PUTNAM PRACTICE—USEFUL TACTICS

1. Symmetry

1.1. Geometric Symmetry.

Problem 1

An equilateral triangle is inscribed in a circle which is inscribed in an equilateral triangle. Without pen and paper, find the ratio of the areas of the two triangles.

Problem 2

(1998) A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

1.2. Algebraic Symmetry.

Problem 3

Let n be an odd integer greater than 1. Let A be an n by n symmetric matrix such that each row and each column of A consists of some permutation of the integers $1, ..., n$. Show that each one of the integers $1, ..., n$ must appear in the main diagonal of A.

Problem 4

(2009) Functions f, g, h are differentiable on some open interval around 0 and satisfy the equations and initial conditions

$$\begin{aligned} f' &= 2f^2gh + \frac{1}{gh}, \qquad f(0) = 1, \\ g' &= fg^2h + \frac{4}{fh}, \qquad g(0) = 1, \\ h' &= 3gfh^2 + \frac{1}{fg}, \qquad h(0) = 1. \end{aligned}$$

Find an explicit formula for $f(x)$, valid in some open interval around 0.

2. Extreme Principle

Problem 5

Let $p(x)$ be a polynomial such that for all x, $p(x) + p'(x) \geq 0$. Does it follow that for all x, $p(x) \geq 0$?

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Problem 6

(2004) Determine all real numbers $a > 0$ for which there exists a nonnegative continuous functions $f(x)$ defined on $[0, a]$ with the property that the region

$$R = \{(x, y); 0 \le x \le a, 0 \le y \le f(x)\}$$

has perimeter k units and area k square units for some real number k.

Problem 7

(2007) Suppose that $f : [0, 1] \to \mathbb{R}$ has a continuous derivative and that $\int_0^1 f(x)dx = 0$. Prove that for every $\alpha \in (0, 1)$,

$$|\int_0^\alpha f(x)dx| \le \frac{1}{8}\text{max}_{0\le x\le 1}|f'(x)|.$$

3. Pigeonhole Principle

Problem 8

Show that any odd number not divisible by 5 must divide some number of the form 10101...01, an alternating string of 1's and 0's.

Problem 9

(2006) Prove that, for every set $X = \{x_1, x_2, ..., x_n\}$ of n real numbers, there exists a non-empty subset S of X and an integer m such that

$$|m + \sum_{s\in S} s| \le \frac{1}{n+1}.$$

Problem 10

(2013) Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

4. Invariants

4.1. **Parity.**

Problem 11

(a) Place a knight on each square of a 7×7 chessboard. Is it possible for each knight to simultaneously make a legal move so that each knight ends up in its own square?

(b) Suppose we have a knight on a 4×4 chessboard. Can it make a sequence of legal moves visiting each square exactly once?

Problem 12

(2002)Let $n \ge 2$ be an integer and T_n the number of non-empty subsets S of $\{1, 2, ..., n\}$ with the property that the average of the elements in S is an integer. Prove that $T_n - n$ is even.

4.2. **Modular Arithmetic.**

Problem 13

(a) Prove that this sequence contains no squares: $11, 111, 1111, 11111, ...$

(b) Prove that if $2n + 1$ and $3n + 1$ are both perfect squares, then n is divisible by 40.

Problem 14

(1995) The number $d_1 d_2 ... d_9$ has nine (not necessarily dis- tinct) decimal digits. The number $e_1 e_2 ... e_9$ is such that each of the nine 9-digit numbers formed by replacing just one of the digits d_i is $d_1 d_2 ... d_9$ by the corresponding digit e_i $(1 \leq i \leq 9)$ is divisible by 7. The number $f_1 f_2 ... f_9$ is related to $e_1 e_2 ... e_9$ is the same way: that is, each of the nine numbers formed by replacing one of the e_i by the corresponding f_i is divisible by 7. Show that, for each i, $d_i - f_i$ is divisible by 7. [For example, if $d_1 d_2 ... d_9 = 199501996$, then e_6 may be 2 or 9, since 199502996 and 199509996 are multiples of 7.]

4.3. **Semi-invariant.**

Problem 15

In a game, each member has at most three enemies. A member cannot be his own enemy, and enmity is mutual. Prove that the members can be divided into two factions such that each member has at most one enemy within his/her faction.

Problem 16

(2008) Start with a finite sequence $a_1, a_2, ..., a_n$ of positive integers. If possible, choose two indices $j < k$ such that a_j does not divide a_k, and replace a_j and a_k by $gcd(a_j, a_k)$ and $lcm(a_j, a_k)$, respectively. Prove that if this process is repeated, it must eventually stop.