. Then we can take d of the 0-current

$$-\sum_j \sum_{q \in \hat{\alpha}_1^j \cap \gamma} \left(\int_{\alpha_1^j} \omega_1 \right) \tilde{z}_2 \cdot \delta_{\overrightarrow{q.o}}$$

to obtain

$$+\sum_j \sum_{q \in \hat{\alpha}_1^j \cap \gamma} \left(\int_{\alpha_1^j} \omega_1 \right) \tilde{z}_2(q)\delta_q - \left\{ \sum_j \sum_{p \in \hat{\alpha}_1^j \cap \gamma} \left(\int_{\alpha_1^j} \omega_1 \right) \delta_{\overrightarrow{q.o}} \right\} \omega_2\,;$$

adding this to (6.1) yields

$$\left\{ z_1 - \sum_j \sum_{p \in \hat{\alpha}_1^j \cap \gamma} (\omega_1)\, \delta_{\overrightarrow{q.o}} \right\} \omega_2 \;=:\; \tilde{z}_1 \cdot \omega_2$$

where $\tilde{z}_1$ is continuous except at o [essentially $(\tilde{z}_1 \circ \varphi)(x) = \int_0^x \varphi^* \omega_1$]. Along γ this is cohomologous to (6.1) by construction, so

$$\int_\gamma R \;=\; \int_\gamma \tilde{z}_1 \cdot \omega_2 \;=\; \int_{\gamma;\, o} \omega_1 \circ \omega_2$$

as desired.

Theorem 6.7. *Let* $\mathcal{S}_{0/\bar{\mathbb{Q}}} \subseteq \mathcal{C}_1 \times \mathcal{C}_2$ *be a curve passing through* $o = (o_1, o_2) \in (\mathcal{C}_1 \times \mathcal{C}_2)(\bar{\mathbb{Q}})$ *with very general point* (p_1, p_2), *and set* $\mathcal{Z} = (p_1,p_2) - (p_1,o_2) - (o_1,p_2) + (o_1,o_2) \in \mathcal{L}^2 CH_0(\mathcal{C}_1 \times \mathcal{C}_{2/\mathbb{C}})$. *Then for* $\gamma \in H_1(\mathcal{S}, \mathbb{Q})$, $\overline{[AJ(\mathfrak{Z})]_1}(\gamma)$ *evaluates to* $\overline{\left(\int_{\tilde{\gamma};\, o} \omega_1^{k_1} \circ \omega_2^{k_2} \right)_{(k_1,k_2) \in \mathsf{K}}} \in \mathbb{C}^{|\mathsf{K}|} / \Lambda_{\{\omega\}}$ *on the basis* $\{\omega\} = \{\omega_1^{k_1} \wedge \omega_2^{k_2}\}$, *where* $\tilde{\gamma}$ *is any closed path* $\ni o$ *and* $\overset{hom}{\equiv} \gamma$.

Example: the Fermat 0-cycle. Specializing to $\mathcal{C}_1 = \mathcal{C}_2 = E$ an elliptic curve, we write $\Omega^1(E) = \mathbb{C}\,\langle dz \rangle$ so that $\omega = dz_1 \wedge dz_2$ generates $\Omega^2(E \times E)$. If we take E to have CM, then $H^2(E \times E) = H^2_{\text{alg}} \oplus H^2_{\text{tr}}$ as HS where $H^2_{tr}(E \times E, \mathbb{C}) = H^{2,0} \oplus H^{0,2}$; this has the effect of making ω's periods $\Lambda_\omega \subseteq \mathbb{C}$ yield an honest 2-dimensional rational lattice.

Setting $E = \overline{\{v^2 = 1 - u^4\}}$ with $o = (0,1)\,[= (u,v)]$ and $dz := \frac{1-i}{4b}\frac{du}{v}$, where $b = \int_0^1 \frac{dt}{\sqrt{1-t^4}}$, we have in fact $\Lambda_{\{dz\}} = \mathbb{Q}\langle 1, i\rangle$. Hence if $\mathcal{S}_0$, $\mathcal{Z}$, γ are as in Thm. 6.7, $\overline{[AJ(\mathfrak{Z})]_1}(\gamma)$ is computed by the number $\int_{\tilde{\gamma};\,o} \pi_1^* dz \circ \pi_2^* dz \in \mathbb{C}/\mathbb{Q}\langle 1, i\rangle$ (integration on $\mathcal{S}$). (It seems remarkable that the iterated integral is well-defined modulo $\mathbb{Q}\langle 1, i\rangle$ on *homology classes*.)

For $\mathcal{S}$ we shall take the Fermat curve $\mathfrak{F} = \overline{\{x^4 + y^4 = 1\}}$, embedded in $E \times E$ by (π_1, π_2) where $\pi_1(x,y) = \left(-\frac{1+i}{\sqrt{2}}\frac{y}{x}, \frac{1}{x^2}\right)$, $\pi_2(x,y) = (-y,\, x^2)$; note $o = (1,0) \in \mathcal{S}$ maps to $o_1 \times o_2 \in E \times E$. We shall also make use of a third map $\pi_3(x,y) = (x,\, y^2)$. Writing $\nu_i = \pi_i^* dz$ we have $\nu_1 = \frac{1}{2b'}\frac{dx}{y^3}$ where $b' = \sqrt{2}b = \int_0^1 \frac{dt}{(1-t^4)^{3/4}}$, $\nu_2 = \frac{1-i}{4b}\frac{x\,dx}{y^3}$, $\nu_3 = \frac{1-i}{4b}\frac{dx}{y^2}$. These span $\Omega^1(\mathfrak{F})$, and so $q \mapsto \left\langle \int_o^q \nu_i \right\rangle_{i=1,2,3}$ induces a map $\mathfrak{F} \overset{AJ_o}{\to} J^1(\mathfrak{F})$ factoring $(\pi_1, \pi_2, \pi_3) : \mathfrak{F} \to E \times E \times E$, hence an isogeny $J^1(\mathfrak{F}) \overset{\mu}{\to} E \times E \times E$.

The computation that follows is almost exactly the same as the one done in [H]: taking $\gamma =$ path (on $\mathfrak{F}$) from $o = (1,0) \to (0,i) \to (i,0) \to (0,1) \to (1,0)$, we use the automorphism $\sigma(x,y) = (\frac{y}{x\sqrt{i}}, \frac{1}{xi})$ to get

$$\int_{\gamma;\,o} \nu_1 \circ \nu_2 \;=\; \int_{\sigma(\gamma);\,(o,-i)} \nu_3 \circ \nu_1 \;=\; 2(1+i) \int_{\overrightarrow{(0,1).o}} \nu_3 \circ \nu_1$$

$$=\; 2(1+i) \int_0^1 \frac{(1-i)}{4b} \frac{dx}{\sqrt{1-x^4}} \circ \frac{1}{2b'} \frac{dx}{(1-x^4)^{3/4}}$$

$$=\; \frac{\int_0^1 \frac{dt}{\sqrt{1-t^4}} \circ \frac{dt}{(1-t^4)^{3/4}}}{2\int_0^1 \frac{dt}{\sqrt{1-t^4}} \cdot \int_0^1 \frac{dt}{(1-t^4)^{3/4}}} \;=:\; \kappa \in \mathbb{R}/\mathbb{Q} \subseteq \mathbb{C}/\mathbb{Q}\langle 1, i\rangle\,.$$

Unfortunately, it appears to be unknown whether κ is irrational; but we can verify the Fermat 0-cycle $\mathcal{Z} \overset{\text{rat}}{\not\equiv} 0$ by other means and so $\kappa \notin \mathbb{Q}$ would be implied by BBC^q.

Bloch has proved (in [B3], using the ℓ-adic Abel-Jacobi map) the algebraic inequivalence of $AJ_o(\mathfrak{F})$ and its mirror image $AJ_o(\mathfrak{F})^-$ as 1-cycles in $J^1(\mathfrak{F})$. Using the isogeny μ and exploiting symmetries of the image of $\mathfrak{F}$ in $E_{(1)} \times E_{(2)} \times E_{(3)}$, one first deduces that $Box^2_{E_3}(\text{im}(\mathfrak{F})) \overset{\text{alg}}{\not\equiv} 0$ there, then that it cannot be $\overset{\text{alg}}{\equiv}$ (*a fortiori* $\overset{\text{rat}}{\equiv}$) to a cycle supported on $E_1 \times E_2 \times \{\text{pts.} \in E_3\}$. Consequently the spread $\bar{\mathfrak{Z}}$ of the Fermat 0-cycle $\mathcal{Z}$, which maps to $Box^2_{E_3}(\text{im}(\mathfrak{F}))$ via $E_1 \times E_2 \times \mathfrak{F} \overset{(\text{id},\text{id},\pi_3)}{\longrightarrow} E_1 \times E_2 \times E_3$, cannot be $\overset{\text{rat}}{\equiv}$ to a cycle supported over a divisor$/\bar{\mathbb{Q}}$ in $\mathfrak{F}$; and so $\mathcal{Z} \overset{\text{rat}}{\not\equiv} 0$.

6.4. **Regulators and relative 0-cycles.** There is one product of "curves" for which one can completely describe the Chow group of 0-cycles and the higher AJ maps. This is the n-fold self-product of the *relative* curve $(\mathbb{P}^1, \{0, \infty\})$, which one may think of *intuitively* (only) as $\mathbb{P}^1$ with $\{0\}$ and $\{\infty\}$ attached — a degenerate elliptic curve. We give just the barest summary of the most striking result and refer the reader to [K2] and [K3] for full details (minus relative $\overset{\text{rat}}{\equiv}$, a discussion of which for these problems is in [K4]).

Set $X^n := (\mathbb{P}^1, \{0, \infty\})^{\times n}$, or equivalently for our purposes, $(\mathbb{P}^n, \bigcup_{i=0}^n \{z_i = 0\})$ (which one can think of as a degenerate CY n-fold). On X^n_L ($L \subseteq \mathbb{C}$) let $\{a_1, \ldots, a_n\}$ represent the relative $\overset{\text{rat}}{\equiv}$-class of $B(a_1, \ldots, a_n) := (\langle a_1 \rangle - \langle 1 \rangle) \times \cdots \times (\langle a_n \rangle - \langle 1 \rangle)$. We make the *ad hoc* definition

$$\mathcal{L}^n CH_0(X^n_L) := \left\{ \{a_1, \ldots, a_n\} \mid a_i \in L^* \ (\forall i) \right\},$$

and observe that this gives Milnor K-theory $K^M_n(L)$ by a result of Totaro ([To]).

Now assume L is finitely generated $/\bar{\mathbb{Q}}$ of trdeg $< n$, so that one has $\bar{\mathbb{Q}}(\mathcal{S}) \underset{\text{ev}}{\overset{\cong}{\to}} L$ for some smooth projective $\mathcal{S}/\bar{\mathbb{Q}}$ of dimension $< n$. There is a map

$$\mathcal{R} : \ K^M_n(\bar{\mathbb{Q}}(\mathcal{S})) \to H^{n-1}(\eta_{\mathcal{S}}, \mathbb{C}/\mathbb{Q}(n))$$

called the Milnor regulator; one defines $\mathcal{R}\{f_1, \ldots, f_n\}$ to be the cohomology class of the $(n-1)$-current

$$\sum_{j=1}^n (-1)^{nj-1} (2\pi\sqrt{-1})^{j-1} \log f_j \,\mathrm{dlog} f_{j+1} \wedge \cdots \wedge \mathrm{dlog} f_n \cdot \delta_{f_1^{-1}(\mathbb{R}^-) \cap \cdots \cap f_{j-1}^{-1}(\mathbb{R}^-)}$$

on $\eta_{\mathcal{S}}$. Obviously one can consider the composition

$$\mathcal{L}^n CH_0(X^n_L) \cong K^M_n(L) \xrightarrow{\text{ev}^{-1}} K^M_n(\bar{\mathbb{Q}}(\mathcal{S})) \xrightarrow{\mathcal{R}} H^{n-1}(\eta_{\mathcal{S}}, \mathbb{C}/\mathbb{Q}(n)).$$

On the other hand, to try computing a higher AJ class for $\mathcal{Z} = B(a_1, \ldots, a_n)$, one would spread this out to $\mathfrak{Z}_{(\mathcal{U})} = B(f_1, \ldots, f_n)$ on $(\mathbb{P}^1, \{0, \infty\})^{\times n} \times \{\mathcal{U} \subseteq \mathcal{S}\}$ by taking $f_i := \text{ev}^{-1}(a_i)$. (Each component of $\mathfrak{Z}$ looks like the graph of n functions over $\mathcal{U}$.) One can then show that $\mathfrak{Z} \overset{\text{hom}}{\equiv} 0$ and obtain a generalized AJ class for $\mathfrak{Z}$, by integrating forms $\mathrm{dlog} z_1 \wedge \cdots \wedge \mathrm{dlog} z_n \wedge \omega_{\mathcal{U}}$ over a limit of chains bounding (in the limit) on $\mathfrak{Z}$. This boils down, by explicit computation and Hodge theory, to integration of the above current over topological cycles on

$\mathcal{S}$. That is, the above composition gives the right generalization[25] of AJ_X^{n-1}, and one would conjecture it is injective.

As a sanity check, one can try out $n = 1$: $\mathcal{R}\{f\} = \log f \in H^0(\eta_{\mathcal{S}}, \mathbb{C}/\mathbb{Q}(1)) \cong \mathbb{C}/(2\pi\sqrt{-1})\mathbb{Q}$, where (since $\dim(\mathcal{S}) = 0$) really $f = \alpha \in \bar{\mathbb{Q}}^*$. The composition sends $(\langle\alpha\rangle - \langle 1\rangle) \mapsto \log\alpha$; so we expect that for $\sum q_i = 0$, $\sum q_i \langle\alpha_i\rangle \overset{\text{rat}}{\equiv} 0$ (mod torsion) $\iff (\Pi\alpha_i^{q_i})^N = 1$ for some $N \in \mathbb{Z}^+$. Since on $(\mathbb{P}^1, \{0,\infty\})_{\bar{\mathbb{Q}}}$, rational equivalence is generated by divisors of functions $F \in \bar{\mathbb{Q}}(\mathbb{P}^1)^*$ with $F(0) = F(\infty) = 1$, this is indeed the case.

