WEAK PRODUCTS OF COMPLETE PICK SPACES.

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ABSTRACT. Let $\mathcal{H}$ be the Drury-Arveson or Dirichlet space of the unit ball of $\mathbb{C}^d$. The weak product $\mathcal{H} \odot \mathcal{H}$ of $\mathcal{H}$ is the collection of all functions h that can be written as $h = \sum_{n=1}^{\infty} f_n g_n$, where $\sum_{n=1}^{\infty} \|f_n\|\|g_n\| < \infty$. We show that $\mathcal{H} \odot \mathcal{H}$ is contained in the Smirnov class of $\mathcal{H}$, i.e. every function in $\mathcal{H} \odot \mathcal{H}$ is a quotient of two multipliers of $\mathcal{H}$, where the function in the denominator can be chosen to be cyclic in $\mathcal{H}$. As a consequence we show that the map $\mathcal{N} \to \operatorname{clos}_{\mathcal{H}\odot\mathcal{H}} \mathcal{N}$ establishes a 1-1 and onto correspondence between the multiplier invariant subspaces of $\mathcal{H}$ and of $\mathcal{H} \odot \mathcal{H}$.

The results hold for many weighted Besov spaces $\mathcal{H}$ in the unit ball of $\mathbb{C}^d$ provided the reproducing kernel has the complete Pick property. One of our main technical lemmas states that for weighted Besov spaces $\mathcal{H}$ that satisfy what we call the multiplier inclusion condition any bounded column multiplication operator $\mathcal{H} \to \oplus_{n=1}^{\infty} \mathcal{H}$ induces a bounded row multiplication operator $\oplus_{n=1}^{\infty} \mathcal{H} \to \mathcal{H}$. For the Drury-Arveson space H_d^2 this leads to an alternate proof of the characterization of interpolating sequences in terms of weak separation and Carleson measure conditions.

1. INTRODUCTION

By a Hilbert (Banach) function space on a set X we mean a Hilbert (Banach) space $\mathcal{B}$ which consists of complex-valued functions on X such that for each point $w \in X$ the point evaluation functional $f \to f(w)$ is bounded on $\mathcal{B}$. Associated with every Banach function space $\mathcal{B}$ we have the collection of multipliers $\operatorname{Mult}(\mathcal{B}) = \{\varphi : X \to \mathbb{C} : \varphi\mathcal{B} \subseteq \mathcal{B}\}$. This is another Banach function space when equipped with the norm $\|\cdot\|_M$

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that equals the operator norm of the induced multiplication operator $M_\varphi : \mathcal{B} \to \mathcal{B}, f \to \varphi f,\ f \in \mathcal{B}, \varphi \in \operatorname{Mult}(\mathcal{B})$. We will call a function f cyclic in $\mathcal{B}$, if the set $\{\varphi f : \varphi \in \operatorname{Mult}(\mathcal{B})\}$ is dense in $\mathcal{B}$.

Each Hilbert function space $\mathcal{H}$ has a reproducing kernel, i.e. a function $k : X \times X \to \mathbb{C}$ such that $f(w) = \langle f, k_w\rangle$ for all $w \in X$. Here $k_w(z) = k(z, w)$. We say a reproducing kernel is normalized, if there is a $z_0 \in X$ such that $k_{z_0} = 1$. A Banach (Hilbert) space of analytic functions will be a Banach (Hilbert) function space which is contained in $\operatorname{Hol}(\Omega)$, the collection of holomorphic functions on the open set $\Omega \subseteq \mathbb{C}^d$ for some $d \in \mathbb{N}$.

In this paper a normalized complete Pick kernel will be a normalized reproducing kernel of the type $k_w(z) = \frac{1}{1-u_w(z)}$, where $u_w(z)$ is positive definite, i.e. for all $n \in \mathbb{N}, z_1, ..., z_n \in X$, and $a_1, ..., a_n \in \mathbb{C}$ we have $\sum_{i,j} a_i\overline{a}_j u_{z_j}(z_i) \geq 0$. An important example of such a complete Pick kernel is the Szegő kernel $k_w(z) = (1 - \overline{w}z)^{-1}$. It is the reproducing kernel for the Hardy space H^2 of the unit disc $\mathbb{D}$, and it is fair to say that the function and operator theories associated with Hilbert function spaces with complete Pick kernels share many properties with the corresponding theories of H^2. We refer the reader to [1] and [5] for some examples of this. In particular, it is a useful fact that Hilbert function spaces $\mathcal{H}$ with a normalized complete Pick kernel are contained in the Smirnov class $N^+(\mathcal{H})$ associated with $\mathcal{H}$, [5], where

$$N^+(\mathcal{H}) = \Big\{ f = \frac{\varphi}{\psi} : \varphi, \psi \in \operatorname{Mult}(\mathcal{H}), \psi \text{ cyclic in } \mathcal{H} \Big\}.$$

At this point we mention two further important examples of spaces with complete Pick kernels where the previous remark applies. The Dirichlet space of the unit disc, $D = \{f \in \operatorname{Hol}(\mathbb{D}) : \int_{\mathbb{D}} |f'|^2 dxdy < \infty\}$, it has reproducing kernel $k_w(z) = \frac{1}{\overline{w}z} \log \frac{1}{1-\overline{w}z}$ (see e.g. [1], Corollary 7.41, for the verification that $k_w(z)$ is a CNP kernel), and the Drury-Arveson space H^2_d of analytic functions in the unit ball of $\mathbb{C}^d$. It is defined by the reproducing kernel $k_w(z) = \frac{1}{1-\langle z,w\rangle}$, $\langle z, w\rangle = \sum_{i=1}^d z_i\overline{w}_i$. See [15] and [25] for indepth information about these spaces.

Let $\mathcal{H}$ be a Hilbert function space on a set X . The weak product of $\mathcal{H}$ is defined by

$$\mathcal{H} \odot \mathcal{H} = \Big\{ \sum_{i=1}^\infty f_i g_i : f_i, g_i \in \mathcal{H}, \sum_{i=1}^\infty \|f_i\|\|g_i\| < \infty \Big\},$$

where the norm on $\mathcal{H} \odot \mathcal{H}$ is given by

$$\|h\|_{\mathcal{H}\odot\mathcal{H}} = \inf \Big\{ \sum_{i=1}^\infty \|f_i\|\|g_i\| : h = \sum_{i=1}^\infty f_i g_i \Big\}.$$

One verifies that $\mathcal{H} \odot \mathcal{H}$ is a Banach function space, [21] for the case of spaces of analytic functions, but the general case can be proved the same way (also see Section 2). It is known that $H^2(\partial\mathbb{B}_d) \odot H^2(\partial\mathbb{B}_d) = H^1(\partial\mathbb{B}_d)$ and $L^2_a(\mathbb{B}_d) \odot L^2_a(\mathbb{B}_d) = L^1_a(\mathbb{B}_d)$, and there are similar results for weighted Bergman spaces, [12]. We think of $\mathcal{H} \odot \mathcal{H}$ an analogue of H^1 for the function theory of the Hilbert function space $\mathcal{H}$. It is known that for many examples of spaces $\mathcal{H}$ one observes an analogue of the H^1-BMO-duality and a connection to Carleson measures and the theory of Hankel operators on $\mathcal{H}$. In this paper we will add to this circle of ideas for Hilbert function spaces with normalized complete Pick kernels. We refer the reader to [12], [7], and [21] for further motivation and details about weak products.

Note that even for the cases where $\mathcal{H}$ equals the Dirichlet or the Drury-Arveson space it is unclear whether there is a simple description of the functions in $\mathcal{H} \odot \mathcal{H}$.

Theorem 1.1. *Let $\mathcal{H}$ be a separable Hilbert function space on the non-empty set X such that the reproducing kernel for $\mathcal{H}$ is a normalized complete Pick kernel. Then*

$$\mathcal{H} \odot \mathcal{H} \subseteq N^+(\mathcal{H} \odot \mathcal{H}),$$

where

$$N^+(\mathcal{H} \odot \mathcal{H}) = \left\{\frac{\varphi}{\psi} : \varphi \in \operatorname{Mult}(\mathcal{H} \odot \mathcal{H}), \psi \in \operatorname{Mult}(\mathcal{H}), \psi \textit{ cyclic in } \mathcal{H}\right\}$$

Beurling's theorem implies that the nonzero multiplier invariant subspaces of H^2 and H^1 are given by φH^2 and φH^1 for some inner function φ. It turns out that for the spaces $\mathcal{H}$ under consideration a similarly close relationship exists between the multiplier invariant subspaces of $\mathcal{H}$ and $\mathcal{H} \odot \mathcal{H}$. In [17] and [22] it was shown that if $\mathcal{H} = D$ or $\mathcal{H} = H^2_d$, then for every multiplier invariant subspace $\mathcal{M}$ of $\mathcal{H}$ we have $\mathcal{M} = \mathcal{H} \cap \operatorname{clos}_{\mathcal{H}\odot\mathcal{H}} \mathcal{M}$. In this paper we will refine this type of connection and we will see that it holds for a much wider class of complete Pick spaces.

One easily checks that for any Hilbert function space the contractive inclusion $\operatorname{Mult}(\mathcal{H}) \subseteq \operatorname{Mult}(\mathcal{H} \odot \mathcal{H})$ holds. For certain first order weighted Besov spaces (including the Dirichlet space of the unit disc and the Drury-Arveson space H^2_d for $d \leq 3$) it was shown in [23] that $\operatorname{Mult}(\mathcal{H}) = \operatorname{Mult}(\mathcal{H}\odot\mathcal{H})$, but we do not know whether such an equality holds in a more general setting.

Theorem 1.2. *Let $\mathcal{H}$ be a separable Hilbert function space on the non-empty set X such that the reproducing kernel for $\mathcal{H}$ is a normalized complete Pick kernel.*

Then $\mathcal{M} \cap \mathrm{Mult}(\mathcal{H} \odot \mathcal{H})$ *is dense in* $\mathcal{M}$ *for every* $\mathrm{Mult}(\mathcal{H} \odot \mathcal{H})$*-invariant subspace* $\mathcal{M}$ *of* $\mathcal{H} \odot \mathcal{H}$.

In particular, this implies that every non-zero multiplier invariant subspace of $\mathcal{H} \odot \mathcal{H}$ contains a non-zero multiplier. It follows that if $\mathrm{Mult}(\mathcal{H} \odot \mathcal{H})$ contains no zero-divisors (for instance if $\mathrm{Mult}(\mathcal{H} \odot \mathcal{H})$ consists of analytic functions on a domain in $\mathbb{C}^d$), then the lattice of multiplier invariant subspaces of $\mathcal{H} \odot \mathcal{H}$ is cellularly indecomposable, i.e. whenever $\mathcal{M}, \mathcal{N}$ are such invariant subspaces with $\mathcal{M} \neq (0), \mathcal{N} \neq (0)$, then $\mathcal{M} \cap \mathcal{N} \neq (0)$. For $\mathcal{H} = D$ (the Dirichlet space of the unit disc) this together with Proposition 3.6 of [20] provides a new proof of the theorem of Luo's that says that all nonzero multiplier invariant subspaces $\mathcal{M}$ of $D \odot D$ have index 1, i.e. they satisfy $\dim \mathcal{M} \ominus z\mathcal{M} = 1$, [16]. Our Theorem also shows that the same results hold in other weighted Dirichlet spaces of the unit disc.

Under a technical hypothesis that is satisfied for many weighted Besov spaces in the unit ball of $\mathbb{C}^d$ we obtain a strengthened version of the previous theorems. Every sequence $\Phi = \{\varphi_1, \varphi_2, ...\} \subseteq \mathrm{Mult}(\mathcal{H})$ of multipliers of a Hilbert function space $\mathcal{H}$ can be used to define a column operator $\Phi^C : h \to (\varphi_1 h, \varphi_2 h, ...)^T$ and a row operator $\Phi^R : (h_1, h_2, ...)^T \to \sum_{i \geq 1} \varphi_i h_i$. Here we have used $(h_1, ...)^T$ to denote a transpose of a row vector. We write $M^C(\mathcal{H})$ for the set of bounded column multiplication operators $\mathcal{H} \to \bigoplus_{n=1}^\infty \mathcal{H}$ and $M^R(\mathcal{H})$ for the set of bounded row multiplication operators $\bigoplus_{n=1}^\infty \mathcal{H} \to \mathcal{H}$.