7. BEHAVIOR OF HIGHER ABEL-JACOBI CLASSES IN FAMILIES

7.1. **Topological invariants for higher normal functions.** Given a smooth family of d-dimensional projective varieties $\mathfrak{X}_{\mathcal{U}} \xrightarrow{\pi} \mathcal{U}\,[\subseteq \mathcal{S}]$ over an affine base, with $\mathfrak{Z}_{(\mathcal{U})} \in Z^d(\mathfrak{X}_{\mathcal{U}})$ such that $\mathcal{Z}_s := \mathfrak{Z}\,|_{X_s} \overset{\text{hom}}{\equiv} 0$ $(\forall\, s \in \mathcal{U})$, one can define the *normal function* $\nu_{\mathfrak{Z}} \in \Gamma(\mathcal{U},\, \mathcal{J}^d_{\mathfrak{X}_{\mathcal{U}}/\mathcal{U}})$ by $\nu_{\mathfrak{Z}}(s) := Alb_{X_s}(\mathcal{Z}_s) \in J^d(X_s)$. Using the connecting homomorphism δ associated to the exact sequence $0 \to \mathcal{H}^{2d-1}_{\mathfrak{X}_{\mathcal{U}}/\mathcal{U},\,\mathbb{Q}} \to \mathcal{H}^{2d-1}_{\mathfrak{X}_{\mathcal{U}}/\mathcal{U}} \Big/ \mathcal{F}^d\mathcal{H}^{2d-1}_{\mathfrak{X}_{\mathcal{U}}/\mathcal{U}} \to \mathcal{J}^d_{\mathfrak{X}_{\mathcal{U}}/\mathcal{U}} \to 0$, one obtains the topological invariant $\delta\nu_{\mathfrak{Z}} \in H^1(\mathcal{U}, \mathcal{H}^{2d-1}_{\mathfrak{X}_{\mathcal{U}}/\mathcal{U},\,\mathbb{Q}})$ of the normal function. In terms of the notation of §4.2, $\nu_{\mathfrak{Z}}$ is the family $\{[AJ(\mathfrak{Z}|_{X_s})]_0\}$ while $\delta\nu_{\mathfrak{Z}}$ is essentially $[\mathfrak{Z}]_1$; so $[\mathfrak{Z}]_1 \neq 0 \implies \{[AJ(\mathfrak{Z}|_{X_s})]_0\} \neq 0$ and $[\mathfrak{Z}]_1$ is an invariant of the family.

Below we will generalize this to show (for $i \geq 2$) $[\mathfrak{Z}]_i$ may be viewed (in some cases) as an invariant of a family of $(i-1)^{\text{st}}$-higher-AJ classes. First we explain (partly following [K1]) how such a family appears in terms of spreads and field extensions. Let K/L be a transcendental extension of subfields of $\mathbb{C}$ finitely generated $/\bar{\mathbb{Q}}$. Then there exist $\mathcal{S}/\bar{\mathbb{Q}}$ with $s_0 \in \mathcal{S}(\mathbb{C})$ such that $\text{ev}_{s_0} : \bar{\mathbb{Q}}(\mathcal{S}) \xrightarrow{\cong} K$, and $\mathcal{M}/\bar{\mathbb{Q}}$ with $\bar{\rho} : \mathcal{S} \to \mathcal{M}$ such that $\text{ev}_{\rho(s_0)} : \bar{\mathbb{Q}}(\mathcal{M}) \xrightarrow{\cong} L$; write $\mu_0 := \rho(s_0) \in \mathcal{M}(\mathbb{C})$. We get a foliation of $\mathcal{S}$ by subvarieties $S_\mu := \rho^{-1}(\mu) \overset{\iota_\mu}{\hookrightarrow} \mathcal{S}$ of dimension $\text{trdeg}(K/L) =: t \geq 1$, and note that S_{μ_0} is defined $/L$ with

$$L(S_{\mu_0}) \underset{\text{ev}_{s_0}}{\overset{\cong}{\to}} K.$$

Finally, it is clear that $\bar{\rho}$ makes sense in the limit as a map $\eta_{\mathcal{S}} \to \eta_{\mathcal{M}}$; we shall denote this by ρ. Note that since this is a limit of affines over an affine base with t-dimensional affine fibers, $R^{t+1}\rho_*\mathbb{Q} = 0$. If $m = \dim\mathcal{M}$ then Leray for ρ yields $H^{t+m}(\eta_{\mathcal{S}}) \cong H^m(\eta_{\mathcal{M}}, R^t\rho_*\mathbb{Q})$.

---

[25] This result was the original motivation for introducing the reduced higher AJ maps of §4.3 and the currents $R_{\mathfrak{Z}}$ in §6.3 computing them.

Next consider X/K and $\mathcal{Z} \in Z^p(X/K)$; then one has the $\bar{\mathbb{Q}}$-spread $\mathfrak{Z} \in Z^p(\mathfrak{X}/\bar{\mathbb{Q}})$ with $\mathfrak{X} \xrightarrow{\pi} \eta_{\mathcal{S}}$ as above. But we don't have to spread clear on down to $\bar{\mathbb{Q}}$. Write $\eta_{S_\mu} := \varprojlim \mathcal{V}$ [$\mathcal{V} \subseteq S_\mu$ affine Zar. op. $/\bar{\mathbb{Q}}(\mu)$]$=\eta_{\mathcal{S}} \cap S_\mu$ and $\mathfrak{X}_\mu := \pi^{-1}(\eta_{S_\mu}) \overset{\mathcal{I}_\mu}{\hookrightarrow} \mathfrak{X}$. Then we can consider instead the (partial) L-spread $\mathfrak{Z}_{\mu_0} \in Z^p(\mathfrak{X}_{\mu_0/L})$, which is nothing other than $\mathcal{I}^*_{\mu_0}(\mathfrak{Z})$ (i.e., $\mathfrak{Z} \cap \mathfrak{X}_{\mu_0}$). We write $\mathcal{L}^\bullet_{K/\bar{\mathbb{Q}}}$, $\psi_i^{K/\bar{\mathbb{Q}}}$ for the $\mathcal{L}^\bullet$ and ψ_i defined on $Z^p(X_K)$ in §4.2 above; one can set up an analogous picture for the partial spread with invariants

$$\psi_{[i]}^{K/L} : [\mathcal{L}^i_{K/L}]\, CH^p(X_K) \to [Gr^i_{\mathcal{L}}]\, \underline{H^{2p}_{\mathcal{D}}}(\mathfrak{X}_{\mu_0}, \mathbb{Q}(p)).$$

(Here the $\mathcal{L}$ on $\underline{H^{2p}_{\mathcal{D}}}$ comes from Leray for $\mathfrak{X}_{\mu_0} \to \eta_{S_{\mu_0}}$.) One has $\psi^{K/L} = \mathcal{I}^*_{\mu_0} \circ \psi^{K/\bar{\mathbb{Q}}}$, $\mathcal{I}^*_{\mu_0}(\mathcal{L}^\bullet) \subseteq \mathcal{L}^\bullet$, hence $\mathcal{L}^\bullet_{K/\bar{\mathbb{Q}}} \subseteq \mathcal{L}^\bullet_{K/L}$ on $CH^p(X_K)$. Thus if $\mathcal{Z} \in \mathcal{L}^i_{K/\bar{\mathbb{Q}}} CH^p(X_K)$ has $\psi_i^{K/L}(\mathcal{Z}) \neq 0$, then trivially $\psi_i^{K/\bar{\mathbb{Q}}}(\mathcal{Z}) \neq 0$. It is a version of the converse that we shall be discussing.

Now the points $\mu \in \eta_{\mathcal{M}}(\mathbb{C})$ correspond to embeddings $\mathrm{ev}_\mu : \bar{\mathbb{Q}}(\mathcal{M}) \hookrightarrow \mathbb{C}$ (with image $=: \bar{\mathbb{Q}}(\mu)$), while the points $s \in \eta_{S_\mu}(\mathbb{C})$ correspond to embeddings $\bar{\mathbb{Q}}(\mathcal{S}) \hookrightarrow \mathbb{C}$ which are ev_μ on $[\rho^*]\bar{\mathbb{Q}}(\mathcal{M})$. Having said this, for $\mu \neq \mu_0$, $\mathfrak{Z}_\mu$ is the partial $\bar{\mathbb{Q}}(\mu)$-spread of $\mathcal{Z}_s \in CH^p(X_{s/\bar{\mathbb{Q}}(s)})$ for any $s \in \eta_{S_\mu}(\mathbb{C})$. Thus we may speak of $\{\mathfrak{Z}_\mu\}_{\mu \in \mathcal{M}(\mathbb{C})}$ as a family of partial spreads, even if they are not all partial spreads of $\mathcal{Z} = \mathcal{Z}_{s_0}$.

So suppose that $\mathcal{Z} \in \mathcal{L}^i_{(K/\bar{\mathbb{Q}})} CH_0(X_K)$ and choose L (or equivalently $\mathcal{M}$) so that $t\,[= \mathrm{trdeg}(K/L)] = i-1$. Clearly for each μ, $[\mathfrak{Z}_\mu]_i = 0$ as $\dim(S_\mu) = t < i$, and so the "higher normal function" $\nu_{\mathfrak{Z}}^{[i-1]}(\mu) := [AJ(\mathfrak{Z}_\mu)]_{i-1}$ is defined. (Note that for $\mu = \mu_0$ this is essentially $\psi_i^{K/L}(\mathcal{Z})$; for general μ, it is equivalent to $\psi_i^{\bar{\mathbb{Q}}(s)/\bar{\mathbb{Q}}(\mu)}(\mathcal{Z}_s)$ where $s \in \eta_{S_\mu}(\mathbb{C})$.)

Expectation:

$(*)_{i+1}:$ $\quad [\mathfrak{Z}]_i \neq 0 \implies$ the function $\nu_{\mathfrak{Z}}^{[i-1]}$ is nonzero (on $\eta_{\mathcal{M}}(\mathbb{C})$) for some choice of L.

That is, $[\mathfrak{Z}]_i$ should be an invariant of $\nu_{\mathfrak{Z}}^{[i-1]}$, its "topological invariant"; this gives a hint for the proof — that a connecting homomorphism δ is involved.

Proposition 7.1. *Expectation (for all i) is* (*a*) *true if X is a surface with a model* $/L$, *and* (*b*) *true modulo GHC if X is of higher dimension with model* $/\bar{\mathbb{Q}}$.

Remark 7.2. In the first instance, the $\bar{\mathbb{Q}}$-spread $\bar{\mathfrak{X}} \to \mathcal{S}$ is a fiber product $\bar{\mathfrak{Y}} \times_{\mathcal{M}} \mathcal{S}$ for some family $\bar{\mathfrak{Y}} \to \mathcal{M}$; hence X_s is constant along fibers of $\mathcal{S} \to \mathcal{M}$.

We prove the second statement (b); the first follows from a similar argument together with the fact that $H^2(\mathcal{U}, \mathcal{F}_h^1 \mathcal{H}^2_{\mathfrak{X}_{\mathcal{U}}/\mathcal{U},\mathbb{Q}}) \hookrightarrow H^2(\mathcal{U}, \mathcal{H}^2_{\mathfrak{X}_{\mathcal{U}}/\mathcal{U},\mathbb{Q}}) \cap F^1 H^2(\mathcal{U}, \mathcal{H}^2_{\mathfrak{X}_{\mathcal{U}}/\mathcal{U}})$ for all $\mathcal{U} \subseteq \mathcal{S}$ s.t. $\mathfrak{X}_{\mathcal{U}} \to \mathcal{U}$ is smooth.

Proof. Observe first that we may essentially reduce to the case where $(s :=)$ $\mathrm{trdeg}(K/\bar{\mathbb{Q}}) = i$. Choose a very general $(s-i)$-fold hyperplane section $\mathcal{S}_0 \subseteq \mathcal{S}$; this will be defined over $L_0[\subseteq K]$ such that $\mathrm{trdeg}(K/L_0) = i$. By affine weak Lefschetz one deduces $\underline{H}^i(\eta_{\mathcal{S}}) \hookrightarrow \underline{H}^i(\eta_{\mathcal{S}_0})$, hence

$$Hom_{\mathrm{MHS}}\left(\mathbb{Q}(-d),\, \underline{H}^i(\eta_{\mathcal{S}}) \otimes H^{2d-i}(X)\right) \hookrightarrow Hom_{\mathrm{MHS}}\left(\mathbb{Q}(-d),\, \underline{H}^i(\eta_{\mathcal{S}_0}) \otimes H^{2d-i}(X)\right)$$

and so $[\mathfrak{Z}]_i \neq 0 \implies [\mathfrak{Z}_0]_i \neq 0$ (where $\mathfrak{Z}_0 := \mathfrak{Z} \cdot (X \times \eta_{\mathcal{S}_0})$). A trivial modification of the argument below (with $\mathfrak{Z}_0$, $\mathcal{S}_0$, L_0 replacing $\mathfrak{Z}$, $\mathcal{S}$, $\bar{\mathbb{Q}}$ and $\mathcal{M}$ of dimension $s-i+1$) then gives the desired conclusion for any L such that $L_0 \overset{\mathrm{trdeg}\,1}{\subseteq} L \subseteq K$.