Theorem 1.3. *Let* $\mathcal{H}$ *be a separable Hilbert function space on the non-empty set* X *such that the reproducing kernel for* $\mathcal{H}$ *is a normalized complete Pick kernel.*

If $M^C(\mathcal{H}) \subseteq M^R(\mathcal{H})$ *continuously, then*

$$\mathcal{H} \odot \mathcal{H} \subseteq N^+(\mathcal{H}).$$

Furthermore, every $\mathrm{Mult}(\mathcal{H})$*-invariant subspace of* $\mathcal{H} \odot \mathcal{H}$ *is* $\mathrm{Mult}(\mathcal{H} \odot \mathcal{H})$*-invariant, and the map*

$$\eta : \mathcal{N} \to \mathrm{clos}_{\mathcal{H} \odot \mathcal{H}} \mathcal{N}$$

establishes a 1-1 and onto correspondence between the multiplier invariant subspaces of $\mathcal{H}$ *and of* $\mathcal{H} \odot \mathcal{H}$. *We have*

(i) $\mathcal{M} = \mathrm{clos}_{\mathcal{H} \odot \mathcal{H}}(\mathcal{M} \cap \mathrm{Mult}(\mathcal{H}))$ *for every multiplier invariant subspace* $\mathcal{M}$ *of* $\mathcal{H} \odot \mathcal{H}$, *and*

(ii) $\mathcal{N} = \mathcal{H} \cap \mathrm{clos}_{\mathcal{H} \odot \mathcal{H}} \mathcal{N}$ *for every multiplier invariant subspace* $\mathcal{N}$ *of* $\mathcal{H}$.

Trent showed that for the Dirichlet space D of the unit disc $\mathbb{D} \subseteq \mathbb{C}$ one has the continuous inclusion $M^C(D) \subseteq M^R(D)$, but that $M^R(D) \nsubseteq M^C(D)$. In fact, he showed that for the Dirichlet space the norm of the inclusion is at most $\sqrt{18}$, see Lemma 1 of [26]. We will establish a generalization of Trent's Theorem to many weighted Besov spaces in the unit ball $\mathbb{B}_d$ of $\mathbb{C}^d$.

A non-negative integrable function ω on $\mathbb{B}_d$ is called an admissible weight, if the weighted Bergman space $L^2_a(\omega) = L^2(\omega dV) \cap \mathrm{Hol}(\mathbb{B}_d)$ is closed in $L^2(\omega dV)$, and if point evaluations $f \to f(z)$ are bounded on $L^2_a(\omega)$ for each $z \in \mathbb{B}_d$. Here V is used to denote Lebesgue measure on $\mathbb{C}^d$ restricted to $\mathbb{B}_d$, normalized so that $V(\mathbb{B}_d) = 1$. Radial weights are non-negative integrable functions such that for each $0 \leq r < 1$ the value $\omega(rz)$ is independent of $z \in \partial\mathbb{B}_d$, and one easily checks that a radial weight is admissible, if and only if $\int_{|z|>t} \omega dV > 0$ for each $t \in [0, 1)$. If ω is radial, then we have

$$\|f\|^2_{L^2_a(\omega)} = \int_{\mathbb{B}_d} |f|^2 \omega dV = \sum_{n\geq 0} \|f_n\|^2_{L^2_a(\omega)},$$

where $f = \sum_{n\geq 0} f_n$ is the decomposition of the analytic function f into a sum of homogeneous polynomials f_n of degree n.

Let $R = \sum_{i=1}^d z_i \frac{\partial}{\partial z_i}$ denote the radial derivative operator, then $Rf = \sum_{n\geq 1} n f_n$. More generally, for each nonzero $t \in \mathbb{R}$ we may consider the "fractional" transformation $R^t : \sum_{n\geq 0} f_n \to \sum_{n\geq 1} n^t f_n$.

For a positive integer N and an admissible weight ω we define

$$B^N_\omega = \{f \in \mathrm{Hol}(\mathbb{B}_d) : R^N f \in L^2_a(\omega)\},$$

$$\|f\|^2_{B^N_\omega} = \|\omega\|_{L^1(V)} |f(0)|^2 + \int_{\mathbb{B}_d} |R^N f|^2 \omega dV.$$

We also write $B^0_\omega = L^2_a(\omega)$. A space $\mathcal{H}$ of analytic functions that occurs as one of the spaces B^N_ω for an admissible weight ω and a non-negative integer N will be called a weighted Besov space.

If ω is an admissible radial weight, then the spaces B^N_ω are part of a one-parameter family of spaces defined for $s \in \mathbb{R}$ by

$$\|f\|^2_{B^s_\omega} = \|f_0\|^2_{L^2_a(\omega)} + \sum_{n\geq 1} n^{2s} \|f_n\|^2_{L^2_a(\omega)} < \infty, \tag{1.1}$$

where as above $f = \sum_{n\geq 0} f_n \in \mathrm{Hol}(\mathbb{B}_d)$.

If $\omega(z) = 1$, $s \in \mathbb{R}$, and $f \in \mathrm{Hol}(\mathbb{B}_d)$, then $f \in B^s_\omega$ if and only if $R^s f \in L^2_a$, the unweighted Bergman space. Thus, in this case the collection B^s_ω consists of standard weighted Bergman or Besov spaces. We have $B^{d/2}_{\mathbf{1}} = H^2_d$, the Drury-Arveson space, $B^{1/2}_{\mathbf{1}} = H^2(\partial\mathbb{B}_d)$, the Hardy space of the Ball, and for $s < 1/2$ we obtain the weighted Bergman spaces $B^s_{\mathbf{1}} = L^2_a((1-|z|^2)^{-2s}dV)$, where all equalities are understood to mean equality of spaces with equivalence of norms. These spaces have been extensively studied in the literature. We refer the reader to [27], where the L^p-analogues of these spaces were considered as well. If $d = 1$ and $s = 1$, then $B^1_{\mathbf{1}} = D$, the classical Dirichlet space of the unit disc. More generally, if $d = 1$ and $s > 1/2$, then these spaces are usually referred to as Dirichlet-type spaces, see [9].

If $\omega_\alpha(z) = (1-|z|^2)^\alpha$ for some $\alpha > -1$, then ω_α is called a standard weight, and we obtain the same spaces as for $\omega = 1$, but with a shift in indices: $B^s_{\omega_\alpha} = B^{s-\frac{\alpha}{2}}_{\mathbf{1}}$. This can be verified by using polar coordinates in (1.1) and the asymptotics $\int_0^1 t^n(1-t)^\alpha dt = \frac{\Gamma(n+1)\Gamma(\alpha+1)}{\Gamma(n+\alpha+2)} \approx n^{-\alpha-1}$, which follows e.g. from Stirling's formula. In particular, it follows that $B^s_{\mathbf{1}}$ is a weighted Besov space for any $s \in \mathbb{R}$.

Definition 1.4. *Let ω be an admissible weight on $\mathbb{B}_d$ and let $N \in \mathbb{N}$. We say that B^N_ω satisfies the multiplier inclusion condition, if*

$$M^C(B^N_\omega) \subseteq M^C(B^{N-1}_\omega) \subseteq \cdots \subseteq M^C(B^0_\omega)$$

with continuous inclusions.

Theorem 1.5. *Let $N \in \mathbb{N}$ and let ω be an admissible weight such that B^N_ω satisfies the multiplier inclusion condition.*

Then $M^C(B^N_\omega) \subseteq M^R(B^N_\omega)$ and there is a $c > 0$ such that

$$\|\Phi^R\|_{B^N_\omega} \le c\|\Phi^C\|_{B^N_\omega}$$

for all $\Phi \in M^C(B^N_\omega)$.

It is known and easy to verify that $M^C(L^2_a(\omega)) = M^R(L^2_a(\omega)) = H^\infty(\ell_2)$, where

$$H^\infty(\ell_2) = \Big\{(\varphi_1, \varphi_2, ...) : \varphi_j \in H^\infty \text{ and } \sup_{z\in\mathbb{B}_d} \sum_j |\varphi_j(z)|^2 < \infty\Big\}.$$

Similarly, it is a standard fact that for each $n \in \mathbb{N}$ one has $M^C(B^n_\omega) \subseteq H^\infty(\ell_2)$. Thus for $N = 1$ every weighted Besov space satisfies the multiplier inclusion condition, and hence $M^C(B^1_\omega) \subseteq M^R(B^1_\omega)$ holds for all admissible weights.

In Section 4 we will provide a short and elementary proof that shows that every Hilbert space of analytic functions in $\mathbb{B}_d$ whose reproducing kernel is of the type $k_w(z) = (1-\langle z, w\rangle)^\alpha$, $\alpha < 0$, satisfies the multiplier inclusion condition. This includes the Drury-Arveson space.

A second approach to the multiplier inclusion condition is via complex interpolation. Indeed, if the spaces $\{B^n_\omega\}$ and $\{B^n_\omega \otimes \ell_2\}$, $n =$

$0, 1, ..., N$, are part of interpolation scales $\{B^s_\omega\}$ and $\{B^s_\omega \otimes \ell_2\}$, $0 \le s \le N, s \in \mathbb{R}$, obtained by the complex method, then the functorial property of the interpolation implies that the hypothesis of Theorem 1.5 reduces to $M^C(B^N_\omega) \subseteq M^C(B^0_\omega) = H^\infty(\ell_2)$. See [8] for information about the complex method.

Thus, our theorem implies that if the spaces $\{B^N_\omega\}_{N\in\mathbb{N}_0}$ and $\{B^N_\omega \otimes \ell_2\}_{N\in\mathbb{N}_0}$ are part of interpolation scales obtained by the complex method, then every bounded column multiplication operator on B^N_ω induces a bounded row operator.

For standard weights and more generally for weights that satisfy a Bekollé-Bonami condition it was shown in [11] and [10] that $\{B^s_\omega\}_{s\in\mathbb{R}}$ is an interpolation scale, and that the spaces satisfy the scalar version of the multiplier inclusion condition (for standard weights also see [27]). The full column operator multiplier inclusion condition follows similarly in those cases. In [4] we similarly show that in fact for every admissible radial measure and every $s \in \mathbb{R}$ the space B^s_ω satisfies multiplier inclusion condition and the conclusion of Theorem 1.5 holds. In that paper we also show that if a radial measure ω satisfies that for some $\alpha > -1$ the ratio $\omega(z)/(1 - |z|^2)^\alpha$ is nondecreasing for $t_0 < |z| < 1$, then B^s_ω is a complete Pick space, whenever $s \ge (\alpha + d)/2$. By a complete Pick space we mean a Hilbert function space $\mathcal{H}$ such that there is a norm on $\mathcal{H}$ that is equivalent to the original one, and such that the reproducing kernel for one of the norms is a normalized complete Pick kernel.

As another application of Theorem 1.5 we mention that it provides a new proof of the main result of [3] in the case where the complete Pick space is a radially weighted Besov space. Indeed, in [3] the proof of the characterization of the interpolating sequences for all spaces with complete Pick kernel is based on the Marcus-Spielman-Srivastava theorem [18], but in Remark 3.7 and Theorem 3.8 of [3] it is explained how an application of Theorem 1.5 provides an alternate proof. We particularly point out that this approach provides a very direct proof for the case of the Drury-Arveson space on a finite dimensional ball. For this only the results of Section 4 are needed.

This paper is organized as follows. In Section 2 we prove that the weak product always carries a weak* topology such that point evaluations are weak*-continuous, and we review the connection between $(\mathcal{H} \odot \mathcal{H})^*$ and Hankel operators. Corollary 3.4 contains Theorem 1.1 and the Smirnov class inclusions of Theorem 1.3, while in Theorem 3.3 we have provided a technical version which gives more information.