So assume $\dim \mathcal{S} = i$, $[\mathfrak{Z}]_i \neq 0$, and choose *any* $L \subseteq K$ with $\mathrm{trdeg}(L/\bar{\mathbb{Q}}) = 1 (= \dim \mathcal{M})$; we prove $\{[AJ(\mathfrak{Z}_\mu)]^{tr}_{i-1}\} \neq 0$. We will require the following notation for sheaves of MHS over $\eta_{\mathcal{M}}$: $\mathcal{H}^{i-1}_{\eta_{\mathcal{S}}/\eta_{\mathcal{M}},\mathbb{Q}} = R^{i-1}\rho_* \mathbb{Q}$, $\mathcal{H}^{i-1}_{\eta_{\mathcal{S}}/\eta_{\mathcal{M}}} = R^{i-1}\rho_* \mathbb{C} \otimes \mathcal{O}_{\eta_{\mathcal{M}}}$, with $\underline{\mathcal{H}}^{i-1}$ denoting lowest weight; and $\overline{\mathcal{H}}^{2d-i}_{X,\mathbb{Q}}$ (resp. $\overline{\mathcal{H}}^{2d-i}_{X}$) denotes the *constant* sheaf $\frac{H^{2d-i}(X)}{F_h^{d-i+1}\{\mathrm{num}\}}$ (resp. $\frac{H^{2d-i}(X,\mathbb{C})}{F_h^{d-i+1}\{\mathrm{num}\}}$). Consider the diagram

$$\begin{tikzcd}
\mathcal{L}^i_{(K/\bar{\mathbb{Q}})} CH_0(X_K) \arrow[d, "\cong"] & \\
\mathcal{L}^i CH^d(X \times \eta_{\mathcal{S}/\bar{\mathbb{Q}}}) \arrow[d, "{[\,\cdot\,]_i}"] \arrow[dr, "{\{[AJ(\,\cdot\,_\mu)]^{tr}_{i-1}\}}"] & \\
Hom_{\mathrm{MHS}}\left(\mathbb{Q}(-d), \underline{H}^i(\eta_{\mathcal{S}}) \otimes H^{2d-i}(X)\right) \arrow[d, hook, "(\dagger)"] & H^0\left(\eta_{\mathcal{M}}, Ext^1_{\mathrm{MHS}(\eta_{\mathcal{M}})}\left(\mathbb{Q}(-d), \underline{\mathcal{H}}^{i-1}_{\eta_{\mathcal{S}}/\eta_{\mathcal{M}},\mathbb{Q}} \otimes \overline{\mathcal{H}}^{2d-i}_{X,\mathbb{Q}}\right)\right) \arrow[d, "\delta"] \\
\underline{H}^i(\eta_{\mathcal{S}}) \otimes \frac{H^{2d-i}(X)}{F_h^{d-i+1} H^{2d-i}} \arrow[d, hook, "\text{Leray for } \rho"] & H^1\left(\eta_{\mathcal{M}}, \underline{H}^{i-1}_{\eta_{\mathcal{S}}/\eta_{\mathcal{M}},\mathbb{Q}} \otimes \overline{H}^{2d-i}_{X,\mathbb{Q}}\right) \arrow[d] \\
H^1\left(\eta_{\mathcal{M}}, R^{i-1}\rho_* \mathbb{Q}\right) \otimes \frac{H^{2d-i}(X)}{F_h^{d-i+1} H^{2d-i}} \arrow[r, leftrightarrow, "\cong"] & H^1\left(\eta_{\mathcal{M}}, \mathcal{H}^{i-1}_{\eta_{\mathcal{S}}/\eta_{\mathcal{M}},\mathbb{Q}} \otimes \overline{\mathcal{H}}^{2d-i}_{X,\mathbb{Q}}\right)
\end{tikzcd}$$

where $\mathrm{GHC}(1,i,\mathcal{S}) \implies$ injectivity of (†) by [K1, Lemma II.2] and δ is induced by the short exact sequence

$$0 \to \underline{\mathcal{H}}^{i-1}_{\eta_{\mathcal{S}}/\eta_{\mathcal{M}},\mathbb{Q}} \otimes \overline{\mathcal{H}}^{2d-i}_{X,\mathbb{Q}} \to \frac{\underline{\mathcal{H}}^{i-1}_{\eta_{\mathcal{S}}/\eta_{\mathcal{M}}} \otimes \overline{\mathcal{H}}^{2d-i}_{X}}{\mathcal{F}^d(\text{num})} \to Ext^1_{\text{MHS}(\eta_{\mathcal{M}})}\left(\mathbb{Q}(-d), \underline{\mathcal{H}}^{i-1}_{\eta_{\mathcal{S}}/\eta_{\mathcal{M}},\mathbb{Q}} \otimes \overline{\mathcal{H}^{2d-i}_{X,\mathbb{Q}}}\right) \to 0$$

on $\eta_{\mathcal{M}}$.

To complete the argument, one needs the $\{[AJ(\mathfrak{Z}_\mu)]^{tr}_{i-1}\}$ to be obtained as quotients of $AJ(\overline{\mathfrak{Z}_\mu})$ as in §4.2 (corollary 4.6). This holds if $\overline{\mathfrak{Z}_\mu}$ can be made $\overset{\text{hom}}{\equiv} 0$, which follows from $[[\mathfrak{Z}_\mu]_0 = \cdots = [\mathfrak{Z}_\mu]_{i-1} = 0 \implies]$ $\mathfrak{Z}_\mu \overset{\text{hom}}{\equiv} 0$ provided one assumes [GHC $\implies$] HC for X. In this case, the diagram then commutes because (quite generally) the topological invariant of a (1-D) family of AJ classes (of cycles on a family of projective varieties over $\mathcal{M}$ affine) gives the fundamental class of the family of cycles; we use this combined with the fiberwise Künneth decomposition (for $X \times S_\mu$). The desired conclusion follows at once. $\square$

We note that in case X is a surface (hence $i = 2$) defined $/\bar{\mathbb{Q}}$, no conjectures need to be assumed: $\mathrm{GHC}(1,2,\mathcal{S})$ is just Lefschetz$(1,1)$ for $\mathcal{S}$, and $[\mathfrak{Z}_\mu]_0 = [\mathfrak{Z}_\mu]_1 = [\mathfrak{Z}_\mu]_2 = 0 \implies \exists\ \overline{\mathfrak{Z}_\mu} \overset{\text{hom}}{\equiv} 0$ can be deduced from Lefschetz $(1,1)$ for X; see argument in §6.3.

7.2. **Idle speculations on the Leray filtration.** There is an interesting relationship between the "expectation" above and a conjectural description of the Leray filtration on $\underline{\mathcal{H}^{2d}_{\mathcal{D}}}(\mathfrak{X},\mathbb{Q}(d))$ (and so ultimately on $CH^d(X_{/K})$). Always Leray is for $\pi : \mathfrak{X}_{\mathcal{U}} \to \mathcal{U}$ (or, in the limit, $\mathfrak{X} \to \eta_{\mathcal{S}}$) unless otherwise indicated.

For ordinary cohomology a coarse description of Leray uses the filtration on forms from §5.3,

$$\mathcal{L}^i H^{2d}(\mathfrak{X}_{\mathcal{U}},\mathbb{C}) = \operatorname{im}\left\{\mathbb{H}^{2d}(\mathfrak{X}_{\mathcal{U}},\mathcal{L}^i\Omega^\bullet_{\mathfrak{X}_{\mathcal{U}}}) \to \mathbb{H}^{2d}(\mathfrak{X}_{\mathcal{U}},\Omega^\bullet_{\mathfrak{X}_{\mathcal{U}}}) \cong H^{2d}(\mathfrak{X}_{\mathcal{U}},\mathbb{C})\right\}.$$

A more invariant version defined for $\mathbb{Q}$-coefficients uses the canonical filtration $\tau_{\leq\bullet}$, defined on a complex $K^\bullet$ by

$$\tau_{\leq\ell}K^j := \left\{\begin{array}{ll} K^j\,, & j < \ell \\ (\ker d \subseteq K^\ell), & j = \ell \\ 0\,, & j > \ell \end{array}\right\}.$$

Writing $\mathcal{K}^\bullet$ for a π_*-acyclic resolution of $\mathbb{Q}$ (by sheaves on $\mathfrak{X}_{\mathcal{U}}$),

$$\mathcal{L}^i H^{2d}(\mathfrak{X}_{\mathcal{U}},\mathbb{Q}) = \operatorname{im}\left\{\mathbb{H}^{2d}(\mathcal{U},\tau_{\leq 2d-i}\pi_*\mathcal{K}^\bullet) \to \mathbb{H}^{2d}(\mathcal{U},\pi_*\mathcal{K}^\bullet) \cong H^{2d}(\mathfrak{X}_{\mathcal{U}},\mathbb{Q})\right\}.$$

Finally, there is the following description due to Arapura [Ar] (worked out with Nori):

$$\mathcal{L}^i H^{2d}(\mathfrak{X}_{\mathcal{U}}, \mathbb{Q}) = \bigcap_{\mathcal{V} \subseteq \mathcal{U}} \ker \left\{ H^{2d}(\mathfrak{X}_{\mathcal{U}}, \mathbb{Q}) \to H^{2d}(\mathfrak{X}_{\mathcal{V}}, \mathbb{Q}) \right\},$$

where the intersection is over $(i-2)$-dimensional smooth subvarieties $\mathcal{V} \subseteq \mathcal{U}$ defined $/\mathbb{C}$. In the limit this becomes

$$\mathcal{L}^i H^{2d}(\mathfrak{X}, \mathbb{Q}) = \bigcap_{\mathcal{T} \subseteq \mathcal{S}} \ker \left\{ H^{2d}(\mathfrak{X}, \mathbb{Q}) \to H^{2d}(\mathfrak{X}_{\eta_{\mathcal{T}}}, \mathbb{Q}) \right\}$$

where $\mathcal{T}$ must be *very general* (of dim. $i-2$) in $\mathcal{S}$ for the restriction from $\eta_{\mathcal{S}}$ to $\eta_{\mathcal{T}}$ to make sense. (No rational function $\in \bar{\mathbb{Q}}(\mathcal{S})^*$ restricts to 0 on $\mathcal{T}$.) In terms of fields, if $\bar{\mathbb{Q}}(\mathcal{S}) \xrightarrow{\cong} K \subseteq \mathbb{C}$ is fixed, the choice of $\eta_{\mathcal{T}} \subseteq \eta_{\mathcal{S}}$ is essentially equivalent to a choice of $L \subseteq K$ s.t. $\mathrm{trdeg}(K/L) = i-2$ (which gives $\mathcal{M}$ with $\mathcal{S} \xrightarrow{\rho} \mathcal{M}$ of rel. dim. $i-2$) together with another independent embedding $L \hookrightarrow \mathbb{C}$ (which gives a choice of point $\mu \in \eta_{\mathcal{M}}(\mathbb{C})$, hence fiber of ρ). Alternatively, one could choose freely another complex embedding $\tilde{K}$ of $\bar{\mathbb{Q}}(\mathcal{S})$, then just choose $L \subseteq \tilde{K}$.

Does a similar description hold for lowest-weight absolute Hodge cohomology, viz.

$$(\sharp)_i : \quad \mathcal{L}^i \underline{H_{\mathcal{D}}^{2d}}(\mathfrak{X}, \mathbb{Q}(d)) \overset{(?)}{=} \bigcap_{\substack{\mathcal{T} \subseteq \mathcal{S} \text{ v.g.} \\ (i-2)\text{-dim'l.}}} \ker \left\{ \underline{H_{\mathcal{D}}^{2d}}(\mathfrak{X}, \mathbb{Q}(d)) \overset{\mathcal{I}_{\eta_{\mathcal{T}}}^*}{\hookrightarrow} \underline{H_{\mathcal{D}}^{2d}}(\mathfrak{X}_{\eta_{\mathcal{T}}}, \mathbb{Q}(d)) \right\},$$

at least for the image $\left(\underline{H_{\mathcal{D}}^{2d}}(\mathfrak{X}, \mathbb{Q}(d)) \right)_{\mathrm{alg}}$ of the cycle map $\underline{c_{\mathcal{D}}}$? Modulo the following Christmas Wish we can show it is at least equivalent to the Expectation above:

Conjecture $(CW)_i$**:** If $\mathcal{Z} \in \mathcal{L}^{i-1}_{(K/\bar{\mathbb{Q}})} CH^d(X_{/K})$ with $[\mathfrak{Z}]_{i-1} = 0$ and $[AJ(\mathfrak{Z}|_{\mathcal{T}})]_{i-2} = 0 \in Gr_{\mathcal{L}}^{i-2} \underline{J^d}(\mathfrak{X}_{\eta_{\mathcal{T}}})$ for all $\mathcal{T} \subseteq \mathcal{S}$ v.g. of dim. $(i-2)$, then $[AJ(\mathfrak{Z})]_{i-2} = 0 \in Gr_{\mathcal{L}}^{i-2} \underline{J^d}(\mathfrak{X})$.

Remark 7.3. It is easy to see that under these circumstances $[AJ(\mathfrak{Z})]_{i-2}^{tr} = 0$, but the extra little bit seems nontrivial.

Proposition 7.4. $(\sharp)_i^{alg} \implies (*)_i$*; and assuming* $(CW)_{\leq i}$, $(*)_{\leq i} \implies (\sharp)_{\leq i}^{alg}$.

Proof. Consider the following five conditions on a cycle $\mathcal{Z} \in CH^d(X_K)$.

$(A)_i$: $\mathcal{Z} \in \mathcal{L}^i_{(K/\bar{\mathbb{Q}})} CH^d(X_K)$;

$(B)_i$: $\psi(\mathcal{Z}) \in \mathcal{L}^i \underline{H_{\mathcal{D}}}^{2d}(\mathfrak{X}, \mathbb{Q}(d))$;

$(C)_i$: $\forall \left\{\mathcal{T} \subseteq \mathcal{S} \text{ v.g. } (i-2)\text{-dim'l.}\right\}, \mathcal{I}^*_{\eta_{\mathcal{T}}}(\psi(\mathcal{Z})) = 0 \in \underline{H_{\mathcal{D}}}^{2d}(\mathfrak{X}_{\eta_{\mathcal{T}}}, \mathbb{Q}(d))$;

$(D)_i$: $\forall \left\{\mathcal{M}/\bar{\mathbb{Q}},\ \rho : \mathcal{S} \to \mathcal{M} \text{ rel. dim.} i-2\right\}, 0 = \{[AJ(\mathfrak{z}_\mu)]_{i-2}\}_{\mu \in \eta_{\mathcal{M}}(\mathbb{C})}$;

$(E)_i$: $[\mathfrak{z}]_{i-1} = 0$ and $[AJ(\mathfrak{z})]_{i-2} = 0$.

(Note that $(D)_i$ and $(E)_i$ require an auxiliary condition, for example $(A)_{i-1}$, in order to have meaning.) Clearly $(A)_i$, $(B)_i$, and $(E)_{\leq i}$ are equivalent; while $(C)_i \implies (D)_i$ and $(A)_{i-1} + (D)_i \implies (C)_i$ from the above discussion. Since $\mathcal{L}^i \underline{H_{\mathcal{D}}}^{2d}(\mathfrak{X}_{\eta_{\mathcal{T}}}, \mathbb{Q}(d)) = 0$ and $\mathcal{I}^*_{\eta_{\mathcal{T}}}(\mathcal{L}^\bullet) \subseteq \mathcal{L}^\bullet$, $(B)_i \implies (C)_i$; the converse $(C)_i \implies (B)_i$ is the nontrivial content of $(\sharp)_i^{\text{alg}}$. Trivially $(A)_{i-1} + (E)_i \implies (D)_i$; the reverse $(A)_{i-1} + (D)_i \implies (E)_i$ is equivalent to $(*)_i + (CW)_i$.

So for the first statement let $(\sharp)_i^{\text{alg}}$ be given; then $(A)_{i-1} + (D)_i \implies (C)_i \overset{(\sharp)_i^{\text{alg}}}{\implies} (B)_i \implies (E)_i$ so $(*)_i$ holds.

For the "converse", inductively assume $(*)_{\leq i-1} + (CW)_{\leq i-1} \implies \{(C)_{\leq i-1} \implies (A)_{\leq i-1}\} + (\sharp)^{\text{alg}}_{\leq i-1}$. Then if $(*)_{\leq i} + (CW)_{\leq i}$, we have $(C)_i \implies (C)_{\leq i} \overset{\text{ind. hyp.}}{\implies} (D)_{\leq i} + (A)_{\leq i-1} \overset{(*)_i + (CW)_i}{\implies} (E)_{\leq i} \implies (A)_i + (B)_i$. $\square$

If it held, $(\sharp)^{\text{alg}}$ would offer an interesting interpretation of $\mathcal{L}^\bullet$ on $CH^d(X_K)$. Let $\iota : K \overset{\cong}{\to} \tilde{K} \subseteq \mathbb{C}$ be *any* embedding of K (even if K comes already equipped with one) that is the identity on $\bar{\mathbb{Q}}$, and $L \subseteq K$ be any subfield $[\supset \bar{\mathbb{Q}}]$ with $t = \operatorname{trdeg}(\tilde{K}/L) = i - 2$. Write $\psi_K^{\tilde{K}/L}$ for the composition $\psi^{\tilde{K}/L} \circ \iota^*$:

$$CH^d(X_K) \xrightarrow[\iota^*]{\cong} CH^d(X_{\tilde{K}}) \xrightarrow[\text{part. sprd.}]{\cong} CH^d(\mathfrak{X}_{\eta_{\mathcal{T}}/L}) \xrightarrow[c_{\mathcal{D}}]{} \underline{H_{\mathcal{D}}}^{2d}(\mathfrak{X}_{\eta_{\mathcal{T}}}, \mathbb{Q}(d));$$

then $(\sharp)_i^{\text{alg}} \implies$

$$\mathcal{L}^i CH^d(X_K) = \bigcap_{\substack{\iota, \tilde{K}, L \\ [t=i-2]}} \ker(\psi_K^{\tilde{K}/L}).$$

Note that the $c_{\mathcal{D}}$ in the composition is *not* supposed to be injective (unless $L = \bar{\mathbb{Q}}$).

7.3. **Specialization to lower transcendence degree.** We now derive a powerful technique for producing 0-cycles with nontrivial higher AJ invariant (due to [GGP] for surfaces and M. Saito [mS] in the general case). Here is the idea: given $\mathcal{Z} \in \mathcal{L}^i CH_0(X_{/K})$ we can lift its $\bar{\mathbb{Q}}$-spread $\mathfrak{Z}$ to $\bar{\mathfrak{Z}} \in CH^d(\bar{\mathfrak{X}}_{/\bar{\mathbb{Q}}})$, and restrict along the inclusion $\mathcal{I}_{\mathcal{T}} : \mathfrak{X}_{\mathcal{T}} \hookrightarrow \mathfrak{X}_{\mathcal{S}} = \bar{\mathfrak{X}}$ (for $\mathcal{T} \subseteq \mathcal{S}$ a smooth $\bar{\mathbb{Q}}$-subvariety) to obtain $\overline{\mathfrak{Z}_{\mathcal{T}}} \in CH^d(\mathfrak{X}_{\mathcal{T}/\bar{\mathbb{Q}}}) \mapsto \mathfrak{Z}_{\mathcal{T}} \in CH^d(\mathfrak{X}_{\eta_{\mathcal{T}}/\bar{\mathbb{Q}}}) \overset{(\mathrm{I}^{-1})^*}{\longmapsto} \mathcal{Z}_{\mathcal{T}} \in CH_0(X_{/K_{\mathcal{T}}})$ where $\mathrm{I} : \bar{\mathbb{Q}}(\mathcal{T}) \overset{\cong}{\to} K_{\mathcal{T}} \, [\subseteq \mathbb{C}]$. We call this a $K_{\mathcal{T}}$-specialization of $\mathcal{Z}$, and it is not a well-defined operation (nor is $\mathfrak{X}_{\eta_{\mathcal{T}}}$ a subvariety of $\mathfrak{X}_{(\eta_{\mathcal{S}})}$, nor is there directly a map of function fields $\bar{\mathbb{Q}}(\mathcal{S}) \to \bar{\mathbb{Q}}(\mathcal{T})$). This is in contrast to the much nicer situation (§§7.1 − 2) when $\mathcal{T} \subseteq \mathcal{S}$ is very general, because here $\mathfrak{Z} \cap \mathfrak{X}_{\eta_{\mathcal{T}}}$ makes no sense. Hence the lift from $\mathfrak{Z}$ to $\bar{\mathfrak{Z}}$, which is not well-defined, plays an essential role in the passage from $\mathfrak{Z}$ to $\mathfrak{Z}_{\mathcal{T}}$.

However, the map $\underline{H^*}(\eta_{\mathcal{S}}) \to \underline{H^*}(\eta_{\mathcal{T}})$ *is* well-defined; consequently so are the maps of invariants $[\,\cdot\,]_j$ and $[AJ(\,\cdot\,)]_j^{tr}$ for the specialization of spreads from $\mathfrak{X}$ to $\mathfrak{X}_{\eta_{\mathcal{T}}}$. In fact, using a version of affine weak Lefschetz, one shows these are injective for $j \leq \dim(\mathcal{T})$ (cf. [K1, Lemma 3]). This of course does nothing for us unless e.g. both $[\mathfrak{Z}]_i$ *and* $[\mathfrak{Z}_{\mathcal{T}}]_i$ are defined, i.e. $\mathfrak{Z}$ and $\mathfrak{Z}_{\mathcal{T}}$ belong to $\mathcal{L}^i$ of $CH_0(X_{/K})$ and $CH_0(X_{/K_{\mathcal{T}}})$ (resp.). Even when $X/\bar{\mathbb{Q}}$, $\mathcal{Z} \in \mathcal{L}^i$ does not necessarily imply[26] $\mathcal{Z}_{\mathcal{T}} \in \mathcal{L}^i$ — one needs the stronger condition $c_{\mathcal{D}}(\bar{\mathfrak{Z}}) \in \mathcal{L}^i_{\mathcal{S}} H^{2d}_{\mathcal{D}}(X \times \mathcal{S}, \mathbb{Q}(d))$,[27] which may be obtained from $\underline{c_{\mathcal{D}}}(\mathfrak{Z}) \in \mathcal{L}^i \underline{H^{2d}_{\mathcal{D}}}(X \times \eta_{\mathcal{S}}, \mathbb{Q}(d))$ $[\Leftarrow \mathcal{Z} \in \mathcal{L}^i]$ only after assuming HC (see [mS, sec. 1.6]). One "exception" would be if $\mathcal{Z} = \langle \Delta_{X,\,2d-i} \rangle_* \mathcal{Z}$ is the image of a Chow-Künneth component; then choosing $\bar{\mathfrak{Z}}$ with $c_{\mathcal{D}}(\bar{\mathfrak{Z}}) \in \mathcal{L}^i_{\mathcal{S}}$ is trivial.

If one does assume HC, it is easy to specialize $\mathcal{Z} \in \mathcal{L}^i CH_0(X_{/K})$ with $[\mathfrak{Z}]_i \neq 0$ to a cycle $\mathcal{Z}_{\mathcal{T}}$ of $\mathrm{trdeg}_{/\bar{\mathbb{Q}}}(K_{\mathcal{T}}) = \dim(\mathcal{T}) = i$ with $[\mathfrak{Z}_{\mathcal{T}}]_i \neq 0$. One has to go much deeper to find a transcendence-degree-$(i-1)$ specialization (necessarily $[\mathfrak{Z}_{\mathcal{T}}]_i = 0$) such that $[AJ(\mathfrak{Z}_{\mathcal{T}})]^{tr}_{i-1} \neq 0$. This is now our objective. Intuitively, we want a version of $(\sharp)^{\mathrm{alg}}_i$ with $\mathcal{T}/\bar{\mathbb{Q}}$ replacing $\mathcal{T}$ very general; unfortunately, in this case $\mathcal{I}^*_{\eta_{\mathcal{T}}}$ is not well-defined.

A better analogy is $(*)_{i+1}$, but with $\nu_{\mathfrak{Z}}^{[i-1]}(\mu) \neq 0$ *for some* $\mu \in \mathcal{M}(\bar{\mathbb{Q}})$, instead of $\mu \in \mathcal{M}(\mathbb{C})$ very general. Here now is a precise statement of what holds (where we take $\mathrm{trdeg}(K/k) = i$ since we have some idea of how to specialize *to* this situation):

Proposition 7.5. *Let X be defined over a number field k, $\mathcal{Z} \in \mathcal{L}^i CH_0(X_K)$ with $K \cong k(\mathcal{S})$ f.g.$_{/\bar{\mathbb{Q}}}$ of trdeg. i and $[\mathfrak{Z}]_i \neq 0$. Take a lift $\bar{\mathfrak{Z}} \in$*

---

[26] The difficulty here is with the full invariants $[AJ(\cdot)]_j$ for $j < i-1$ (as opposed to the quotient indicated by "tr") — it does *not* seem clear that $[AJ(\mathfrak{Z})]_j = 0 \implies [AJ(\mathfrak{Z}_{\mathcal{T}})]_j = 0$.

[27] $\mathcal{L}^i_{\mathcal{S}}$ denotes Leray for $X \times \mathcal{S} \to \mathcal{S}$

$CH^d(X \times \mathcal{S}_{/k})$ of $\mathfrak{Z}$ with $c_{\mathcal{D}}(\bar{\mathfrak{Z}}) \in \mathcal{L}^i_{\mathcal{S}}$ (assuming HC), and a Lefschetz pencil[28] *$\mathcal{S} \xrightarrow{\bar{\rho}} \mathbb{P}^1_k$ with fibers $S_\mu := \bar{\rho}^{-1}(\mu)$. Assume $\mathcal{L}^\bullet = F^\bullet_{\mathcal{M}_\ell}$ (or BBC^q) and the GHC. Then $\exists\ \mu \in \mathbb{P}^1(\bar{\mathbb{Q}})$ s.t. $[AJ(\mathfrak{Z}_\mu)]_{i-1} \neq 0$, where $\mathfrak{Z}_\mu$ is the restriction to $X \times \eta_{S_\mu}$ of $\bar{\mathfrak{Z}} \cdot (X \times S_\mu)$. So we get a specialized cycle $\mathcal{Z}_\mu \in \mathcal{L}^i CH_0(X_{/\bar{\mathbb{Q}}(S_\mu)}) \setminus \mathcal{L}^{i+1}$ of trdeg. $i-1$.*

Proof. We work in the category $(\mathcal{M} =)\mathcal{M}_\ell$ unless otherwise stated, see §5.5(ii) (and also [mS] for additional details in what follows).

Pick $\mathsf{U} \subseteq \mathbb{P}^1_k$ affine Zariski open and sufficiently small that $\mathcal{S}_{\mathsf{U}} \xrightarrow{\rho} \mathsf{U}$ has smooth fibers; let $\mu \in \mathsf{U}(k)$. Set $\mathcal{H} := H^i(X) \cong H^{2d-i}(X)^\vee$, $\underline{\mathcal{H}} := \frac{H^i(X)}{F^1_h} \cong \left(\frac{H^{2d-i}(X)}{F^{d-i+1}_h}\right)^\vee$; pulling back along the structure morphism $\mathsf{U} \xrightarrow{a} \operatorname{Spec} k$ yields $\mathcal{H}_{\mathsf{U}} := a^*\mathcal{H}$, $\underline{\mathcal{H}}_{\mathsf{U}} := a^*\underline{\mathcal{H}}$.

In $\mathcal{M}(\mathsf{U})$ we have by semisimplicity a decomposition

$$R^{i-1}\rho_*\mathbb{Q} \;=\; L' \oplus L'' \;=\; \text{variable} \oplus \text{constant}$$

where $H^0(\mathsf{U}, L') = 0$ and $W_i H^1(\mathsf{U}, L'') = W_1 H^1(\mathsf{U}) \otimes L''_\mu = 0$ (since $\mathsf{U} \subseteq \mathbb{P}^1$). Hence one has noncanonically

$$H^i(\mathcal{S}_{\mathsf{U}}) \cong H^0(\mathsf{U}, R^i\rho_*\mathbb{Q}) \oplus H^1(\mathsf{U}, L') \oplus H^1(\mathsf{U}, L'')$$

and (using in addition that the fibers of ρ are $(i-1)$-dimensional)

$$W_i H^i(\mathcal{S}_{\mathsf{U}}) \cong W_i H^1(\mathsf{U}, L').$$

Moreover, the short exact sequence $\mathcal{N}^1(R^{i-1}\rho_*\mathbb{Q}) \cap L' =: \mathcal{N}^1 L' \to L' \to L'/\mathcal{N}^1 L'$ splits and $H^1(\mathsf{U}, \mathcal{N}^1 L') \subseteq F^1_h H^1(\mathsf{U}, L') \implies$

$$\big(W_i H^i(\mathcal{S}_{\mathsf{U}})\big)\big/\, F^1_h \;\cong\; \big(W_i H^1(\mathsf{U},\, L'/\mathcal{N}^1 L')\big)\big/\, F^1_h\,.$$

Now $W_i\underline{\mathcal{H}}/F^1_h \;=\; \underline{\mathcal{H}}$ and so

$$Hom_{\mathcal{M}}\left(\underline{\mathcal{H}},\, H^i(\mathcal{S}_{\mathsf{U}})\right) \;\cong\; Hom_{\mathcal{M}}\left(\underline{\mathcal{H}},\, H^1(\mathsf{U}, L'/\mathcal{N}^1 L')\right). \tag{7.1}$$

For any object $\mathcal{G} \in \mathcal{M}(\mathsf{U})$ one has a spectral sequence[29]

$$E_2^{p,q} \;=\; Ext^p_{\mathcal{M}}\left(\underline{\mathcal{H}},\, H^q(\mathsf{U}, \mathcal{G})\right) \;\implies\; Ext^{p+q}_{\mathcal{M}(\mathsf{U})}\left(\underline{\mathcal{H}}_{\mathsf{U}},\, \mathcal{G}\right),$$

and in particular an exact sequence

$$0 \to Ext^1_{\mathcal{M}}\left(\underline{\mathcal{H}},\, H^0(\mathsf{U}, \mathcal{G})\right) \to Ext^1_{\mathcal{M}(\mathsf{U})}\left(\underline{\mathcal{H}}_{\mathsf{U}},\, \mathcal{G}\right) \to$$

---

[28] Here either $\mathcal{S}$ itself is suitably blown up or this is really $\mathcal{S} \xleftarrow{b} \mathfrak{S} \xrightarrow{\bar{\rho}} \mathbb{P}^1_k$. It makes no difference since we restrict immediately to open $\mathsf{U} \subseteq \mathbb{P}^1_k$; in fact, the pencil need not even be Lefschetz since all we use is that $\mathcal{S}_{\mathsf{U}} \xrightarrow{\rho} \mathsf{U}$ is smooth projective.