In Section 3.2 we have proved Theorem 1.2 and the parts of Theorem 1.3 that give information about the invariant subspaces. Corollary 3.8 states that if a complete Pick space $\mathcal{H}$ satisfies the condition $M^C(\mathcal{H}) \subseteq M^R(\mathcal{H})$, then all multiplier invariant subspaces of $\mathcal{H}$ are equal to a countable intersection of null spaces of bounded Hankel operators. This extends results of [17] and [22]. Section 4 is independent of the results of Sections 2 and 3, it contains our results on weighted Besov spaces. Theorem 1.5 will be a special case of Theorem 4.2, where operators between possibly different spaces are considered. In [26] Trent has provided an example that shows that $M^R(D) \nsubseteq M^C(D)$. Since Trent's example does not immediately generalize from the Dirichlet space to the Drury-Arveson space, we have provided an example of a bounded row multiplication operator on $H^2_d, d > 1$, that does not induce a bounded column operator, see Section 4.2.

2. Background on the weak product of a Hilbert function space

In [21] some general results about the weak product $\mathcal{H} \odot \mathcal{H}$ and its dual were shown for the case when $\mathcal{H}$ is a Hilbert space of analytic functions. In particular, it was shown that $\mathcal{H} \odot \mathcal{H}$ is always a Banach function space and that its dual can be identified with a space of Hankel operators, provided the space $\mathcal{H}$ satisfied a certain extra hypothesis. This makes it reasonable to conjecture that $\mathcal{H} \odot \mathcal{H}$ has an isometric predual which can be identified with a space of compact Hankel operators. We will now show that indeed for any Hilbert function space $\mathcal{H}$ the weak product has an isometric predual and if $\text{Mult}(\mathcal{H})$ is densely contained in $\mathcal{H}$, then the dual and predual of $\mathcal{H} \odot \mathcal{H}$ can each be identified with Hankel operators defined by symbol sets that are contained in $\mathcal{H}$.

2.1. The predual of $\mathcal{H} \odot \mathcal{H}$.

Theorem 2.1. *Let $\mathcal{H}$ be a Hilbert function space on a set X. Then the weak product $\mathcal{H} \odot \mathcal{H}$ is a Banach function space on X. It is isometrically isomorphic to the dual of a Banach space in such a way that point evaluations on $\mathcal{H} \odot \mathcal{H}$ are weak* continuous with respect to the duality.*

Proof. Let $\mathcal{H} \otimes_\pi \mathcal{H}$ denote the Banach space projective tensor product of $\mathcal{H}$ with itself, that is, the completion of the algebraic tensor product with respect to the norm

$$||h|| = \inf \Big\{ \sum_{n=1}^{n} ||f_n|| \, ||g_n|| : h = \sum_{n=1}^{n} f_n \otimes g_n \Big\}.$$

Every element u of $\mathcal{H} \otimes_\pi \mathcal{H}$ can in fact be written in the form

$$u = \sum_{n=1}^\infty f_n \otimes g_n \quad \text{with} \sum_{n=1}^\infty ||f_n|| \, ||g_n|| < \infty,$$

see e.g. [24]. In the following we will use the Hilbert space of complex conjugates

$$\overline{\mathcal{H}} = \{\overline{f} : f \in \mathcal{H}\},$$

which has inner product given by $\langle \overline{f}, \overline{g} \rangle = \langle g, f \rangle$ and can be isometrically identified with the dual $\mathcal{H}^*$ via the correspondence $\overline{f} \to L_{\overline{f}}$, $L_{\overline{f}}(g) = \langle g, f \rangle_{\mathcal{H}}$ for $g \in \mathcal{H}$.

By definition of the weak product, the map

$$\rho : \mathcal{H} \otimes_\pi \mathcal{H} \to \mathcal{H} \odot \mathcal{H}, \quad \rho\Big(\sum_{n=1}^\infty f_n \otimes g_n \Big)(z) = \sum_{n=1}^\infty f_n(z) g_n(z)$$

is a quotient map. For $z \in X$, let $E_z = \overline{k}_z \otimes \overline{k}_z \in (\mathcal{H} \otimes_\pi \mathcal{H})^*$ denote the functional of evaluation at z. Then

$$\ker \rho = \bigcap_{z \in X} \ker E_z,$$

thus ρ induces an isometric isomorphism $(\mathcal{H} \otimes_\pi \mathcal{H}) / \ker \rho \cong \mathcal{H} \odot \mathcal{H}$.

Let $\mathcal{C}_1(\overline{\mathcal{H}}, \mathcal{H})$ denote the space of all trace class operators from $\overline{\mathcal{H}}$ to $\mathcal{H}$. Then $\mathcal{H} \otimes_\pi \mathcal{H}$ can be isometrically identified with $\mathcal{C}_1(\overline{\mathcal{H}}, \mathcal{H})$ via the map

$$\Phi : \mathcal{H} \otimes_\pi \mathcal{H} \to \mathcal{C}_1(\overline{\mathcal{H}}, \mathcal{H}), \quad \Phi(f \otimes g)(\overline{h}) = \langle g, h \rangle f.$$

On the other hand, $\mathcal{C}_1(\overline{\mathcal{H}}, \mathcal{H})$ is the dual space of the space of compact operators from $\mathcal{H}$ to $\overline{\mathcal{H}}$ via trace duality. Thus, $\mathcal{H} \otimes_\pi \mathcal{H}$ becomes a dual space in this way, and every functional of the form $\overline{f} \otimes \overline{g}$ on $\mathcal{H} \otimes_\pi \mathcal{H}$ for $\overline{f}, \overline{g} \in \overline{\mathcal{H}}$ is weak-$*$ continuous. In particular, it follows that $\ker \rho$ is weak-$*$ closed and thus $\mathcal{H} \odot \mathcal{H}$ can be identified with the dual of ${}^\perp \ker \rho$. Since $\overline{k}_z \otimes \overline{k}_z$ belongs to ${}^\perp \ker \rho$ for each z, point evaluations on $\mathcal{H} \odot \mathcal{H}$ are weak-$*$ continuous with respect to this duality. ■

It follows from the Hahn-Banach theorem that the linear span of the point evaluations is dense in the predual of $\mathcal{H} \odot \mathcal{H}$. Using the uniform boundedness principle, we therefore obtain the following standard corollary.

Corollary 2.2. *Let $\mathcal{H}$ be a Hilbert function space on a set X, and let $h_n \in \mathcal{H} \odot \mathcal{H}$ be a sequence of functions. Then the following are equivalent for a function h on X:*

(a) $h \in \mathcal{H} \odot \mathcal{H}$ and $h_n \to h$ in the weak topology given by the previous theorem,*

(b) $\|h_n\|_{\mathcal{H}\odot\mathcal{H}} \le C$ *and* $h_n(z) \to h(z)$ *for all* $z \in X$.

We further remark that if $\mathcal{H}$ is separable, then so are $\mathcal{H}\odot\mathcal{H}$ and its predual. It follows that in this case the closed unit ball of $\mathcal{H}\odot\mathcal{H}$ is compact metrizable in the weak* topology.

2.2. **The Connection to Hankel operators.** Since the dual space of $\mathcal{C}_1(\overline{\mathcal{H}}, \mathcal{H})$ is the space $\mathcal{B}(\mathcal{H}, \overline{\mathcal{H}})$ via trace duality and since $\mathcal{H}\otimes_\pi\mathcal{H} \cong \mathcal{C}_1(\overline{\mathcal{H}}, \mathcal{H})$, every $T \in \mathcal{B}(\mathcal{H}, \overline{\mathcal{H}})$ defines a linear functional on $\mathcal{H}\otimes_\pi\mathcal{H}$ by

$$f \otimes g \to \langle g, \overline{Tf}\rangle. \tag{2.1}$$

Let

$$\rho : \mathcal{H}\otimes_\pi\mathcal{H} \to \mathcal{H}\odot\mathcal{H}$$

be the quotient map from the proof of Theorem 2.1. Then

$$(\mathcal{H}\odot\mathcal{H})^* \cong (\ker\rho)^\perp \subseteq (\mathcal{H}\otimes_\pi\mathcal{H})^* \cong \mathcal{B}(\mathcal{H}, \overline{\mathcal{H}}).$$

We will now see that if the multipliers are dense in $\mathcal{H}$, then the operators in $(\ker\rho)^\perp$ can be considered to be little Hankel operators, each of which is identified with a symbol from the space $\mathcal{H}$.

Lemma 2.3. *If* $T \in (\ker\rho)^\perp$, *then*
(a) $T^*\overline{f} = \overline{Tf}$ *for every* $f \in \mathcal{H}$, *and*
(b) $TM_\varphi = M^*_{\overline{\varphi}}T$ *for every* $\varphi \in \mathrm{Mult}(\mathcal{H})$.

Furthermore, if $\mathrm{Mult}(\mathcal{H})$ *is densely contained in* $\mathcal{H}$, *then for* $T \in (\ker\rho)^\perp$ *we have* $T = 0$ *if and only if* $T1 = 0$.

Proof. Let $T \in (\ker\rho)^\perp$. Then for $f, g \in \mathcal{H}$ and $\varphi \in \mathrm{Mult}(\mathcal{H})$ we have $f\otimes\varphi g - g\otimes\varphi f \in \ker\rho$, hence $\langle\varphi g, \overline{Tf}\rangle = \langle\varphi f, \overline{Tg}\rangle = \langle Tg, \overline{\varphi f}\rangle_{\overline{\mathcal{H}}} = \langle g, T^*\overline{\varphi f}\rangle$. Thus $T^*\overline{\varphi f} = M^*_\varphi\overline{Tf}$, and (a) follows by taking $\varphi = 1$.

Next we substitute (a) into $T^*M_{\overline{\varphi}}\overline{f} = T^*\overline{\varphi f} = M^*_\varphi\overline{Tf}$, then (b) follows by taking adjoints.

The remaining part of the Lemma follows from (b). ∎

Thus, if $\mathrm{Mult}(\mathcal{H})$ is densely contained in $\mathcal{H}$, then an operator in $T \in (\ker\rho)^\perp$ is uniquely associated with the function $T1$, and T intertwines multiplication operators and adjoints of multiplication operators, hence T deserves to be called a Hankel operator.

Definition 2.4. *Let* $\mathcal{H}$ *be a Hilbert function space such that* $\mathrm{Mult}(\mathcal{H})$ *is densely contained in* $\mathcal{H}$. *Then define*

$$\mathrm{Han}(\mathcal{H}) = \{\overline{T1} \in \mathcal{H} : T \in (\ker\rho)^\perp\}.$$

For $b \in \mathrm{Han}(\mathcal{H})$ *we write* $H_b \in \mathcal{B}(\mathcal{H}, \overline{\mathcal{H}})$ *for the unique operator in* $(\ker\rho)^\perp$ *that satisfies* $H_b1 = \overline{b}$, *and we set* $\|b\|_{\mathrm{Han}(\mathcal{H})} = \|H_b\|$.

Furthermore, we define

$$\mathrm{Han}_0(\mathcal{H}) = \{b \in \mathrm{Han}(\mathcal{H}) : H_b \textit{ is compact } \}.$$

With these definitions we have that $\mathrm{Han}_0(\mathcal{H})$ is isometrically isomorphic to ${}^\perp(\ker \rho)$, and the following Theorem holds.

Theorem 2.5. *Let $\mathcal{H}$ be a Hilbert function space such that* $\mathrm{Mult}(\mathcal{H})$ *is densely contained in $\mathcal{H}$. Then the following conjugate linear isometric isomorphisms hold:*
(a) $\mathrm{Han}_0(\mathcal{H})^* \cong \mathcal{H} \odot \mathcal{H}$ *and*
(b) $(\mathcal{H} \odot \mathcal{H})^* \cong \mathrm{Han}(\mathcal{H})$.
If $b \in \mathrm{Han}(\mathcal{H})$, then the associated linear functional L_b satisfies

$$L_b(\varphi f) = \langle \varphi f, b\rangle = \langle f, \overline{H_b\varphi}\rangle = \langle \varphi, \overline{H_b f}\rangle$$

for every $f \in \mathcal{H}$ and $\varphi \in \mathrm{Mult}(\mathcal{H})$.