[29] Grothendieck spectral sequence for derived functors of $Hom_{\mathcal{M}_\ell(\mathsf{U})}(\underline{\mathcal{H}}_{\mathsf{U}}, \text{—}) = Hom_{\mathcal{M}_\ell}(\underline{\mathcal{H}}, \Gamma(\mathsf{U}, \text{—}))$; $\mathcal{M}_\ell(\mathsf{U})$ is sheaves of $\mathcal{M}_\ell$-objects.

$$Hom_{\mathcal{M}}\left(\underline{\mathcal{H}},\, H^1(\mathsf{U},\mathcal{G})\right) \to Ext^2_{\mathcal{M}}\left(\underline{\mathcal{H}},\, H^0(\mathsf{U},\mathcal{G})\right).$$

Applying this to $\mathcal{G} = L'/\mathcal{N}^1L'$, with the observation $H^0(\mathsf{U}, L'/\mathcal{N}^1L') = 0$, gives

$$(7.2) \qquad Hom_{\mathcal{M}}\left(\underline{\mathcal{H}},\, H^1(\mathsf{U}, L'/\mathcal{N}^1L')\right) \cong Ext^1_{\mathcal{M}(\mathsf{U})}\left(\underline{\mathcal{H}}_{\mathsf{U}},\, L'/\mathcal{N}^1L'\right).$$

Referring to §5.5(ii), consider the following diagram

$$(7.3)$$

$$\begin{tikzcd}
 & Hom_{\mathcal{M}}\left(\underline{\mathcal{H}},\, H^i(\mathcal{S}_{\mathsf{U}})\right) \arrow[d, leftrightarrow, "\cong"] \\
 & Hom_{\mathcal{M}}\left(\mathbb{Q}(-d),\, \underline{H}^i(\mathcal{S}_{\mathsf{U}}) \otimes \frac{H^{2d-i}(X)}{F_h^{d-i+1}}\right) \arrow[d, twoheadrightarrow] \\
Hom_{\mathcal{M}}\left(\mathbb{Q}(-d),\, \underline{H}^i(\eta_{\mathcal{S}}) \otimes H^{2d-i}(X)\right) \arrow[r] \arrow[d] & Hom_{\mathcal{M}}\left(\mathbb{Q}(-d),\, \underline{H}^i(\eta_{\mathcal{S}}) \otimes \frac{H^{2d-i}(X)}{F_h^{d-i+1}}\right) \arrow[d] \\
Hom_{\text{MHS}}\left(\mathbb{Q}(-d),\, \underline{H}^i(\eta_{\mathcal{S}}) \otimes H^{2d-i}(X)\right) \arrow[r] & Hom_{\text{MHS}}\left(\mathbb{Q}(-d),\, \underline{H}^i(\eta_{\mathcal{S}}) \otimes \frac{H^{2d-i}(X)}{F_h^{d-i+1}}\right)
\end{tikzcd}$$

in the lower left corner of which lies $[\mathfrak{Z}]_i$. Since we are assuming $\mathcal{L}^i = F^i_{\mathcal{M}}$ on $CH_0(X_K)$, $\mathcal{Z}$ has a class $[\mathfrak{Z}]_i^{\mathcal{M}} \mapsto [\mathfrak{Z}]_i$ (in the diagram); and since we are assuming GHC, the horizontal arrows are injective. Since $[\mathfrak{Z}]_i \neq 0$ by hypothesis, we get a nonzero element $\widetilde{[\mathfrak{Z}]_i^{\mathcal{M}}} \in Hom_{\mathcal{M}}(\underline{\mathcal{H}}, H^i(\mathcal{S}_{\mathsf{U}}))$, hence one in $Ext^1_{\mathcal{M}_\ell(\mathsf{U})}(\underline{\mathcal{H}}_{\mathsf{U}}, L'/\mathcal{N}^1L')$ by (7.1) and (7.2).

The crucial step is that $\mu \in \mathsf{U}(k)$ can be chosen so that the specialization of this nontrivial extension class to $Ext^1_{\mathcal{M}_\ell}(\underline{\mathcal{H}},\, L'_\mu/N^1L'_\mu)$ is still nonzero.[30] (The fiber S_μ over μ is a $\bar{\mathbb{Q}}$-member of the pencil.) This is really a statement about the underlying ℓ-adic structure of the

---

[30] Technical note: *a priori* one gets a nonzero specialized class only in $Ext^1_{\mathcal{M}_\ell}(\underline{\mathcal{H}},\, L'_\mu/(\mathcal{N}^1L')_\mu)$, where $(\mathcal{N}^1L')_\mu$ may be a proper submodule of $N^1(L'_\mu)$. The choice of μ (see [mS, sec. 2.1]) allows $N^1(L'_\mu)$ to be extended to a subsheaf $L'_1 \subseteq L'$ on a finite cover of U. Repeating the arguments leading to (7.1), (7.2) with L'_1 replacing $\mathcal{N}^1L'$ (as L'_1 is likewise $\subseteq F^1_hL'$) shows that $Ext^1_{\mathcal{M}_\ell(\mathsf{U})}(\underline{\mathcal{H}}_{\mathsf{U}},\, L'/\mathcal{N}^1L') \overset{\cong}{\to} Ext^1_{\mathcal{M}_\ell(\mathsf{U})}(\underline{\mathcal{H}}_{\mathsf{U}},\, L'/L'_1)$. One then argues that the image of our extension class in the latter has nonzero specialization (at the same μ)

$\mathcal{M}_\ell$-objects, *not the MHS*: i.e., nontrivial extensions of ℓ-adic sheaves on U may be specialized to nontrivial extensions of ℓ-adic $Gal(\bar{k}/k)$ representations. This follows from the Hilbert irreducibility theorem, see [mS], [Te], [GGP] for details.

Now $\underline{H^{i-1}}(\eta_{S_\mu})$ is just the stalk of $(R^{i-1}\rho_*\mathbb{Q})/\mathcal{N}^1$ at μ, so $L'_\mu/N^1L'_\mu$ is a sub-$\mathcal{M}_\ell$-object, and

$$\begin{aligned} Ext^1_{\mathcal{M}_\ell}\left(\underline{\mathcal{H}},\, L'_\mu/N^1L'_\mu\right) &\hookrightarrow Ext^1_{\mathcal{M}_\ell}\left(\underline{\mathcal{H}},\, \underline{H^{i-1}}(\eta_{S_\mu})\right) \\ &\cong Ext^1_{\mathcal{M}_\ell}\left(\mathbb{Q}(-d),\, \underline{H^{i-1}}(\eta_{S_\mu}) \otimes \frac{H^{2d-i}(X)}{F_h^{d-i+1}}\right) \\ &\hookrightarrow Ext^1_{\mathcal{M}_\ell}\left(\mathbb{Q}(-d),\, H^{i-1}(\eta_{S_\mu}) \otimes \frac{H^{2d-i}(X)}{F_h^{d-i+1}}\right). \end{aligned}$$

Let ξ_μ denote the ultimate (nonzero) image of our class in the last term. Since we are assuming (G)HC, the specialized cycle $\mathcal{Z}_\mu$ lies in $\mathcal{L}^i$, hence (by the other assumption) $F^i_\mathcal{M}$; obviously $[\mathfrak{Z}_\mu]_i$ and $[\mathfrak{Z}_\mu]_i^\mathcal{M}$ are 0. Therefore $\mathcal{Z}_\mu$ has an invariant "$[AJ(\mathfrak{Z}_\mu)]_i^\mathcal{M}$" in $Gr^1_\mathcal{E} Gr^i_{\mathcal{L}_{\mathcal{M}_\ell}} H^{2d}_{\mathcal{D},\mathcal{M}_\ell}\left(X \times \eta_{S_\mu}/k,\, \mathbb{Q}(d)\right)$ *also* mapping to ξ_μ. Hence $[AJ(\mathfrak{Z}_\mu)]^\mathcal{M}_{i-1} \neq 0$ and $\mathcal{Z}_\mu \notin F^{i+1}_\mathcal{M}$. Using our assumption on the filtrations once more ($F^{i+1}_\mathcal{M} = \mathcal{L}^{i+1}$), we get $\mathcal{Z}_\mu \notin \mathcal{L}^{i+1}$; *therefore* $[AJ(\mathfrak{Z}_\mu)]_{i-1} \neq 0$. Without the assumption there is no reason why this should be so! $\square$

Now suppose $X/\bar{\mathbb{Q}}$ is a smooth (d-dimensional) complete intersection in $\mathbb{P}^N$ with a nontrivial holomorphic d-form. The diagonal Δ_X has a Chow-Künneth decomposition $/\bar{\mathbb{Q}}$ and the 0-cycle $\mathcal{Z} = p - o$, for $p \in X(\mathbb{C})$ very general and $o/\bar{\mathbb{Q}}$, satisfies $\mathcal{Z} = \langle \Delta_{X,d} \rangle_* \mathcal{Z}$. Therefore $\mathcal{Z}$ and any specializations (*without* using HC) lie in $F^d_\mathcal{S} CH_0(X_{/\mathbb{C}})$, *a fortiori* in $F^d_\mathcal{M}$ and $\mathcal{L}^d$. Furthermore, taking $\mathcal{S} = X$, $\mathfrak{Z}$ induces a nontrivial map $\Omega^i(X) \to \Omega^i(\mathcal{S})$ and hence one need not use GHC to get the nonzero classes on the right-hand side of the diagram (7.3). So by the proof of Prop. (7.5) above, assuming *no* conjectures, we have $\mathcal{Z}_\mu \notin F^{d+1}_\mathcal{M}$. Taking $N = d + r$, we have proved the following

Theorem 7.6. [with no conjectural assumptions] *Let $X \subseteq \mathbb{P}^{d+r}$ be a smooth complete intersection defined $/\bar{\mathbb{Q}}$, of multidegree $(D_1, \ldots, D_r)$ with $\sum D_j \geq d + r + 1$, and let $o \in X(\bar{\mathbb{Q}})$. Then there exists a $\bar{\mathbb{Q}}$-hyperplane section $Y \overset{\iota_Y}{\hookrightarrow} X$ containing o, such that if $q \in Y(\mathbb{C})$ is very general then $\iota_Y(q) - o \overset{rat}{\not\equiv} 0$ in $\mathcal{L}^d CH_0(X_{/\mathbb{C}})$.*

---

in $Ext^1_{\mathcal{M}_\ell}(\underline{\mathcal{H}},\, L'_\mu/L'_{1,\mu} = L'_\mu/N^1L'_\mu)$. See the last 2 paragraphs of [mS, sec. 2.3] for details.

We emphasize that under the identification $\mathcal{S} = X$, $Y = S_\mu$ and $\mathcal{Z}_\mu = \iota_Y(q) - o$. (Moreover, $\mathcal{S}[\leftarrow \mathfrak{S}] \xrightarrow{\bar{\rho}} \mathbb{P}^1$ is taken to be a pencil of hyperplane sections with base locus containing o.) Incidentally, it is obvious that $q - o \overset{\text{rat}}{\not\equiv} 0$ on Y, since $\Omega^{d-1}(Y) \neq \{0\}$ and the spread of q induces the identity $\Omega^{d-1}(Y) \to \Omega^{d-1}(S_\mu)$. So $q - o$ gives a nonzero element of $Gr_{\mathcal{L}}^{d-1} CH_0(Y_{/\mathbb{C}})$. What make things hard on X is that fact that $\iota_Y(q) - o$ is of trdeg. $d-1$ and $\Omega^{d-1}(X) = \{0\}$.

In the Proposition and Theorem the spread base $\mathcal{S}$ has been fibered over a *rational* curve (unlike the $\mathcal{M}$ in §7.1); what if the curve (say, $\mathcal{C}$) is of positive genus? The only result in this direction takes $\mathcal{S} = S \times \mathcal{C}$, and is adapted from [LS]:

Theorem (Lewis): Let $\mathcal{C}$, S, X be smooth projective $/\bar{\mathbb{Q}}$ of resp. dimensions 1, $i-1$, d. Take $\mathcal{Z} \in \mathcal{L}^i CH_0(X_K)$ to be a cycle (of trdeg. i) with complete $\bar{\mathbb{Q}}$-spread $\bar{\mathfrak{Z}} \in CH^d(\mathcal{C} \times S \times X_{/\bar{\mathbb{Q}}})$ inducing a nontrivial map $\Omega^i(X) \to \Omega^i(\mathcal{C} \times S)$. Then there exists $\gamma \in CH_0^{\text{hom}}(\mathcal{C}_{/\bar{\mathbb{Q}}})$ s.t. writing $\bar{\mathfrak{Z}}_\gamma := \pi_{S \times X} \left\{ \pi_{\mathcal{C}}^*(\gamma) \cdot \bar{\mathfrak{Z}} \right\}$, we have $[AJ(\mathfrak{Z}_\gamma)]_{i-1}^{tr} \neq 0$. Pulling $\mathfrak{Z}_\gamma$ back along the inclusion $\{s_0\} \times X \hookrightarrow S \times X$ for $s_0 \in S(\mathbb{C})$ v.g., we get the specialized cycle $\mathcal{Z}_\gamma \in \mathcal{L}^i CH_0(X_{/K_0}) \setminus \mathcal{L}^{i+1}$ of trdeg. $i-1$.

The crucial step in the proof (which plays the role of the Hilbert-Terasoma argument) is the existence of a $\bar{\mathbb{Q}}$-point of infinite rank on any abelian variety $/\bar{\mathbb{Q}}$ of positive dimension, applied to a certain sub-abelian-variety of $CH_0^{\text{hom}}(\mathcal{C}_{/\bar{\mathbb{Q}}})$. A similar result for symmetric squares of curves, together with the proof of this fact, can be found in [GGP].

8. Applications to Calabi-Yau varieties

In this final section we further illustrate the use of theorems from §§4, 6, and 7, on terrain probably quite familiar to readers of this volume. The results here are in the nature of straightforward corollaries rather than profound new theorems.

For us a "CY n-fold" shall mean a smooth projective variety X (of dimension n, defined over a subfield of $\mathbb{C}$), such that $X_{\mathbb{C}}^{(an)}$ has trivial canonical bundle and $H^0(\Omega_X^k)$ vanishes for $0 < k < n$. So all CY's (and also K3 surfaces) in this section are algebraic (in fact, they will all be defined $/\bar{\mathbb{Q}}$). Our main purpose will be to construct 0-cycles in $\ker(Alb_X)$ which are not rationally equivalent to zero, hence (as we work $\otimes\mathbb{Q}$) of infinite order in $CH_0(X_{\mathbb{C}})$.

8.1. **Projective complete intersections.** Let $X_{/\bar{\mathbb{Q}}} \subseteq \mathbb{P}^{n+r}$ be smooth of multidegree $(D_1, \ldots, D_r)$, $\sum D_i =: D = n + r + 1$. This is a CY n-fold, a basic example being the Fermat quintic 3-fold in $\mathbb{P}^4$. The diagonal class $[\Delta_X] \in [H^{2n}(X \times X, \mathbb{Q})$ has algebraic Künneth components

$$\tilde{\Delta}_{X,j} = \begin{cases} 0 & j \text{ odd}, \neq n \\ \frac{1}{D} H_X^{n-i} \times H_X^i & j \text{ even} = 2i, \neq n \\ \Delta_X - \sum_{k \neq n} \tilde{\Delta}_{X,k} & j = n \end{cases} \quad \text{in}\, CH_0(X \times X_{/\bar{\mathbb{Q}}}),$$

where (say) $H_X^i := X \cap \{z_1 = 0\} \cap \cdots \cap \{z_i = 0\}$. Now let K be finitely generated $/\bar{\mathbb{Q}}$ and $\mathcal{Z} \in CH_0^{\text{hom}}(X_K)\,[= \mathcal{L}^1 CH_0]$; then Proposition 4.3($b$) implies that $\mathcal{Z} - \sum_{2n > j > n} \left\langle \tilde{\Delta}_{X,j} \right\rangle_* \mathcal{Z} \in \mathcal{L}^n CH_0(X_K)$. But $H_X^{n-i} \cdot \mathcal{Z} = 0$ for $n > i$, and so $\mathcal{Z} \in \mathcal{L}^n CH_0$. Consequently, any difference of two points (say, of $X(\mathbb{C})$) already belongs to $\mathcal{L}^n CH^n(X_{\mathbb{C}})$.

For a cycle of transcendence degree n, take $\mathcal{Z} = p - o$, where $p \in X(\mathbb{C})$ is very general and $o \in X(\bar{\mathbb{Q}})$. We can take as complete spread $\bar{\mathfrak{Z}} = \Delta_X - \{o\} \times X$ in $X \times (\mathcal{S} = X)$, and clearly $[\mathfrak{Z}]_n^* : \Omega^n(X) \to \Omega^n(\mathcal{S})$ is nontrivial; hence $[\mathfrak{Z}]_n \neq 0 \implies \mathcal{Z} \overset{\text{rat}}{\not\equiv} 0$.

To obtain a cycle of transcendence degree $n-1$, one considers a pencil

$$\mathbb{P}^1 \xleftarrow{f} \tilde{X} \xrightarrow{b} X$$

of hyperplane sections with $o \in$ base-locus; write $Y_\mu := b(f^{-1}(\mu)) \overset{\mathcal{I}^\mu}{\hookrightarrow} X$, $q \in Y_\mu(\mathbb{C})$ very general, ${}'\mathcal{Z}_\mu := (\mathcal{I}_*^\mu(q) - o) \in Z_0(X_{\mathbb{C}})$ for any $\mu \in \mathbb{P}^1(\bar{\mathbb{Q}})$. One may think of these as "general" cycles on "special" hyperplane sections. By theorem 7.6, there exists $\mu_0 \in \mathbb{P}^1(\bar{\mathbb{Q}})$ such that ${}'\mathcal{Z}_{\mu_0} \overset{\text{rat}}{\not\equiv} 0$ (and under conjectural conditions, also $[AJ({}'\mathfrak{Z}_{\mu_0})]_{n-1} \neq 0$); the theorem does not tell *which* μ_0 will suffice.

Cycles of transcendence degree $n-2$ in $CH_0^{\text{hom}} = \mathcal{L}^n$ should, assuming BBC^q, be $\overset{\text{rat}}{\equiv} 0$ (modulo torsion!).

Clearly transcendence degree $n-1$ is the interesting case; this is what we seek now to extend to a different set of examples, specifically for $n = 3$.

8.2. **Borcea-Voisin threefolds.** We first recall the construction (due to [Bo1, V2]), which will begin with two ingredients:

(I). an elliptic curve $E/\bar{\mathbb{Q}}$ with $o \in E(\bar{\mathbb{Q}})$, $dz \in \Omega^1(E)$, and involution ι fixing o. One has $\iota^*(dz) = -dz$, fixed-point set $FP(\iota) = \{4 \text{ 2-torsion points}\} =: E^{[2]}$, and $E \xrightarrow{\pi^\iota} (E/\iota) \cong \mathbb{P}^1$. Write $\mathbb{T}$ for the

π^ι-image of the torsion points of E relative to o (which all belong to $E(\bar{\mathbb{Q}})$), and $\mathbb{T}_{[2]}$ for $\pi^\iota(E^{[2]})$.

(II). a (smooth) K3 surface $S/\bar{\mathbb{Q}}$ with $\omega \in \Omega^2(S)$ and involution $\jmath$ with $\jmath^*(\omega) = -\omega$. One has $\mathcal{C} := FP(\jmath)$ either empty or a disjoint union of smooth curves (essentially clear from the fact that ω is nowhere-vanishing; see [Bo1] for requirements on their genera). Quotienting by the action of $\jmath$ yields a (branched) double cover $S \xrightarrow{\pi^\jmath} (S/\jmath) =: \mathcal{R}$. Plenty of exampes were tabulated by Nikulin (again see [Bo1]); we just mention a few:

(a). If $\mathcal{C} = \emptyset$ then $S \xrightarrow{\pi^\jmath} \mathcal{R}$ is an étale cover of an Enriques surface. [Otherwise $\mathcal{R}$ is rational (birational to $\mathbb{P}^2$).]

(b). $S \xrightarrow{\pi^\jmath} \mathbb{P}^2 = \mathcal{R}$ the double cover branched along a smooth sextic (genus 10) curve ($= \mathcal{C}$).

(c). Let $\hat{S}$ be the double cover of $\mathbb{P}^2$ branched over 6 lines in general position; this is singular over the lines' 15 intersection points. Blowing $\hat{S}$ up at these (and lifting the involution associated to $\hat{S} \xrightarrow{2} \mathbb{P}^2$) yields $S \xrightarrow{\pi^\jmath} \mathcal{R}$ ($S =$K3, $\mathcal{R}$ rational with $h^{1,1}(\mathcal{R}) = 16$) branched over 6 *disjoint* rational curves. If the 6 lines in $\mathbb{P}^2$ were common tangents to a conic Q, then S is the Kummer surface[31] associated to the Jacobian of the (genus 2) double cover of Q branched over the 6 points of tangency ([Be]).

(d). If E_2, E_3 are elliptic curves then $(E_2 \times E_3/\{1, (\iota_2, \iota_3)\}) =: \hat{S}$ is the (singular) 2-cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched over 8 $\mathbb{P}^1$'s intersecting in 16 points; blowing these points up yields the Kummer surface S of $E_2 \times E_3$, with $S \xrightarrow{\pi^\jmath} \mathcal{R}$ branched over 8 disjoint rational curves (and $h^{1,1}(\mathcal{R}) = 18$). Here $\jmath$ is induced from $(\iota_2, 1) \sim (1, \iota_3)$ on $\hat{S}$.

Now the involution $(\iota, \jmath)$ on $E \times S$ has FP-set $E^{[2]} \times \mathcal{C}$, which is a disjoint union of curves (e.g., $4 \times 6 = 24$ rational curves if S comes from case (c) above). Under the quotient, $FP(\iota, \jmath)$ becomes the singular set of $\hat{X} = E \times S/(\iota, \jmath)$. All we use from the Borcea-Voisin construction is that the blow-up of $\hat{X}$ along these curves gives a (smooth) CY 3-fold $X/\bar{\mathbb{Q}}$. (In particular, $dz \wedge \omega$ is $(\iota, \jmath)^*$-invariant, so descends to $\Omega \in \Omega^3(X)$.)

---

[31] Recall that given an abelian variety A of dimension 2 with involution $\sigma = -id$, the Kummer surface associated to A is the resolution of $A/\{1, \sigma\}$ obtained by blowing up at the 16 singular points arising from the 2-torsion points of A.

For any $\xi \in \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{T}_{[2]}$, one has a diagram of analytic varieties $/\mathbb{C}$:[32]

$$\begin{tikzcd}
E \times S \arrow[dr, two heads, "\theta"] & & X \arrow[dl, two heads, "b"'] \arrow[dd, "f"] & Y_\xi := f^{-1}(\xi) \cong S \arrow[l, hook', "\mathcal{I}^\xi"] \arrow[dd] \\
& \frac{E \times S}{(\iota, \jmath)} = \hat{X} \arrow[dl, two heads] \arrow[dr, two heads] & & \\
\mathcal{R} = S/\jmath & & E/\iota \cong \mathbb{P}^1 & \{\xi\} \arrow[l, hook']
\end{tikzcd}$$

In analogy to §8.1, we want to show the following

Claim. There exists $\mu \in \mathbb{P}^1(\bar{\mathbb{Q}})$ such that (with $q \in Y_\mu(\mathbb{C})$ very general, $q_0 \in Y_\mu(\bar{\mathbb{Q}})$ any algebraic point) $Z_\mu := \mathcal{I}^\mu_*(q - q_0) \overset{\text{rat}}{\not\equiv} 0$ on X.

Here Proposition 7.5 (i.e., in the form of a corollary like Theorem 7.6) is no longer appropriate, and we shall use the Lewis result.

Since b^{-1} is well-defined on $\theta\left((E \setminus E^{[2]}) \times S\right)$, we have a map $\theta^{\setminus 2} : (E \setminus E^{[2]}) \times S \to X$. Let $q_0 \in S(\bar{\mathbb{Q}})$ be any algebraic point (which need not avoid $\mathcal{C}$), and write $\theta^{\setminus 2}_{q_0}$ for the composite of $\theta^{\setminus 2}$ with the internal projection $\pi^{q_0} : E \times S \to E \times \{q_0\} \hookrightarrow E \times S$ (restricted to $(E \setminus E^{[2]}) \times S$). The graphs of $\theta^{\setminus 2}$, $\theta^{\setminus 2}_{q_0}$ embed in $(E \setminus E^{[2]}) \times S \times X$, and taking their closures in $E \times S \times X$ gives correspondences $\Gamma, \Gamma_{q_0} \in Z^3(E \times S \times X)$. Next take $p \in E(\mathbb{C})$, $q \in S(\mathbb{C})$ very general and algebraically independent; and consider $\mathcal{Z}_0 := (p, q) - (o, q) - (p, q_0) + (o, q_0) \in \mathcal{L}^3 CH_0(E \times S_{/\mathbb{C}})$. Obviously $(p) + (\iota(p)) - 2(o) \overset{\text{rat}}{\equiv} 0$; we also need the following:

Lemma. $\jmath^* \jmath_*((q) - (q_0)) = (q) - (q_0) + (\jmath(q)) - (\jmath(q_0)) \overset{\textit{rat}}{\equiv} 0.$

Proof. $\mathcal{R}$ is either an Enriques surface or a blow-up of $\mathbb{P}^2$ (which replaces points by rational curves). Clearly in the latter case any 2 points are $\overset{\text{rat}}{\equiv}$ on $\mathcal{R}$; this holds also for Enriques by [BKL]. Apply this to $\jmath(q)$ and $\jmath(q_0)$. $\square$

---

[32] In particular $f^{-1}(\xi)$ is smooth. If $\xi \in \mathbb{P}^1(\bar{\mathbb{Q}})$ then it is also a diagram of $\bar{\mathbb{Q}}$-varieties. However if $\xi[\in \mathbb{P}^1(\mathbb{C})]$ is *not* algebraic, the isomorphism $S \cong f^{-1}(\xi)$ is *not* defined over $\bar{\mathbb{Q}}(\xi)$ but over a degree-2 extension isomorphic to $\bar{\mathbb{Q}}(E)$. In this case one must specify whether $\mathcal{I}^\xi$ means the map $S \hookrightarrow X$ (with the degree-2 extension) or $f^{-1}(\xi) \hookrightarrow X$, and for us it will be the former. So for $\xi \in \mathbb{P}^1(\mathbb{C})$, $q \in S(\mathbb{C})$ very general (and algebraically independent in the sense that $\xi \times q \in \mathbb{P}^1 \times S$ is very general), the natural spread base $\mathcal{S}$ for $\mathcal{I}^\xi_*\{q\}$[=general point on special fiber] is not X but $E \times S$ (or a birationally equivalent variety). This is used implicitly below.