Proof. We have explained the isometric isomorphisms above. Let $f \in \mathcal{H}$ and $\varphi \in \mathrm{Mult}(\mathcal{H})$. According to (2.1) and the definition of H_b we have $L_b(\varphi f) = \langle f, \overline{H_b\varphi}\rangle = \langle \varphi, \overline{H_b f}\rangle$. But then Lemma 2.3 implies

$$L_b(\varphi f) = \langle f, H_b^* \overline{\varphi}\rangle = \langle f, M_\varphi^* H_b^* 1\rangle = \langle \varphi f, \overline{H_b 1}\rangle = \langle \varphi f, b\rangle.$$

■

Theorem 2.5 does not address the question of how one can easily identify which functions are in $\mathrm{Han}(\mathcal{H})$. The set

$$\mathcal{X}(\mathcal{H}) = \{b \in \mathcal{H} : \exists C \geq 0 |\langle \varphi f, b\rangle| \leq C ||\varphi||_{\mathcal{H}}\, ||f||_{\mathcal{H}} \forall f \in \mathcal{H}, \varphi \in \mathrm{Mult}(\mathcal{H})\}$$

seems more suited for that question, and Theorem 2.5 implies that $\mathrm{Han}(\mathcal{H}) \subseteq \mathcal{X}(\mathcal{H})$.

We note that Theorems 1.2 and 1.3 of [21] imply that $\mathrm{Han}(B^s_\omega) = \mathcal{X}(B^s_\omega) = (B^s_\omega \otimes B^s_\omega)^*$ for all admissible radial weights ω and all $s \in \mathbb{R}$. Using the main result of [2], we will now show that the equality $\mathrm{Han}(\mathcal{H}) = \mathcal{X}(\mathcal{H})$ also holds whenever $\mathcal{H}$ is a complete Pick space with $M^C(\mathcal{H}) \subseteq M^R(\mathcal{H})$.

Theorem 2.6. *Let $\mathcal{H}$ be a separable Hilbert function space on the nonempty set X, and suppose that the reproducing kernel for $\mathcal{H}$ is a complete Pick kernel, which is normalized at a point $z_0 \in X$.*
If $M^C(\mathcal{H}) \subseteq M^R(\mathcal{H})$, then $\mathrm{Han}(\mathcal{H}) = \mathcal{X}(\mathcal{H})$.

Proof. As mentioned above, by Theorem 2.5 it suffices to show that $\mathcal{X}(\mathcal{H}) \subseteq \mathrm{Han}(\mathcal{H})$. Let $b \in \mathcal{X}(\mathcal{H})$. In order to show that $b \in \mathrm{Han}(\mathcal{H})$, we note that the definition of $\mathcal{X}(\mathcal{H})$ and the universal property of the projective tensor product show that there exists a bounded linear functional L on $\mathcal{H} \otimes_\pi \mathcal{H}$ such that

$$L(f \otimes \varphi) = \langle f\varphi, b \rangle$$

for all $f \in \mathcal{H}$ and $\varphi \in \operatorname{Mult}(\mathcal{H})$. We claim that $L \in (\ker \rho)^\perp$. Assuming this claim for a moment, we can regard L as a functional on $\mathcal{H} \odot \mathcal{H}$, hence by Theorem 2.5, there exists $c \in \operatorname{Han}(\mathcal{H})$ such that

$$\langle \varphi f, c \rangle = L(\varphi f) = \langle \varphi f, b \rangle$$

for all $f \in \mathcal{H}, \varphi \in \operatorname{Mult}(\mathcal{H})$, so that $b = c \in \operatorname{Han}(\mathcal{H})$.

To prove the claim, let $h \in \ker \rho$ with $||h||_{\mathcal{H} \otimes_\pi \mathcal{H}} < 1$. We wish to show that $L(h) = 0$. To this end, observe that there exist $f_n, g_n \in \mathcal{H}$ with $||f_n|| = ||g_n||$ for all n,

$$h = \sum_{n=1}^\infty f_n \otimes g_n$$

and $\sum_{n=1}^\infty ||f_n||^2 = 1$. By [2, Theorem 1.1], there exist $\psi, \varphi_n \in \operatorname{Mult}(\mathcal{H})$ such that $\psi(z_0) = 0$,

$$||\psi h||^2 + \sum_n ||\varphi_n h||^2 \le ||h||^2 \quad (h \in \mathcal{H})$$

and $f_n = \frac{\varphi_n}{1-\psi}$ for all $n \in \mathbb{N}$. For $r \in (0,1)$, let

$$f_n^{(r)} = \frac{\varphi_n}{1 - r\psi}.$$

Then $[f_n^{(r)}]_{n=1}^\infty \in \operatorname{Mult}(\mathcal{H}, \mathcal{H}(\ell^2))$ for each $r < 1$ by [2, Lemma 3.6 (i)]. Moreover, by the remark at the end of Section 3 of [2], $[f_n^{(r)}]$ converges to $[f_n]$ in the norm of $\mathcal{H}(\ell^2)$ as $r \to 1$. Thus,

$$\begin{aligned} &\Big\| \sum_n f_n^{(r)} \otimes g_n - \sum_n f_n \otimes g_n \Big\|^2_{\mathcal{H} \otimes_\pi \mathcal{H}} \\ \le &\Big(\sum_n ||f_n^{(r)} - f_n||^2 \Big) \Big(\sum_n ||g_n||^2 \Big) \xrightarrow{r \to 1} 0, \end{aligned}$$

so by continuity of L, it suffices to show that

$$L\Big(\sum_n f_n^{(r)} \otimes g_n \Big) = 0$$

for all $r \in (0,1)$.

To see this, fix $r \in (0,1)$. The series $\sum_n f_n^{(r)} \otimes g_n$ converges absolutely in the norm of $\mathcal{H} \otimes_\pi \mathcal{H}$, so that

$$L\Big(\sum_n f_n^{(r)} \otimes g_n \Big) = \sum_n L(f_n^{(r)} \otimes g_n) = \sum_n \langle f_n^{(r)} g_n, b \rangle.$$

Since $[f_n^{(r)}] \in \mathrm{Mult}(\mathcal{H}, \mathcal{H}(\ell^2))$, the assumption $M^C(\mathcal{H}) \subseteq M^R(\mathcal{H})$ implies that the series $\sum_n f_n^{(r)} g_n$ converges in $\mathcal{H}$, so that

$$L\Big(\sum_n f_n^{(r)} \otimes g_n\Big) = \Big\langle \sum_n f_n^{(r)} g_n, b\Big\rangle = 0,$$

where the last equality follows from the observation that since $h \in \ker \rho$, we also have

$$\sum_{n=1}^{\infty} f_n^{(r)}(z) g_n(z) = \frac{1-\psi(z)}{1-r\psi(z)} \sum_{n=1}^{\infty} f_n(z) g_n(z) = 0$$

for all $z \in X$. ■

3. Weak products of complete Pick spaces

3.1. Functions as ratios of multipliers.

Theorem 3.1. *Let $\mathcal{H}$ be a separable Hilbert function space on a set X with reproducing kernel $k_z \neq 0$ for all $z \in X$, and let $\{\Phi_n^C\}_{n\geq 1}, \{\Psi_n^C\}_{n\geq 1} \subseteq M^C(\mathcal{H})$ be sequences of column operators such that*

$$\sum_{n\geq 1} \|\Phi_n^C f\|^2_{\mathcal{H}\otimes \ell_2} \leq \|f\|^2_{\mathcal{H}} \text{ and } \sum_{n\geq 1} \|\Psi_n^C f\|^2_{\mathcal{H}\otimes \ell_2} \leq \|f\|^2_{\mathcal{H}}$$

for all $f \in \mathcal{H}$.

Then for each $n \in \mathbb{N}$ we have $\Psi_n^R \Phi_n^C \in \mathrm{Mult}(\mathcal{H} \odot \mathcal{H})$ and

$$\sum_{n\geq 1} \|\Psi_n^R \Phi_n^C h\|_{\mathcal{H}\odot\mathcal{H}} \leq \|h\|_{\mathcal{H}\odot\mathcal{H}}$$

for all $h \in \mathcal{H} \odot \mathcal{H}$.

Furthermore, if $\mathcal{H}$ satisfies the continuous inclusion $M^C(\mathcal{H}) \subseteq M^R(\mathcal{H})$, then for each $n \in \mathbb{N}$ we have $\Psi_n^R \Phi_n^C \in \mathrm{Mult}(\mathcal{H})$ and there is a $c > 0$ such that for all $f \in \mathcal{H}$ we have

$$\sum_{n\geq 1} \|\Psi_n^R \Phi_n^C f\|^2_{\mathcal{H}} \leq c\|f\|^2_{\mathcal{H}} \text{ and}$$

$$\|\sum_{n=1}^{N} \Psi_n^R \Phi_n^C f\|^2_{\mathcal{H}} \leq c\|f\|^2_{\mathcal{H}} \text{ for each } N \in \mathbb{N}.$$

Proof. For each $n \in \mathbb{N}$ let $\Phi_n = \{\varphi_{ni}\}_{i\geq 1}, \Psi_n = \{\psi_{ni}\}_{i\geq 1}$ define bounded column operators that satisfy the hypothesis of the Theorem. Then for any $f, g \in \mathcal{H}$ we have

$$\sum_{i\geq 1} \|\varphi_{ni}\psi_{ni} f g\|_{\mathcal{H}\odot\mathcal{H}} \leq \sum_{i\geq 1} \|\varphi_{ni} f\| \|\psi_{ni} g\| \leq \|\Phi_n^C f\| \|\Psi_n^C g\|.$$

It is well-known and easy to show that $\|\Phi_n(z)\|_{\ell_2} \le \|\Phi_n^C\|_{M^C(\mathcal{H})}$ and $\|\Psi_n(z)\|_{\ell_2} \le \|\Psi_n^C\|_{M^C(\mathcal{H})}$ for each $z \in X$. Thus $\Psi_n^R(z)\Phi_n^C(z) = \sum_{i\ge1}\psi_{ni}(z)\varphi_{ni}(z)$ converges absolutely, $(\Psi_n^R(z)\Phi_n^C(z))f(z)g(z) = \sum_{i\ge1}(\varphi_{ni}f)(z)(\psi_{ni}g)(z)$, and hence

$$\begin{aligned}\sum_{n\ge1}\|(\Psi_n^R\Phi_n^C)fg\|_{\mathcal{H}\odot\mathcal{H}} &\le \sum_{n,i\ge1}\|\varphi_{ni}\psi_{ni}fg\|_{\mathcal{H}\odot\mathcal{H}}\\ &\le \sum_{n\ge1}\|\Phi_n^C f\|_{\mathcal{H}\otimes\ell^2}\|\Psi_n^C g\|_{\mathcal{H}\otimes\ell^2}\\ &\le \|f\|_{\mathcal{H}}\|g\|_{\mathcal{H}}.\end{aligned}$$

Let $h \in \mathcal{H}\odot\mathcal{H}$ and let $\{f_j\},\{g_j\} \in \bigoplus_{j=1}^\infty \mathcal{H}$ with $h = \sum_{j=1}^\infty f_j g_j$. Then

$$\begin{aligned}\sum_{n\ge1}\|(\Psi_n^R\Phi_n^C)h\|_{\mathcal{H}\odot\mathcal{H}} &\le \sum_{n,j=1}^{\infty}\|(\Psi_n^R\Phi_n^C)f_jg_j\|_{\mathcal{H}\odot\mathcal{H}}\\ &\le \sum_{j=1}^{\infty}\|f_j\|_{\mathcal{H}}\|g_j\|_{\mathcal{H}}.\end{aligned}$$

Taking the infimum over all possible representations $h = \sum_{j=1}^\infty f_j g_j$, we obtain $\sum_{n\ge1}\|(\Psi_n^R\Phi_n^C)h\|_{\mathcal{H}\odot\mathcal{H}} \le \|h\|_{\mathcal{H}\odot\mathcal{H}}$.