It clearly follows that $\mathcal{Z}_0 \overset{\text{rat}}{\equiv} (\iota,\jmath)_*\mathcal{Z}_0$, hence that

$$\begin{aligned}\mathcal{Z}_0 &\overset{\text{rat}}{\equiv} \frac{1}{2}\left(\mathcal{Z}_0 + (\iota,\jmath)_*\mathcal{Z}_0\right)\\ &\overset{\text{rat}}{\equiv} \frac{1}{2}\left\{(p,q)-(p,q_0)+(\iota(p),\jmath(q))-(\iota(p),\jmath(q_0))\right\} \;=:\; \mathcal{Z}_0'.\end{aligned}$$

By [L1, Thm. 1.2(iv)] the action of Γ preserves $\mathcal{L}^\bullet$ and so

$$\mathcal{Z} \;:=\; \Gamma_*\mathcal{Z}_0' \;:=\; \pi_*^X\left\{\pi_{E\times S}^*(\mathcal{Z}_0')\cdot\Gamma\right\} \;=\; \mathcal{I}_*^{\pi^\iota(p)}(q-q_0)$$

belongs to $\mathcal{L}^3CH_0(X_{\mathbb{C}})$. (Note $\pi^\iota(p)\in\mathbb{P}^1(\mathbb{C})\setminus\mathbb{P}^1(\bar{\mathbb{Q}})$.) A choice of complete spread (for $\mathcal{Z}$) is $\Gamma-\Gamma_{q_0} =: \bar{\mathfrak{Z}}$, and its action sends Ω to $dz\wedge\omega$. This shows $\mathcal{Z}\overset{\text{rat}}{\not\equiv} 0$ (a trdeg.-3 cycle).

To get our desired transcendence-degree-2 cycle, observe that Lewis's theorem (§7.3) now guarantees the existence of $\gamma\in CH_0^{\text{hom}}(E_{/\bar{\mathbb{Q}}})$ such that $\mathcal{Z}_\gamma := \bar{\mathfrak{Z}}_\gamma\big|_{\{q\}\times X}$ is $\overset{\text{rat}}{\not\equiv} 0$ (with $[AJ(\mathfrak{Z}_\gamma)]_2\neq 0$). Up to $\overset{\text{rat}}{\equiv}$, γ is of the form $(p_0)-(o)$ for some $p_0\in\{E(\bar{\mathbb{Q}})\setminus\text{torsion}\}$, and noting that $(\Gamma_{q_0})_* = \Gamma_*\circ\pi_*^{q_0}$,

$$\begin{aligned}\mathcal{Z}_{p_0-o} &= \pi_*^X\left\{\pi_C^*(p_0-o)\cdot\pi_S^*(q)\cdot\bar{\mathfrak{Z}}\right\}\\ &= \Gamma_*\left\{(p_0,q)-(o,q)\right\} - (\Gamma_{q_0})_*\left\{(p_0,q)-(o,q)\right\}\\ &= \Gamma_*\left\{(p_0,q)-(o,q)-(p_0,q_0)+(o,q_0)\right\}.\end{aligned}$$

Using the above computation for $\mathcal{Z}_0$ with p_0 replacing p,

$$\mathcal{Z}_{p_0-o}\overset{\text{rat}}{\equiv}\mathcal{I}_*^{\mu}(q-q_0)\,,\qquad \text{where}\,\mu:=\pi^\iota(p_0)\in\mathbb{P}^1(\bar{\mathbb{Q}}).$$

The claim is proved, and moreover $AJ_X^2(\mathcal{I}_*^\mu(q-q_0))\neq 0$. As in §8.1, the proof doesn't say which values of μ suffice; this time we can do better.

Theorem 8.1. *For $X/\bar{\mathbb{Q}}$ a Borcea-Voisin CY 3-fold (as described above), take any algebraic point $q_0\in S(\bar{\mathbb{Q}})$ and* ***any*** *$\mu\in\mathbb{P}(\bar{\mathbb{Q}})\setminus\mathbb{T}$; let $q\in S(\mathbb{C})$ be very general. Then writing $\mathcal{I}^\mu : S\cong f^{-1}(\mu)\hookrightarrow X$, $Z:=\mathcal{I}_*^\mu(q-q_0)$ gives a nontrivial (nontorsion) class in $\mathcal{L}^3CH_0(X_{\mathbb{C}})$.*

Proof. Let p_0 be one of the two points in $(\pi^\iota)^{-1}(\mu)\subseteq E(\bar{\mathbb{Q}})$; $(p_0)-(o)$ is a nontorsion class. As above, $\mathcal{Z}\in\mathcal{L}^3$ follows by writing $\mathcal{Z}$ as the Γ_*-image of the cycle

$$\mathcal{Z}_0' \;:=\; \frac{1}{2}\left\{(p_0,q)-(p_0,q_0)+(\iota(p_0),\jmath(q))-(\iota(p_0),\jmath(q_0))\right\},$$

since this is

$$\overset{\text{rat}}{\equiv}\;\mathcal{Z}_0 \;:=\; (p_0,q)-(o,q)-(p_0,q_0)+(o,q_0)\;\in\;\mathcal{L}^3CH_0(E\times S).$$

Moreover, by Theorem 6.1 (with $j = 2$, $Y_1 = E$, $Y_2 = S$, $\mathcal{W} = p_0 - o$, $\mathcal{V} = q - q_0$), one has $\mathcal{Z}_0 = \mathcal{W} \times \mathcal{V} \overset{\text{rat}}{\not\equiv} 0$; hence $\mathcal{Z}_0' \overset{\text{rat}}{\not\equiv} 0$. But then

$$\theta^*(b_*\mathcal{Z}) \;=\; \theta^*\left(\theta_*\{(p_0, q) - (p_0, q_0)\}\right) \;=\; 2\mathcal{Z}_0'$$

implies $b_*\mathcal{Z} \overset{\text{rat}}{\not\equiv} 0$ on $\hat{X}$; so $\mathcal{Z} \overset{\text{rat}}{\not\equiv} 0$ on X, done. □

Now let E_1, E_2, and E_3 be elliptic curves defined $/\bar{\mathbb{Q}}$ (with origins o); take S to be the Kummer of $E_2 \times E_3$, and X to be the Borcea-Voisin CY of $E_1 \times S$. An alternate construction of X due to [Bo2] proceeds as follows. Write $\Theta : E_1 \times E_2 \times E_3 \to \frac{E_1 \times E_2 \times E_3}{V_4} =: \hat{X}'$ for the quotient by the action of

$$V_4 = \{(1,1,1),\, (\iota_1, \iota_2, 1),\, (\iota_1, 1, \iota_3),\, (1, \iota_2, \iota_3)\}\,.$$

Now $\hat{X}'$ is singular along the 48 rational curves $\Theta(E_1 \times E_2^{[2]} \times E_3^{[2]})$, $\Theta(E_1^{[2]} \times E_2 \times E_3^{[2]})$, and $\Theta(E_1^{[2]} \times E_2^{[2]} \times E_3)$; blowing up *first* at the $\Theta(E_1 \times E_2^{[2]} \times E_3^{[2]})$ then at the rest, yields $X \xrightarrow{B} \hat{X}'$.

Next we take points p, Q_2, Q_3 on E_1, E_2, E_3 (resp.); assume either

(i) p is algebraic and Q_2, Q_3 are (alg. indep.) very general, OR

(ii) Q_2 or Q_3 is algebraic and the other two are (alg. indep.) very general.

Write q, q_0 for points mapping (under $S \twoheadrightarrow \hat{S}$) to the images of (Q_2, Q_3), (Q_2, o) (resp.) under $E_2 \times E_3 \twoheadrightarrow \hat{S}$. Then by Example 6.2 and explicit rational equivalences one has on $E_1 \times E_2 \times E_3$:

$$\begin{aligned} 0 \overset{\text{rat}}{\not\equiv} 4\,Box^3(p,\, Q_2,\, Q_3) &\overset{\text{rat}}{\equiv} \sum_{\sigma \in V_4} \sigma_* Box^3(p,\, Q_2,\, Q_3) \\ &\overset{\text{rat}}{\equiv} \sum_{\sigma \in V_4} \sigma_*\{(p,\, Q_2,\, Q_3) - (p,\, Q_2,\, o)\} \\ &= \Theta^*\left(\Theta_*\{(p,\, Q_2,\, Q_3) - (p,\, Q_2,\, o)\}\right). \end{aligned}$$

This implies on $\hat{X}'$:

$$0 \overset{\text{rat}}{\not\equiv} \Theta_*\{(p,\, Q_2,\, Q_3) - (p,\, Q_2,\, o)\} \;=\; B_*\left\{\mathcal{I}_*^{\pi^\iota(p)}(q - q_0)\right\},$$

hence on X:

$$\mathcal{Z} \;:=\; \mathcal{I}_*^{\pi^\iota(p)}(q - q_0) \overset{\text{rat}}{\not\equiv} 0.$$

In case (i) this is the cycle $\mathcal{Z}_{\pi^\iota(p)}$ of Theorem 8.1 — a transcendence-degree-2 cycle on an algebraic fiber of f, included into X. In case (ii), $\mathcal{Z}$ is a transcendence-degree-1 cycle[33] on a general fiber of f, included into X.

---

[33] i.e., the cycle has trdeg. 1 over the ***field of definition*** of the general fiber.

Remark 8.2. A similar approach also yields nontrivial 0-cycles on the (smooth) CY 3-folds arising from the recent construction in [PR].

8.3. 0**-cycles and differential characters.** To conclude the section we revisit the reduced higher AJ maps of §4.3 for X a CY n-fold defined $/\bar{\mathbb{Q}}$, as there are some nice simplifications in this case. For one thing, the basis $\{\omega\}$ (for $i = n$) is replaced by a single form $\Omega \in \Omega^n(X)$; and so the period lattice is

$$\Lambda = \mathrm{im}\left\{ \int_{(\cdot)} \Omega : \, H_n(X, \mathbb{Q}) \to \mathbb{C} \right\}.$$

Now assume $K \cong \bar{\mathbb{Q}}(\mathcal{S})$ is finitely generated $/\bar{\mathbb{Q}}$ of transcendence degree $n-1$, and note that $Gr^n_{\mathcal{L}}\underline{Hg}^n(X \times \eta_{\mathcal{S}}) = 0$ (as weak Lefschetz $\implies$ $H^n(\mathcal{S}) = N^1H^n(\mathcal{S}) \implies \underline{H^n}(\eta_{\mathcal{S}}) = 0$). This means that AJ^{n-1}_X is defined directly on $\mathcal{L}^nCH_0(X_K)$, and composing this with the projections of §4.3 gives

$$\overline{AJ^{n-1}_X} : \, \mathcal{L}^nCH_0(X_{/K}) \to Hom_{\mathbb{Q}}\left(\underline{H_{n-1}}(\eta_{\mathcal{S}}, \mathbb{Q}), \mathbb{C}/\Lambda \right). \tag{8.1}$$

Using Proposition 4.8 and Remark 4.10(b) we may describe this as follows, for $\langle \mathcal{Z} \rangle \in \mathcal{L}^nCH_0$ and $[\gamma] \in \underline{H_{n-1}}(\eta_{\mathcal{S}}, \mathbb{Q})$ $[\subseteq H_{n-1}(\mathcal{S}, \mathbb{Q})]$. Choosing a complete spread $\bar{\mathfrak{Z}} \in Z^n(X \times \mathcal{S}_{/\bar{\mathbb{Q}}})$ for $\mathcal{Z}$, take $\mathcal{U}_{\bar{\mathfrak{Z}}} \subseteq \mathcal{S}$ Zariski open such that $\bar{\mathfrak{Z}}\big|_{X \times \mathcal{U}_{\bar{\mathfrak{Z}}}}$ has relative dimension 0 over $\mathcal{U}_{\bar{\mathfrak{Z}}}$. We may also select a representative γ_0 of the class $[\gamma] \in H_{n-1}(\mathcal{S}, \mathbb{Q})$ which is supported in $\mathcal{U}_{\bar{\mathfrak{Z}}}$. Then $\pi_X\{\bar{\mathfrak{Z}} \cap (\gamma_0 \times X)\} \in Z^{\mathrm{top}}_{n-1}(X, \mathbb{Q})$ is well-defined and (since $[\bar{\mathfrak{Z}}]_{n-1} = 0$) a boundary, say $= \partial\Gamma$ (Γ a topological n-chain). One then sets

$$\overline{AJ^{n-1}_X(\mathcal{Z})}[\gamma] := \int_{\Gamma} \Omega \, \in \mathbb{C}/\Lambda, \tag{8.2}$$

which is independent of all choices.

A generalization of the Bloch conjecture says that the vanishing of $H^0(\Omega^i_X)$ for $0 < i < n$ should imply $Gr^i_{\mathcal{L}}CH_0(X_{/K}) = 0$ in that range. So $\overline{AJ^{n-1}_X}$ should be defined already on $CH_0^{\mathrm{hom}}(X_{/K})$. Moreover, under the duality $H^{n-1}(\mathcal{S}, \mathbb{Q}) \cong H_{n-1}(\mathcal{S}, \mathbb{Q})^{\vee}$ induced by integration, the dual of $\underline{H_{n-1}}(\eta_{\mathcal{S}}, \mathbb{Q})$ (considered as subgroup of $H_{n-1}(\mathcal{S}, \mathbb{Q})$) is $H^{n-1}(\mathcal{S}, \mathbb{Q})/N^1H^{n-1}(\mathcal{S}, \mathbb{Q})$. So assuming the "GBC", $\overline{AJ^{n-1}_X}$ may be written as a map

$$(8.3) \qquad CH_0^{\mathrm{hom}}(X_{/K}) \longrightarrow \frac{H^{n-1}(\mathcal{S},\mathbb{C})}{N^1H^{n-1}(\mathcal{S},\mathbb{C}) + H^{n-1}(\mathcal{S},\Lambda)}.$$

Example. For the complete intersection CY's considered in §8.1, we *know* (without conjecture) that $\mathcal{L}^n CH_0 = CH_0^{\mathrm{hom}}$. Consider a smooth $\bar{\mathbb{Q}}$-hyperplane section $Y \overset{\iota_Y}{\hookrightarrow} X$ with $q \in Y(\mathbb{C})$ very general, $o \in Y(\bar{\mathbb{Q}})$; take $\mathcal{Z} = \iota_*^Y(q-o)$ (defined $/K \cong \bar{\mathbb{Q}}(Y)$), $\mathcal{S} = Y$, and $\bar{\mathfrak{Z}}$ essentially as in Example 4.2. Then (as in Example 4.9) $\pi_X\left\{\bar{\mathfrak{Z}} \cap (\gamma \times X)\right\} = \iota_Y(\gamma)$ for $\gamma \in Z_{n-1}^{\mathrm{top}}(Y)$ with class $[\gamma] \in \underline{H_{n-1}}(\eta_Y,\mathbb{Q})$, and $\overline{AJ_X^{n-1}}(\mathcal{Z})[\gamma] = \int_\Gamma \Omega$ where $\partial\Gamma = \iota_Y(\gamma)$. If any membrane integral of this form gives a value not equal to a *period* of Ω (an integral over a topological n-cycle), then $\mathcal{Z} \overset{\mathrm{rat}}{\not\equiv} 0$ on X. [Note that since Y has a holomorphic form, $(H^{n-1}(Y)/N^1) \neq 0$; hence $\underline{H_{n-1}}(\eta_Y,\mathbb{Q}) \neq 0$.] However, nonvanishing of such integrals may be hard to establish, as one typically has $\dim_{\mathbb{Q}}(\Lambda) > 2$ (the exception being the case in which the subgroup $H^{n,0}(X) \oplus H^{0,n}(X) \subseteq H^n(X,\mathbb{C})$ is [the complexification of] a subHS).

We have (so far) restricted to the case $\mathrm{trdeg}(K/\bar{\mathbb{Q}}) = n-1$; obviously trdeg $< n-1$ makes $\overline{AJ_X^{n-1}} = 0$. On the other hand, a more complicated picture arises in the event that trdeg $> n-1$. It is worked out in [K2] and we summarize pertinent details for the CY case. Since $Gr_{\mathcal{L}}^n \underline{Hg}^n(X \times \eta_{\mathcal{S}})$ no longer vanishes, $\overline{AJ_X^{n-1}}$ is not defined. Consequently, the $\mathbb{C}/\Lambda$-valued functional $\gamma \mapsto \int_{\partial_X^{-1}\{\bar{\mathfrak{Z}}_*(\cdot)\}} \Omega$ on (admissible) topological cycles (this is essentially the prescription from (8.2)) no longer descends to (admissible) homology classes. Instead, on a boundary $\gamma = \partial\mathcal{G}$ on $\mathcal{S}$, the functional evaluates (mod Λ) to $\int_{\bar{\mathfrak{Z}}_*\mathcal{G}} \Omega = \int_{\mathcal{G}} \{\bar{\mathfrak{Z}}^*\Omega =: \Omega_{\mathcal{Z}}\}$.[34]

This may be formalized as follows: replacing (8.1) is a map

$$(8.4) \qquad \chi_X^n : \mathcal{L}^n CH_0(X_{/K}) \to \varinjlim \underline{H_{\mathcal{D}}^n}(\mathcal{S} \setminus D,\ \Lambda\{n\})$$

(limit over divisors $D \subseteq \mathcal{S}$ defined $/\bar{\mathbb{Q}}$) also factoring through the map $Gr_{\mathcal{L}}^n \psi$ to $Gr_{\mathcal{L}}^n \underline{H_{\mathcal{D}}^{2n}}(X \times \eta_{\mathcal{S}}, \mathbb{Q}(n))$; it sends $\mathcal{Z}$ to the above-described functional. We do not define the right-hand side of (8.4), but simply say that it sits in a short exact sequence

---

[34] Here $\bar{\mathfrak{Z}}^* : \Omega^n(X) \to \Omega^n(\mathcal{S})$ is the induced map of holomorphic forms (see [K2, sec. 2]); this is the same as $[\mathfrak{Z}]_n^*$ in §4.

(8.5)

$$\underline{H^{n-1}}(\eta_{\mathcal{S}}, \mathbb{C}/\Lambda) \xrightarrow{\overline{\beta_{n-1}}} \underline{H^n_{\mathcal{D}}}(\mathcal{S} \setminus D, \Lambda\{n\}) \xrightarrow{\overline{\alpha_n}} F^n \underline{H^n}(\eta_{\mathcal{S}}, \mathbb{C}) \cap \underline{H^n}(\eta_{\mathcal{S}}, \Lambda)$$

and informally explain the terms. In the limit, the left-hand term (of (8.5)) is just the right-hand side of (8.1) (or (8.3)). The right-hand term (also in the limit) identifies with the subset

$$\left\{ \omega \in \Omega^n(\mathcal{S}_{\{\mathbb{C}\}}) \,\middle|\, \int_\gamma \omega \in \Lambda \;\; \forall [\gamma] \in \underline{H_n}(\eta_{\mathcal{S}}, \mathbb{Q}) \right\} \subseteq \Omega^n(\mathcal{S}_{\mathbb{C}}).$$

The middle term may be interpreted as the space of "$\mathbb{C}/\Lambda$-valued differential $(n-1)$-characters on $\mathcal{S} \setminus D$", or functionals $\mathfrak{F}$ on $Z^{\text{top}}_{n-1}(\mathcal{S} \setminus D)$ with an associated holomorphic form $\overline{\alpha_n}(\mathfrak{F})$ such that $\mathfrak{F}(\partial \mathcal{G}) = \int_{\mathcal{G}} \overline{\alpha_n}(\mathfrak{F})$. Hence, $\overline{\alpha_n}\left(\chi^n_X(\mathcal{Z})\right)$ is precisely $\Omega_{\mathcal{Z}}$.

We hope that such differential-geometric descriptions of higher cycle maps (extended also beyond $CH_0(X)$ to all higher Chow groups $CH^p(X, m)$) will ultimately shed some light on the role played by algebraic K-theory and motivic cohomology in the context of mirror symmetry.

REFERENCES

[Ar] D. Arapura, *The Leray spectral sequence is motivic*, to appear in Invent. Math.

[As] M. Asakura, *Arithmetic Hodge structures and nonvanishing of the cycle-class of* 0*-cycles*, K-Theory 27 (2002), no. 3, pp. 273-280.

[Be] A. Beauville, "Surfaces algébriques complexes", Asterisque 54, Soc. Math France, 1978.

[BV] A. Beauville and C. Voisin, *On the Chow ring of a* $K3$ *surface*, J. Alg. Geom. 13 (2004), no. 3, pp. 417-426.

[B1] S. Bloch, *Some elementary theorems about algebraic cycles on Abelian varieties*, Invent. Math. 37 (1976), pp. 215-228.

[B2] ———, "Lectures on algebraic cycles," Duke Univ. Math. Series IV, 1980.

[B3] ———, *Algebraic cycles and values of* L*-functions*, J. Reine Angew. Math. 350 (1984), pp. 94-108.

[BKL] S. Bloch, A. Kas and D. Lieberman, *Zero cycles on surfaces with* $p_g = 0$, Compositio Math. 33 (1976), no.2, 135-145.

[Bo1] C. Borcea, *K3 surfaces with involution and mirror pairs of Calabi-Yau manifolds*, AMS/IP Studies in Advanced Math. 1, 1997, pp. 717-743.

[Bo2] ———, *Calabi-Yau threefolds and complex multiplication*, Essays on mirror manifolds, Internat. Press, Hong Kong, 1992, pp. 489-502.

[De] P. Deligne, *Théorie de Hodge III*, Publ. Math IHES 44 (1974), pp. 5-77.

[Fa] N. Fakhruddin, *Zero-cycles on generic hypersurfaces of large degree*, preprint math.AG/0208180.

[Fu] W. Fulton, “Intersection theory” (2nd edition), Springer-Verlag, Berlin, 1998.

[G1] M. Green, Notes from the Banff conference on algebraic cycles, June 1998.

[G2] ————, *Griffiths' infinitesimal invariant and the Abel-Jacobi map*, J. Differential Geom. 29 (1989), no. 3, 545-555.

[Gr] P. Griffiths, *On the periods of certain rational integrals II*, Ann. of Math. (2) 90 (1969), pp. 496-541.

[GG1] M. Green and P. Griffiths, *Hodge-theoretic invariants for algebraic cycles*, Intl. Math. Res. Not. 2003, no. 9, pp. 477-510.

[GG2] ————, *An interesting* 0*-cycle*, Duke Math. J. 119 (2003), no. 2, 261-313.

[GGP] M. Green, P. Griffiths and K. Paranjape, *Cycles over fields of transcendence degree one*, Michigan Math. J. 52 (2004), no. 1, pp. 181-187.

[GS] P. Griffiths and W. Schmid, *Recent developments in Hodge theory: a discussion of techniques and results*, Discrete subgroups of Lie groups and applications to moduli (Bombay 1973), Oxford Univ. Press, 1975, pp. 31-127.

[H] B. Harris, *Homological versus algebraic equivalence in a Jacobian*, Proc. Nat. Acad. Sci. U.S.A. 80 (1983), no. 4 i., 1157-1158.

[J1] U. Jannsen, *Motivic sheaves and filtrations on Chow groups*, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math. 55(I), AMS, Providence, RI, 1994.

[J2] ————, *Mixed motives and algebraic K-theory*, Lecture Notes in Math. 1400, Springer-Verlag, Berlin, 1990.

[J3] ————, *Continuous étale cohomology*, Math. Ann. 280 (1988), pp. 207-245.

[J4] ————, *Deligne homology, Hodge-$\mathcal{D}$-conjecture, and motives*, Beilinson's Conjectures on Special Values of L-functions, pp. 305-372, Perspect. Math. 4, Academic Press, Boston, MA, 1988.

[K1] M. Kerr, *Exterior products of* 0*-cycles*, preprint, 2005.

[K2] ————, *Higher Abel-Jacobi maps for* 0*-cycles*, preprint, 2003.

[K3] ————, *A regulator formula for Milnor K-groups*, K-Theory 29 (2003), pp. 175-210.

[K4] ————, *Geometric construction of regulator currents with applications to algebraic cycles*, Princeton thesis, 2003.

[Ki] S-I. Kimura, *Chow groups can be finite dimensional in some sense*, preprint.

[Kl] S. Kleiman, *Algebraic cycles and the Weil conjectures*, in “Dix exposés sur la cohomologie des schémas”, North-Holland, Amsterdam, 1968, pp. 359-386.

[L1] J. Lewis, *A filtration on the Chow groups of a complex projective variety*, Compositio Math. 128 (2001), no. 3, pp. 299-322.

[L2] ————, “A survey of the Hodge conjecture” (2nd edition), AMS, Providence, 1999.

[L3] ————, *Lectures on algebraic cycles*, Bol. Soc. Mat. Mexicana (3) 7 (2001), no. 2, 137-192.

[LS] J. Lewis and S. Saito, *Algebraic cycles and Mumford-Griffiths invariants*, preprint, 2004.

[Mu] D. Mumford, *Rational equivalence of* 0*-cycles on surfaces*, J. Math. Kyoto Univ. 9 (1968), pp. 195-204.

[M] J. Murre, *On a conjectural filtration on the Chow group of an algebraic variety (I)*, Indag. Math. (N.S.) 4 (1993), no. 2, 177-188.

[MR] J. Murre and D. Ramakrishnan, *Galois symbols on the square of an elliptic curve*, preprint.

[P] K. Paranjape, *Some spectral sequences for filtered complexes with applications*, J. Algebra 186 (1996), no. 3, 793-806.

[PR] K. Paranjape and D. Ramakrishnan, *Quotients of* E^n *by* $\mathfrak{a}_{n+1}$ *and Calabi-Yau manifolds*, to appear.

[Ra] W. Raskind, *Higher* ℓ*-adic Abel-Jacobi mappings and filtrations on Chow groups*, Duke Math. J. 78 (1995), no.1, pp. 33-57.

[R1] A. Roitman, *Rational equivalence of zero-dimensional cycles*, Math. USSR Sbornik 18 (1972), no. 4, pp. 571-588.

[R2] ———, *The torsion group of* 0*-cycles modulo rational equivalence*, Ann. Math. 111 (1980), pp. 553-569.

[RS] A. Rosenschon and M. Saito, *Cycle map for strictly decomposable cycles*, Amer. J. Math. 125 (2003), no. 4, pp. 773-790.

[mS] M. Saito, *Filtrations on Chow groups and transcendence degree*, Publ. Res. Inst. Math. Sci. 40 (2004), no. 3, pp. 933-948.

[sS] S. Saito, *Higher normal functions and Griffiths groups*, J. Alg. Geom. 11 (2002), no. 1, pp. 161-201.

[Te] T. Terasoma, *Complete intersections with middle Picard number* 1 *defined over* $\mathbb{Q}$, Math. Z. 189 (1985), no. 2, pp. 289-296.

[To] B. Totaro, *Milnor* K*-theory is the simplest part of algebraic* K*-theory*, K-Theory 6 (1992), no.2, pp. 177-189.

[V1] C. Voisin, *Variations de struture de Hodge et zéro-cycles sur les surfaces générales.* (French), Math. Ann. 299 (1994), no. 1, pp. 77-103.

[V2] ———, *Miroirs et involutions sur les surfaces K3*, in "Journeés de Géométrie Algébrique d'Orsay", Astérisque 218, Soc. Math. France, 1993, pp. 273-323.

[V3] ———, *Remarks on zero-cycles of self-products of varieties*, in "Moduli of vector bundles" (Proceedings of the Taniguchi symposium on vector bundles), Maruyama Ed., Dekker, 1996, pp. 265-285.

[V4] ———, "Hodge theory and complex algebraic geometry. II", Cambridge Studies in Advenced Mathematics 77, Cambridge University Press, Cambridge, 2003.

[Vo] V. Voevodsky, *A nilpotence theorem for cycles algebraically equivalent to zero*, Intl. Math. Res. Notices 1995, no. 4, pp. 187-198 (electronic).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO,
CHICAGO, IL, USA 60637
matkerr@math.uchicago.edu