If every bounded column operator on $\mathcal{H}$ induces a bounded row operator and the inclusion has norm $\sqrt{c}$, then since for each n we have $\|\Psi_n^C\| \le 1$ it follows easily that $\Psi_n^R\Phi_n^C \in \text{Mult}(\mathcal{H})$ with $\|\Psi_n^R\Phi_n^C f\|^2 \le c\|\Phi_n^C f\|^2_{\mathcal{H}\otimes\ell_2}$. Thus, an application of the hypothesis finishes the proof of the Theorem. ∎

It was shown in [5], see also [2], that if $\mathcal{H}$ is a Hilbert function space on X with a complete Pick kernel, normalized at a point $z_0 \in X$, then every $f \in \mathcal{H}$ can be written as $f = \frac{\varphi}{1-\psi}$, where $\varphi, \psi \in \text{Mult}(\mathcal{H})$, $||\psi||_{\text{Mult}(\mathcal{H})} \le 1$ and $\psi(z_0) = 1$. In this case, $|\psi(z)| < 1$ for all $z \in X$, hence $\frac{\varphi}{1-\psi}$ is defined on X, see Lemma 2.2 of [5]. [2] also contains a vector version of this result. The following lemma is an analogue of this result for the weak product $\mathcal{H}\odot\mathcal{H}$.

Lemma 3.2. *Let $\mathcal{H}$ be a separable Hilbert function space on the nonempty set X, and suppose that the reproducing kernel for $\mathcal{H}$ is a complete Pick kernel, which is normalized at a point $z_0 \in X$.*

If $\{h_n\}_{n\ge1} \subseteq \mathcal{H}\odot\mathcal{H}$, $\sum_{n\ge1}\|h_n\|_{\mathcal{H}\odot\mathcal{H}} < 1$, then there are $\psi \in$ Mult$(\mathcal{H})$ *and $\{\Phi_n^C\}_{n\ge1} \subseteq M^C(\mathcal{H})$ such that*

(a) $\|\psi\|_{\text{Mult}(\mathcal{H})} \le 1$ and $\psi(z_0) = 0$,

(b) $\sum_{n\ge1}\|\Phi_n^C u\|^2_{\mathcal{H}\otimes\ell_2} \le \|u\|^2_{\mathcal{H}}$ for all $u \in \mathcal{H}$,

(c) for each $n \in \mathbb{N}$ $h_n = \frac{\varphi_n}{(1-\psi)^2}$ with $\varphi_n = \Phi_n^R \Phi_n^C \in \text{Mult}(\mathcal{H} \odot \mathcal{H})$ and

$$\sum_{n\geq 1} \|\varphi_n h\|_{\mathcal{H}\odot\mathcal{H}} \leq \|h\|_{\mathcal{H}\odot\mathcal{H}} \text{ for all } h \in \mathcal{H} \odot \mathcal{H}.$$

(d) If additionally it is true that $M^C(\mathcal{H}) \subseteq M^R(\mathcal{H})$, then there is a $c > 0$ such that

$$\sum_{n\geq 1} \|\varphi_n f\|_{\mathcal{H}}^2 \leq c\|f\|_{\mathcal{H}}^2$$

for each $f \in \mathcal{H}$. Furthermore, for each an $N \in \mathbb{N}$ we have

$$\|\sum_{n=1}^{N} \varphi_n\|_{\text{Mult}(\mathcal{H})} \leq c.$$

Proof. Note that if $f, g \in \mathcal{H}$ with $\|f\| = \|g\|$, then $fg = \left(\frac{f+g}{2}\right)^2 - \left(\frac{f-g}{2}\right)^2$ with $\|f\|\|g\| = \|\frac{f+g}{2}\|^2 + \|\frac{f-g}{2}\|^2$ by the parallelogram law. Thus, for any $h \in \mathcal{H} \odot \mathcal{H}$ and any $\varepsilon > 0$ there are $f_j \in \mathcal{H}$ with $\sum_{j\geq 1} \|f_j\|_{\mathcal{H}}^2 \leq \|h\|_{\mathcal{H}\odot\mathcal{H}} + \varepsilon$ and $h = \sum_{j\geq 1} f_j^2$. Hence for each n we choose a sequence $\{f_{nj}\}_{j\geq 1} \subseteq \mathcal{H}$ such that $h_n = \sum_{j\geq 1} f_{nj}^2$ and $\sum_{n,j\geq 1} \|f_{nj}\|_{\mathcal{H}}^2 \leq 1$.

Then by Theorem 1.1 of [2], there there are contractive multipliers ψ, $\{\varphi_{nj}\}_{n,j\geq 1}$ such that $\psi(z_0) = 0$, $\|\psi g\|^2 + \sum_{n,j\geq 1} \|\varphi_{nj} g\|^2 \leq \|g\|^2$ for all $g \in \mathcal{H}$, and $f_{nj} = \frac{\varphi_{nj}}{1-\psi}$ for all $n, j \geq 1$. Then $h_n = \frac{\varphi_n}{(1-\psi)^2}$ with $\varphi_n = \sum_{j\geq 1} \varphi_{nj}^2 = \Phi_n^R \Phi_n^C$, where $\Phi_n = \{\varphi_{nj}\}_{j\geq 1}$. Thus the lemma follows from Theorem 3.1 with $\Psi_n = \Phi_n$. ∎

The following Theorem is a slight refinement of the previous Lemma in the case of a single function.

Theorem 3.3. *Let $\mathcal{H}$ be a separable Hilbert function space on the non-empty set X, and suppose that the reproducing kernel for $\mathcal{H}$ is a complete Pick kernel, which is normalized at a point $z_0 \in X$.*

If $h \in \mathcal{H} \odot \mathcal{H}$, then there are $\varphi \in \text{Mult}(\mathcal{H} \odot \mathcal{H})$, $\|\varphi\|_{\text{Mult}(\mathcal{H}\odot\mathcal{H})} \leq \|h\|_{\mathcal{H}\odot\mathcal{H}}$ and $\psi \in \text{Mult}(\mathcal{H})$, $\|\psi\|_{\text{Mult}(\mathcal{H})} \leq 1$, $\psi(z_0) = 0$ such that $h = \frac{\varphi}{(1-\psi)^2}$.

Proof. We assume $\|h\|_{\mathcal{H}\odot\mathcal{H}} = 1$. For each $m \in \mathbb{N}$ we apply the single function version of Lemma 3.2 with $(1 - \frac{1}{m+1})h$ and thus obtain functions φ_m, ψ_m with $\|\psi_m\|_{\text{Mult}(\mathcal{H})} \leq 1$, $\psi_m(z_0) = 0$, $\|\varphi_m\|_{\text{Mult}(\mathcal{H}\odot\mathcal{H})} \leq 1 + 1/m$, and $h = \frac{\varphi_m}{1-\psi_m}$. It follows from the hypothesis that $\mathcal{H}$ and $\mathcal{H} \odot \mathcal{H}$ are separable, thus we can assume that there are subsequences such that $\varphi_{m_j} \to \varphi$ in the weak* topology of $\mathcal{H} \odot \mathcal{H}$ and $\psi_{m_j} \to \psi$ weakly in $\mathcal{H}$.

Weak* and weak convergence imply pointwise convergence, and hence the norm bound and Corollary 2.2 imply $\varphi_{m_j} g \to \varphi g$ weak* in $\mathcal{H} \odot \mathcal{H}$

for each $g \in \mathcal{H} \odot \mathcal{H}$. Since this is valid for all $g \in \mathcal{H} \odot \mathcal{H}$ we conclude $\|\varphi\|_{\text{Mult}(\mathcal{H}\odot\mathcal{H})} \le 1$. Similarly $\|\psi\|_{\text{Mult}(\mathcal{H})} \le 1$. Since $\psi(z_0) = 0$ Lemma 2.2 of [5] implies that $|\psi(z)| < 1$ for all $z \in X$. Thus $\varphi/(1-\psi)^2$ is well-defined and in that case it clearly must equal h. ∎

Corollary 3.4. *Let $\mathcal{H}$ be a separable Hilbert function space on the non-empty set X, and suppose that the reproducing kernel for $\mathcal{H}$ is a normalized complete Pick kernel. Then*

$$\mathcal{H} \odot \mathcal{H} \subseteq N^+(\mathcal{H} \odot \mathcal{H}).$$

If additionally $M^C(\mathcal{H}) \subseteq M^R(\mathcal{H})$, then

$$\mathcal{H} \odot \mathcal{H} \subseteq N^+(\mathcal{H}).$$

Proof. Suppose the reproducing kernel is normalized at the point $z_0 \in X$. Let $h \in \mathcal{H}\odot\mathcal{H}$. By Lemma 3.2 we have $h = \frac{\varphi}{(1-\psi)^2}$ for $\psi \in \text{Mult}(\mathcal{H})$, $\Phi^C \in M^C(\mathcal{H})$ with $\|\psi\|_{\text{Mult}(\mathcal{H})} \le 1$, $\psi(z_0) = 0$, and $\varphi = \Phi^R\Phi^C \in \text{Mult}(\mathcal{H} \odot \mathcal{H})$ by Theorem 3.1.

It now follows immediately from Lemma 2.3 of [5] that $1-\psi$ is cyclic in $\mathcal{H}$. Furthermore, it is easy to see that products of cyclic multipliers are cyclic, hence the corollary follows. ∎

3.2. **Multiplier Invariant subspaces.** Recall that if $\mathcal{M}$ is a closed subspace of a Banach function space $\mathcal{B}$, then we say $\mathcal{M}$ is multiplier invariant, if $\varphi\mathcal{M} \subseteq \mathcal{M}$ for all multipliers $\varphi \in \text{Mult}(\mathcal{B})$. If $f \in \mathcal{B}$, then we write $[f]_{\mathcal{B}}$ for the smallest multiplier invariant subspace of $\mathcal{B}$ that contains f, i.e.

$$[f]_{\mathcal{B}} = \text{clos}_{\mathcal{B}}\{\varphi f : \varphi \in \text{Mult}(\mathcal{B})\}.$$

Thus, since $\text{Mult}(\mathcal{H}) \subseteq \text{Mult}(\mathcal{H} \odot \mathcal{H})$ we have

$$\text{clos}_{\mathcal{H}\odot\mathcal{H}}\{\varphi f : \varphi \in \text{Mult}(\mathcal{H})\} \subseteq [f]_{\mathcal{H}\odot\mathcal{H}}.$$

We will start this section by showing that for complete Pick spaces $\mathcal{H}$ with $M^C(\mathcal{H}) \subseteq M^R(\mathcal{H})$ these two sets are always the same.

Lemma 3.5. *Let $\mathcal{H}$ be a separable Hilbert function space on a set X. If $h_1, h_2 \in \mathcal{H} \odot \mathcal{H}$ and $\psi_n \in \text{Mult}(\mathcal{H})$ with*

(i) $\psi_n h_2 \in \text{clos}_{\mathcal{H}\odot\mathcal{H}}\{uh_1 : u \in \text{Mult}(\mathcal{H})\}$ *for each n,*
(ii) $\psi_n(z) \to 1$ *for each $z \in X$ and*
(iii) $\|\psi_n\|_{\text{Mult}(\mathcal{H})} \le C$ *for each n,*

then $h_2 \in \text{clos}_{\mathcal{H}\odot\mathcal{H}}\{uh_1 : u \in \text{Mult}(\mathcal{H})\} \subseteq [h_1]_{\mathcal{H}\odot\mathcal{H}}$.

Proof. Let M be the convex hull of $\{\psi_n : n \in \mathbb{N}\}$ inside of $\text{Mult}(\mathcal{H})$. It follows from assumptions (ii) and (iii) that 1 belongs to the WOT-closure of M. By convexity of M, there is a sequence (φ_n) in M

that converges to 1 in the strong operator topology of $\mathcal{H}$. It is then straightforward to check that the sequence $(\varphi_n h_2)$ converges to h_2 in the norm of $\mathcal{H} \odot \mathcal{H}$, so assumption (i) implies that $h_2 \in \operatorname{clos}_{\mathcal{H}\odot\mathcal{H}}\{uh_1 : u \in \operatorname{Mult}(\mathcal{H})\}$. ■

Lemma 3.6. *Let $\mathcal{H}$ be a separable Hilbert function space on a set X, and let $z_0 \in X$.*

(a) If $f, g \in \mathcal{H}$, $\psi \in \operatorname{Mult}(\mathcal{H})$, $\|\psi\|_{\operatorname{Mult}(\mathcal{H})} \leq 1$, $\psi(z_0) = 0$ such that $f = \frac{g}{1-\psi}$, then $[f]_{\mathcal{H}} = [g]_{\mathcal{H}}$.

(b) If $h, g \in \mathcal{H} \odot \mathcal{H}$, $\psi \in \operatorname{Mult}(\mathcal{H})$, $\|\psi\|_{\operatorname{Mult}(\mathcal{H})} \leq 1$, $\psi(z_0) = 0$ such that $h = \frac{g}{(1-\psi)^2}$, then $h \in \operatorname{clos}_{\mathcal{H}\odot\mathcal{H}}\{ug : u \in \operatorname{Mult}(\mathcal{H})\}$ and $[h]_{\mathcal{H}\odot\mathcal{H}} = [g]_{\mathcal{H}\odot\mathcal{H}}$.

Proof. (a) Let $f, g \in \mathcal{H}$, $\psi \in \operatorname{Mult}(\mathcal{H})$, $\|\psi\|_{\operatorname{Mult}(\mathcal{H})} \leq 1$, $\psi(z_0) = 0$ such that $f = \frac{g}{1-\psi}$. Then $g = (1-\psi)f \in [f]_{\mathcal{H}}$. Thus $[g]_{\mathcal{H}} \subseteq [f]_{\mathcal{H}}$.

Let $0 < r < 1$, then $1/(1 - r\psi) \in \operatorname{Mult}(\mathcal{H})$ and $\frac{g}{1-r\psi} \in [g]_{\mathcal{H}}$. A short calculation shows that $\|\frac{g}{1-r\psi} - f\| = \|\frac{(1-r)\psi}{1-r\psi} f\| \leq \|f\|$. Thus, it follows that $\frac{g}{1-r\psi}$ converges weakly to f as $r \to 1^-$. Hence $f \in [g]_{\mathcal{H}}$. This proves (a).

(b) Let $h, g \in \mathcal{H} \odot \mathcal{H}$, $\psi \in \operatorname{Mult}(\mathcal{H})$, $\|\psi\|_{\operatorname{Mult}(\mathcal{H})} \leq 1$, $\psi(z_0) = 0$ such that $h = \frac{g}{(1-\psi)^2}$. As in (a) the inclusion $g = (1-\psi)^2 h \in [h]_{\mathcal{H}\odot\mathcal{H}}$ is trivial. For $0 < r < 1$ we have $\|\frac{1-\psi}{1-r\psi}\|_{\operatorname{Mult}(\mathcal{H})} = \|1 - \frac{(1-r)\psi}{1-r\psi}\|_{\operatorname{Mult}(\mathcal{H})} \leq 2$, hence (b) now follows from Lemma 3.5 with $\psi_n = \left(\frac{1-\psi}{1-r_n\psi}\right)^2$ for $r_n \to 1^-$ since $\psi_n h = \frac{g}{(1-r_n\psi)^2}$ and $\frac{1}{(1-r_n\psi)^2} \in \operatorname{Mult}(\mathcal{H})$ for each n. ■

Theorem 3.7. *Let $\mathcal{H}$ be a separable Hilbert function space on the non-empty set X, and suppose that the reproducing kernel for $\mathcal{H}$ is a complete Pick kernel, which is normalized at a point $z_0 \in X$.*

(a) Then $\mathcal{M} \cap \operatorname{Mult}(\mathcal{H} \odot \mathcal{H})$ is dense in $\mathcal{M}$ for every multiplier invariant subspace $\mathcal{M}$ of $\mathcal{H} \odot \mathcal{H}$. If $\operatorname{Mult}(\mathcal{H} \odot \mathcal{H})$ has no zero-divisors, the lattice of multiplier invariant subspaces of $\mathcal{H} \odot \mathcal{H}$ is cellularly indecomposable, i.e. whenever $\mathcal{M}, \mathcal{N}$ are such invariant subspaces with $\mathcal{M} \neq (0), \mathcal{N} \neq (0)$, then $\mathcal{M} \cap \mathcal{N} \neq (0)$.

(b) If additionally $M^C(\mathcal{H}) \subseteq M^R(\mathcal{H})$, then every $\operatorname{Mult}(\mathcal{H})$-invariant subspace of $\mathcal{H} \odot \mathcal{H}$ is $\operatorname{Mult}(\mathcal{H} \odot \mathcal{H})$-invariant and the map

$$\eta : \mathcal{N} \to \operatorname{clos}_{\mathcal{H}\odot\mathcal{H}} \mathcal{N}$$

establishes a 1-1 and onto correspondence between the multiplier invariant subspaces of $\mathcal{H}$ and of $\mathcal{H} \odot \mathcal{H}$. We have

(i) *$\mathcal{M} = \operatorname{clos}_{\mathcal{H}\odot\mathcal{H}}(\mathcal{M} \cap \operatorname{Mult}(\mathcal{H}))$ for every multiplier invariant subspace $\mathcal{M}$ of $\mathcal{H} \odot \mathcal{H}$, and*

(ii) $\mathcal{N} = \mathcal{H} \cap \operatorname{clos}_{\mathcal{H}\odot\mathcal{H}} \mathcal{N}$ *for every multiplier invariant subspace* $\mathcal{N}$ *of* $\mathcal{H}$.

It is clear from (i) and (ii) that $\eta^{-1}(\mathcal{M}) = \mathcal{H} \cap \mathcal{M}$. Furthermore, it is easy to see that η preserves spans and intersections. We note that it follows from Corollary 2.7 of [14] and Corollary 5.3 of [13] that for a normalized complete Pick kernel the weak*closed ideals of $\operatorname{Mult}(\mathcal{H})$ are in 1-1 and onto correspondence with the multiplier invariant subspaces of $\mathcal{H}$. An alternate proof of this fact is in [6], and the current proof of (ii) is inspired by that approach.

Proof. (a) Let $\mathcal{M}$ be a multiplier invariant subspace of $\mathcal{H} \odot \mathcal{H}$, and let $h \in \mathcal{M}$. Then by Lemma 3.2 $h = \varphi/(1-\psi)^2$ for some $\varphi \in \operatorname{Mult}(\mathcal{H} \odot \mathcal{H})$ and $\psi \in \operatorname{Mult}(\mathcal{H})$ with $\psi(z_0) = 0$. Then by Lemma 3.6 we have $h \in [\varphi]_{\mathcal{H}\odot\mathcal{H}} = [h]_{\mathcal{H}\odot\mathcal{H}} \subseteq \mathcal{M}$ and hence there is a sequence $u_n \in \operatorname{Mult}(\mathcal{H} \odot \mathcal{H})$ such that $u_n\varphi \to h$. Clearly $u_n\varphi \in \mathcal{M} \cap \operatorname{Mult}(\mathcal{H} \odot \mathcal{H})$. This proves the first part of (a) and the second part of (a) easily follows from this.

(b) Now suppose that $M^C(\mathcal{H}) \subseteq M^R(\mathcal{H})$. In order to show that every $\operatorname{Mult}(\mathcal{H})$-invariant subspace of $\mathcal{H} \odot \mathcal{H}$ is $\operatorname{Mult}(\mathcal{H} \odot \mathcal{H})$-invariant it suffices to take $h \in \mathcal{H} \odot \mathcal{H}$ and $u \in \operatorname{Mult}(\mathcal{H} \odot \mathcal{H})$ and show that $uh \in \operatorname{clos}_{\mathcal{H}\odot\mathcal{H}}\{vh : v \in \operatorname{Mult}(\mathcal{H})\}$.

Since the reproducing kernel of $\mathcal{H}$ is normalized, $\mathcal{H}$ contains the constant functions, so we must have $u \in \mathcal{H}\odot\mathcal{H}$ and hence by the hypothesis and Lemma 3.2 $u = \frac{\varphi}{(1-\psi)^2}$ for some $\varphi, \psi \in \operatorname{Mult}(\mathcal{H})$, $\|\psi\|_{\operatorname{Mult}(\mathcal{H})} \le 1$ and $\psi(z_0) = 0$. Then $uh = \frac{\varphi h}{(1-\psi)^2}$ and Lemma 3.6 (b) implies that

$$uh \in \operatorname{clos}_{\mathcal{H}\odot\mathcal{H}}\{v\varphi h : v \in \operatorname{Mult}(\mathcal{H})\} \subseteq \operatorname{clos}_{\mathcal{H}\odot\mathcal{H}}\{vh : v \in \operatorname{Mult}(\mathcal{H})\}.$$

This establishes the first part of (b). Furthermore, we note that if $\mathcal{M}$ is a multiplier invariant subspace of $\mathcal{H} \odot \mathcal{H}$, then $\mathcal{H} \cap \mathcal{M}$ is a multiplier invariant subspace of $\mathcal{H}$ and

$$\operatorname{clos}_{\mathcal{H}\odot\mathcal{H}}(\mathcal{M} \cap \operatorname{Mult}(\mathcal{H})) \subseteq \operatorname{clos}_{\mathcal{H}\odot\mathcal{H}}(\mathcal{M} \cap \mathcal{H}) \subseteq \mathcal{M}.$$

Thus, statement (i) will show that η is onto and statement (ii) will show it is 1-1.

(i) The fact that $\mathcal{M} \cap \operatorname{Mult}(\mathcal{H})$ is dense in $\mathcal{M}$ for every multiplier invariant subspace $\mathcal{M}$ of $\mathcal{H} \odot \mathcal{H}$ follows as in (a), except that now any $h \in \mathcal{H} \odot \mathcal{H}$ is of the form $h = \frac{\varphi}{(1-\psi)^2}$ with $\varphi \in \operatorname{Mult}(\mathcal{H})$. This proves (i).

(ii) Let $\mathcal{N}$ be a multiplier invariant subspace of $\mathcal{H}$, we have to show that $\mathcal{H} \cap \operatorname{clos}_{\mathcal{H}\odot\mathcal{H}} \mathcal{N} \subseteq \mathcal{N}$. To this end let $f_n \in \mathcal{N}$ and $f \in \mathcal{H}$ with $f_n \to f$ in $\mathcal{H} \odot \mathcal{H}$. We have to show that $f \in \mathcal{N}$. By possibly considering a subsequence we may assume that $\sum_{n\ge1} \|f_{n+1} - f_n\|_{\mathcal{H}\odot\mathcal{H}} < 1$. Now we apply Lemma 3.2 with $h_n = f_{n+1} - f_n$. Thus there are $\psi \in \mathrm{Mult}(\mathcal{H})$ and $\{\Phi_n^C\}_{n\geq 1} \subseteq M^C(\mathcal{H})$ such that

(a) $\|\psi\|_{\mathrm{Mult}(\mathcal{H})} \leq 1$ and $\psi(z_0) = 0$,

(b) $\sum_{n\geq 1} \|\Phi_n^C u\|^2_{\mathcal{H}\otimes\ell_2} \leq \|u\|^2_{\mathcal{H}}$ for all $u \in \mathcal{H}$,

(c) for each $n \in \mathbb{N}$ $h_n = \frac{\varphi_n}{(1-\psi)^2}$ with $\varphi_n \in \mathrm{Mult}(\mathcal{H})$ and

$$\|\sum_{n=1}^{N} \varphi_n\|_{\mathrm{Mult}(\mathcal{H})} \leq c$$

for each $N \in \mathbb{N}$.

Set $g_1 = (1-\psi)(f - f_1)$ and $g_2 = (1-\psi)g_1$. Then $g_1, g_2 \in \mathcal{H}$ and by Lemma 3.6 (a) it suffices to prove that $g_2 \in \mathcal{N}$. But $g_2 = \sum_{n\geq 1} \varphi_n$ with $\sum_{n=1}^N \varphi_n = \sum_{n=1}^N (1-\psi)^2(f_{n+1} - f_n) \in \mathcal{N}$ for each N. Since $1 \in \mathcal{H}$ condition (c) from above implies that the partial sums $\sum_{n=1}^N \varphi_n$ converge weakly in $\mathcal{H}$ to g_2. Thus $g_2 \in \mathcal{N}$. ■

The following Corollary was known for the Dirichlet D of the unit disc and Drury-Arveson space H^2_d of the finite dimensional ball $\mathbb{B}_d$, see [17], [22].

Corollary 3.8. *Let $\mathcal{H}$ be a separable Hilbert function space on the non-empty set X, and suppose that the reproducing kernel for $\mathcal{H}$ is a complete Pick kernel, which is normalized at a point $z_0 \in X$.*

If $M^C(\mathcal{H}) \subseteq M^R(\mathcal{H})$, then for every multiplier invariant subspace $\mathcal{M}$ of $\mathcal{H}$, there is a sequence $\{b_n\}$ of symbols of bounded Hankel operators such that

$$\mathcal{M} = \bigcap_n \ker H_{b_n}.$$

Proof. Since we are assuming that $\mathcal{H}$ has a normalized complete Pick kernel it follows that $\mathrm{Mult}(\mathcal{H})$ is dense in $\mathcal{H}$. Thus, by Theorem 2.5 the dual of $\mathcal{H} \odot \mathcal{H}$ can be identified with $\mathrm{Han}(\mathcal{H})$, the set of symbols of bounded Hankel operators $\mathcal{H} \to \overline{\mathcal{H}}$. The duality is given by the inner product of $\mathcal{H}$, and we have

$$\langle \varphi f, b\rangle = \langle \varphi, \overline{H_b f}\rangle \quad \text{for all } f \in \mathcal{H}, \varphi \in \mathrm{Mult}(\mathcal{H}), b \in \mathrm{Han}(\mathcal{H}).$$

Let $\mathcal{N} = \mathrm{clos}_{\mathcal{H}\odot\mathcal{H}} \mathcal{M} \subseteq \mathcal{H} \odot \mathcal{H}$ and consider the annihilator $\mathcal{N}^\perp$ of $\mathcal{N}$ in $\mathrm{Han}(\mathcal{H})$. Since $\mathrm{Han}(\mathcal{H}) \subseteq \mathcal{H}$ and because of the particular form of the duality, it is easy to see that $\mathcal{N}^\perp \subseteq \mathcal{M}^\perp$. Furthermore, if $f \in \mathcal{H}$ with $f \perp \mathcal{N}^\perp$, then $f \in \mathcal{N} \cap \mathcal{H} = \mathcal{M}$ by Theorem 3.7. Hence $\mathcal{N}^\perp$ is dense in $\mathcal{M}^\perp$ in the topology of $\mathcal{H}$, and hence there is a countable set $\{b_n\} \subseteq \mathcal{N}^\perp$ such that $\{b_n\}$ is dense in $\mathcal{M}^\perp$.

We claim that $\mathcal{M} = \bigcap_n \ker H_{b_n}$. If $f \in \mathcal{M}$, then for each $\varphi \in \operatorname{Mult}(\mathcal{H})$ we have $\langle \varphi, \overline{H_{b_n} f} \rangle = \langle \varphi f, b_n \rangle = 0$ for each n since $b_n \in \mathcal{N}^\perp \subseteq \mathcal{M}^\perp$. Thus $H_{b_n} f = 0$ for each n, and hence $\mathcal{M} \subseteq \bigcap_n \ker H_{b_n}$.

Note that $H_b^* 1 = b$ for every $b \in \operatorname{Han}(\mathcal{H})$. Thus, by the choice of the b_n's

$$\mathcal{M}^\perp = \bigvee_n \{b_n\} \subseteq \bigvee_n \operatorname{ran} H_{b_n}^* = \left(\bigcap_n \ker H_{b_n} \right)^\perp.$$

This conclude the proof of the Corollary. ∎

4. Column operators between weighted Besov spaces

4.1. **Multiplier estimates for weighted Besov spaces.** Let ω be an arbitrary admissible weight, and let $N \in \mathbb{N}$. The admissibility of ω implies that $L^2_a(\omega) = B^0_\omega$ is a Hilbert space of analytic functions on $\mathbb{B}_d$. By use of the identity $f(z) = f(0) + \int_0^1 Rf(tz) \frac{dt}{t}$ one shows that there is an absolute constant $C > 0$ such that $|f(z)| \le |f(0)| + C \sup_{\lambda \in \mathbb{C}, |\lambda| \le 1} |Rf(\lambda z)|$ for any $f \in \operatorname{Hol}(\mathbb{B}_d)$. With this estimate one easily establishes that each B^N_ω is also a Hilbert function space on $\mathbb{B}_d$, whenever ω is admissible.

Then in order to check whether an analytic function $\varphi \in \operatorname{Mult}(B^N_\omega)$ we must check that there is $C > 0$ with $\int_{\mathbb{B}_d} |R^N(\varphi f)|^2 \omega dV \le C \|f\|^2_{B^N_\omega}$ for all $f \in B^N_\omega$. By the Leibnitz rule for the nth derivative of a product and the triangle inequality we have

$$\int_{\mathbb{B}_d} |R^N(\varphi f)|^2 \omega dV \le c \sum_{k=0}^N \int_{\mathbb{B}_d} |(R^k \varphi) R^{N-k} f|^2 \omega dV. \tag{4.1}$$

For standard weights ω and for so-called Bekollé-Bonami weights it has been shown in [19] and [10] that the right hand side of this is bounded by $c\|f\|^2_{B^N_\omega}$, if and only if the terms of the sum corresponding to $k = 0$ and $k = N$ are bounded by $c\|f\|^2_{B^N_\omega}$, and that these two conditions together characterize $\operatorname{Mult}(B^N_\omega)$. Note that these conditions can be equivalently expressed as $\varphi \in H^\infty$ and $|R^N \varphi|^2 \omega dV$ is a B^N_ω-Carleson measure. We will show in [4] that the same is true for all admissible radial weights. In fact, it is a rather short argument that shows that a bound on the left hand side of (4.1) implies a bound on the right hand side of (4.1), and this argument is valid for all weighted Besov spaces B^N_ω that satisfy a scalar version of the multiplier inclusion condition (see Definition 1.4). It turns out that the vector-valued versions of these results are true as well, and that will be an important ingredient in the proof of Theorem 4.2. We start by setting up the notation that we will use. A part of this involves extending the definitions given in the Introduction to multipliers between different spaces.

If $\mathcal{E}$ is a separable Hilbert space and if $\mathcal{H}$ is a reproducing kernel Hilbert space on $\mathbb{B}_d$, then we will identify $\mathcal{H} \otimes \mathcal{E}$ with a space $\mathcal{H}(\mathcal{E})$ of $\mathcal{E}$-valued functions on $\mathbb{B}_d$, where the identification is given by $f(z)x \cong f(z) \otimes x$ for $f \in \mathcal{H}$ and $x \in \mathcal{E}$. We will use the notations $\mathcal{H} \otimes \mathcal{E}$ and $\mathcal{H}(\mathcal{E})$ interchangeably.

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces of analytic functions. We will write

$$\mathrm{Mult}(\mathcal{H}, \mathcal{K}) = \{\varphi : \varphi\mathcal{H} \subseteq \mathcal{K}\}.$$

Then any sequence $\Phi = \{\varphi_1, \varphi_2, ...\} \subseteq \mathrm{Mult}(\mathcal{H}, \mathcal{K})$ of multipliers can be used to define a column operator $\Phi^C : h \to (\varphi_1 h, \varphi_2 h, ...)^T$ and a row operator $\Phi^R : (h_1, h_2, ...)^T \to \sum_{i \geq 1} \varphi_i h_i$. Here we have used $(h_1, ...)^T$ to denote a transpose of a row vector. We write $M^C(\mathcal{H}, \mathcal{K})$ for the set of bounded column multiplication operators $\mathcal{H} \to \bigoplus_{n=1}^{\infty} \mathcal{K}$ and $M^R(\mathcal{H}, \mathcal{K})$ for the set of bounded row multiplication operators $\bigoplus_{n=1}^{\infty} \mathcal{H} \to \mathcal{K}$. Thus $\Phi^C \in M^C(\mathcal{H}, \mathcal{K})$ if and only if there is a $c > 0$ such that

$$\sum_{j=1}^{\infty} \|\varphi_j h\|_{\mathcal{K}}^2 \leq c\|h\|_{\mathcal{H}}^2 \text{ for all } h \in \mathcal{H},$$

and $\Phi^R \in M^R(\mathcal{H}, \mathcal{K})$ if and only if there is a $c > 0$ such that

$$\|\sum_{j=1}^{\infty} \varphi_j h_j\|_{\mathcal{K}}^2 \leq c \sum_{j=1}^{\infty} \|h_j\|_{\mathcal{H}}^2 \text{ for all } h_j \in \mathcal{H}.$$

We will write $\|\Phi^C\|_{(\mathcal{H},\mathcal{K})}$ and $\|\Phi^R\|_{(\mathcal{H},\mathcal{K})}$ for the norms of these operators. Note that by considering the components of $\Phi \in \mathrm{Mult}(\mathcal{H}, \mathcal{K}(\ell_2))$ with respect to the standard orthonormal basis of ℓ_2, we obtain an identification between $M^C(\mathcal{H}, \mathcal{K})$ and $\mathrm{Mult}(\mathcal{H}, \mathcal{K}(\ell_2))$. In the remainder of this paper we will frequently pass back and forth between these different viewpoints. We just need to remember that when we are given $\Phi \in \mathrm{Mult}(\mathcal{H}, \mathcal{K}(\ell_2))$ and we want to consider a row operator induced by Φ that we have to fix a particular orthonormal basis.

We will now set up the framework for the spaces for which our results hold. Recall from the Introduction that a Hilbert space $\mathcal{H}$ of functions on $\mathbb{B}_d$ will be called a *weighted Besov space*, if there is an admissible weight ω on $\mathbb{B}_d$ and a nonnegative integer N such that $\mathcal{H} = B_\omega^N$ with equivalence of norms. Note that it is possible for a weighted Besov space to have $\mathcal{H} = B_\omega^N = B_{\tilde{\omega}}^K$ for $N \neq K$ and $\omega \neq \tilde{\omega}$. In fact, in [4] we will show that for each admissible radial weight ω there is a one parameter family $\{\omega_s\}_{s \geq 0}$ of admissible weights such that $B_\omega^N = B_{\omega_s}^{N-s}$ for all $s \geq 0$.

Since the weight function ω is integrable it follows that $H^\infty(\mathbb{B}_d) \subseteq B^0_\omega$, and hence any weighted Besov space contains the polynomials. This implies in particular that $k_z(z) \neq 0$ for all $z \in \mathbb{B}_d$, whenever k is the reproducing kernel of any weighted Besov space.

Let $\mathcal{H}$ and $\mathcal{K}$ be weighted Besov spaces. We say that the pair $(\mathcal{H}, \mathcal{K})$ satisfies the *multiplier inclusion condition*, if there are admissible weights ω and $\tilde{\omega}$ and $N \in \mathbb{N}$ such that $\mathcal{H} = B^N_\omega$, $\mathcal{K} = B^N_{\tilde{\omega}}$ and

$$\operatorname{Mult}(B^N_\omega, B^N_{\tilde{\omega}}(\ell_2)) \subseteq \operatorname{Mult}(B^{N-1}_\omega, B^{N-1}_{\tilde{\omega}}(\ell_2)) \subseteq \cdots \subseteq \operatorname{Mult}(B^0_\omega, B^0_{\tilde{\omega}}(\ell_2)),$$

and if the inclusions are continuous, i.e. whenever $1 \leq n \leq N$, then there is a $c > 0$ such that for all $\Phi \in \operatorname{Mult}(B^n_\omega, B^n_{\tilde{\omega}}(\ell_2))$ we have

$$\|\Phi\|_{\operatorname{Mult}(B^{n-1}_\omega, B^{n-1}_{\tilde{\omega}}(\ell_2))} \leq c \|\Phi\|_{\operatorname{Mult}(B^n_\omega, B^n_{\tilde{\omega}}(\ell_2))} \tag{4.2}$$

We will say that the pair $(B^N_\omega, B^N_{\tilde{\omega}})$ satisfies the scalar multiplier inclusion condition, if

$$\operatorname{Mult}(B^N_\omega, B^N_{\tilde{\omega}}) \subseteq \operatorname{Mult}(B^{N-1}_\omega, B^{N-1}_{\tilde{\omega}}) \subseteq \cdots \subseteq \operatorname{Mult}(B^0_\omega, B^0_{\tilde{\omega}})$$

with continuous inclusions.

We mentioned in the Introduction that a well-known approach to verify that a weighted Besov space satisfies the multiplier inclusion condition uses the functorial property of the complex interpolation method. At this point we will also indicate a somewhat more elementary method that can be used to verify that a pair $(\mathcal{H}, \mathcal{K})$ satisfies the multiplier inclusion condition (4.2). This method works sometimes, when simple formulas for the reproducing kernels of the spaces are known. We will prove that $(B^s_{\mathbf{1}}, B^t_{\mathbf{1}})$ satisfies the multiplier inclusion condition, whenever $t \leq s < (d+1)/2$. This includes the interesting case where $t = s = d/2$ and the spaces both equal the Drury-Arveson space H^2_d.

We mentioned before that for $s < (d+1)/2$ the space $B^s_{\mathbf{1}}$ has reproducing kernel $k^s_w(z) = \frac{1}{(1-\langle z, w\rangle)^{d+1-2s}}$ (up to equivalence of norms). Furthermore, for $s < 1/2$ one has $B^s_{\mathbf{1}} = L^2_a((1-|z|^2)^{-2s})$ (see e.g. [22], Section 2). Choose a positive integer $N > s - 1/2$ and set $\omega(z) = (1-|z|^2)^{2(N-s)}$ and $\tilde{\omega}(z) = (1-|z|^2)^{2(N-t)}$. Then $\omega, \tilde{\omega}$ are admissible weights and since $s - N < 1/2$ we have $B^{s-N}_{\mathbf{1}} = L^2_a(\omega)$. This implies $B^s_{\mathbf{1}} = B^N_\omega$. Similarly, $B^t_{\mathbf{1}} = B^N_{\tilde{\omega}}$, and it suffices to show that continuous inclusions of the type

$$\operatorname{Mult}(B^s_{\mathbf{1}}, B^t_{\mathbf{1}}(\ell_2)) \subseteq \operatorname{Mult}(B^{s-1}_{\mathbf{1}}, B^{t-1}_{\mathbf{1}}(\ell_2))$$

hold for all $t \leq s < (d+1)/2$. We establish the scalar version

$$\operatorname{Mult}(B^s_{\mathbf{1}}, B^t_{\mathbf{1}}) \subseteq \operatorname{Mult}(B^{s-1}_{\mathbf{1}}, B^{t-1}_{\mathbf{1}}).$$

Let $\varphi \in \text{Mult}(B^s_{\mathbf{1}}, B^t_{\mathbf{1}})$ of multiplier norm ≤ 1. Then $k^t_w(z) - \varphi(z)\overline{\varphi(w)}k^s_w(z)$ is positive definite. Note that $(1-\langle z,w\rangle)^{-2}$ is positive definite and $(1-\langle z,w\rangle)^{-2}k^s_w(z) = k^{s-1}_w(z)$ and $(1-\langle z,w\rangle)^{-2}k^t_w(z) = k^{t-1}_w(z)$. Thus by the Schur product theorem we can conclude that $k^{t-1}_w(z) - \varphi(z)\overline{\varphi(w)}k^{s-1}_w(z)$ is positive definite. That is equivalent to saying that φ is a contractive multiplier from $B^{s-1}_{\mathbf{1}}$ into $B^{t-1}_{\mathbf{1}}$.

Thus the scalar multiplier inclusion condition (4.2) holds, and the column vector version can be shown similarly.

The following lemma says that the multiplier inclusion condition (4.2) implies that given a bound on the left hand side of inequality (4.1) one also has a bound on the right hand side of that inequality.

Lemma 4.1. *Let ω and $\tilde{\omega}$ be admissible weights, let $N \in \mathbb{N}$, and suppose that $(B^N_\omega, B^N_{\tilde{\omega}})$ satisfies the multiplier inclusion condition (4.2).*

Then there is a $c > 0$ such that whenever $\Phi = \{\varphi_i\}_{i\in\mathbb{N}} \in M^C(B^N_\omega, B^N_{\tilde{\omega}})$ satisfies $\|\Phi^C\|_{(B^N_\omega, B^N_{\tilde{\omega}})} \leq 1$, then for all integers $j,k \geq 0$ with $j+k \leq N$ and each $h \in \mathcal{B}^N_\omega$ we have

$$\sum_{i=1}^\infty \|(R^j\varphi_i)R^k h\|^2_{B^{N-(j+k)}_{\tilde{\omega}}} \leq c\|h\|^2_{B^N_\omega}.$$

Of particular interest is the case $k = 0$. In compact form it implies that under the hypothesis of the Lemma there is $c > 0$ such that for each j with $0 \leq j \leq N$

$$\|R^j\Phi\|_{\text{Mult}(B^N_\omega, B^{N-j}_{\tilde{\omega}}(\ell_2))} \leq c\|\Phi\|_{\text{Mult}(B^N_\omega, B^N_{\tilde{\omega}}(\ell_2))}.$$

Proof. Suppose $\|\Phi\|_{\text{Mult}(B^N_\omega, B^N_{\tilde{\omega}}(\ell_2))} = \|\Phi^C\|_{(B^N_\omega, B^N_{\tilde{\omega}})} \leq 1$. The multiplier inclusion condition (4.2) implies that there is $c > 0$ such that for each $0 \leq k \leq n$ we have $\|\Phi\|_{\text{Mult}(B^{N-k}_\omega, B^{N-k}_{\tilde{\omega}}(\ell_2))} \leq c$ and hence for each $h \in B^N_\omega$ we have

$$\sum_{i=1}^\infty \|\varphi_i R^k h\|^2_{B^{N-k}_{\tilde{\omega}}} \leq c\|R^k h\|^2_{B^{N-k}_\omega} \lesssim c\|h\|^2_{B^N_\omega}.$$

Thus the Lemma holds for $j = 0$ and any $0 \leq k \leq N$, and we claim that the case of $j > 0$ can be reduced to the case of $j = 0$.

If $j > 0$ and $j + k \leq N$, then

$$(R^j\varphi_i)R^k h = R((R^{j-1}\varphi_i)R^k h) - (R^{j-1}\varphi_i)R^{k+1}h$$

and hence

$$\sum_{i=1}^{\infty} \|(R^j\varphi_i)R^k h\|^2_{B^{N-(j+k)}_{\tilde{\omega}}}$$

$$\leq 2\sum_{i=1}^{\infty} \|R((R^{j-1}\varphi_i)R^k h)\|^2_{B^{N-(j+k)}_{\tilde{\omega}}} + \|(R^{j-1}\varphi_i)R^{k+1}h\|^2_{B^{N-(j+k)}_{\tilde{\omega}}}$$

$$\lesssim \sum_{i=1}^{\infty} \|(R^{j-1}\varphi_i)R^k h\|^2_{B^{N-(j-1+k)}_{\tilde{\omega}}} + \|(R^{j-1}\varphi_i)R^{k+1}h\|^2_{B^{N-((j-1)+k+1)}_{\tilde{\omega}}}.$$

This means that the proof of the Lemma for $j > 0$ has been reduced to the case of $j - 1$. Hence finitely many iterations of this argument conclude the proof. ■

Theorem 4.2. *Let $\mathcal{H}$ and $\mathcal{K}$ be weighted Besov spaces such that $(\mathcal{H}, \mathcal{K})$ satisfies the multiplier inclusion condition (4.2).*

Then there is a $c > 0$ such that

$$\|\Phi^R\|_{(\mathcal{H},\mathcal{K})} \leq c\|\Phi^C\|_{(\mathcal{H},\mathcal{K})}$$

for all $\Phi = \{\varphi_1, \varphi_2, ...\} \in M^C(\mathcal{H}, \mathcal{K})$.

Proof. By hypothesis we may choose $N \in \mathbb{N}$ and admissible weights $\omega, \tilde{\omega}$ so that $\mathcal{H} = B^N_\omega$, $\mathcal{K} = B^N_{\tilde{\omega}}$, and so that Lemma 4.1 applies.

Let $\Phi = \{\varphi_i\}$ be a sequence of analytic functions on $\mathbb{B}_d$ with

$$\|\Phi^C h\|^2_{\mathcal{K}(\ell_2)} = \sum_{i=1}^{\infty} \|\varphi_i h\|^2_{\mathcal{K}} \leq \|h\|^2_{\mathcal{H}} \text{ for all } h \in \mathcal{H}.$$

We have to show the existence of $c > 0$ such that

$$\|\sum_{j=1}^{\infty} \varphi_j h_j\|^2_{\mathcal{K}} \leq c\sum_{j=1}^{\infty} \|h_j\|^2_{\mathcal{H}} \text{ for all } \{h_j\} \in \bigoplus_{j=1}^{\infty} \mathcal{H}.$$

We have

$$\|f\|^2_{\mathcal{K}} \approx |f(0)|^2 + \|R^N f\|^2_{L^2_a(\tilde{\omega})}.$$

Let k^1_z be the reproducing kernel for $\mathcal{H}$ and k^2_z be the reproducing kernel for $\mathcal{K}$. Then

$$\sum_{i=1} |\varphi_i(z)|^2 = \sum_{i=1} \frac{|\langle \varphi_i k^1_z, k^2_z\rangle_{\mathcal{K}}|^2}{\|k^1_z\|^4_{\mathcal{H}}} \leq \frac{\|k^2_z\|^2_{\mathcal{K}}}{\|k^1_z\|^2_{\mathcal{H}}}.$$

This implies that for any $\{h_j\} \in \bigoplus_{j=1}^{\infty} \mathcal{H}$

$$\sum_{j=1}^{\infty} |\varphi_j(z)h_j(z)| \leq \frac{\|k^2_z\|_{\mathcal{K}}}{\|k^1_z\|_{\mathcal{H}}} \left(\sum_{j=1}^{\infty} |h_j(z)|^2\right)^{1/2} \leq \|k^2_z\|_{\mathcal{K}} \left(\sum_{j=1}^{\infty} \|h_j\|^2_{\mathcal{H}}\right)^{1/2}.$$

Taking $z = 0$ we note that it suffices to show that

$$\|R^N(\sum_{j=1}^{\infty} \varphi_j h_j)\|^2_{L^2_a(\tilde{\omega})} \leq C \sum_{j=1}^{\infty} \|h_j\|^2_{\mathcal{H}}.$$

By the above the series $\sum_{j=1}^{\infty} \varphi_j h_j$ converges uniformly on compact subsets of $\mathbb{B}_d$, thus by analyticity and the Leibnitz rule we have

$$\begin{aligned}
|R^N\left(\sum_{j=1}^{\infty} \varphi_j h_j\right)|^2 &= |\sum_{k=0}^{N} \binom{N}{k} \sum_{j=1}^{\infty} (R^k \varphi_j)(R^{N-k} h_j)|^2 \\
&\leq c \sum_{k=0}^{N} \left(\sum_{j=1}^{\infty} |R^k \varphi_j||R^{N-k} h_j|\right)^2 \\
&= c \sum_{k=0}^{N} \sum_{j=1,i=1}^{\infty} |R^k \varphi_j||R^{N-k} h_j||R^k \varphi_i||R^{N-k} h_i| \\
&\leq c \sum_{k=0}^{N} \sum_{j=1,i=1}^{\infty} |R^k \varphi_j R^{N-k} h_i|^2.
\end{aligned}$$

Now we integrate both sides of the inequality over $\mathbb{B}_d$ against the measure $\tilde{\omega} dV$ and obtain

$$\begin{aligned}
\|R^N\left(\sum_{j=1}^{\infty} \varphi_j h_j\right)\|^2_{L^2_a(\tilde{\omega})} &\leq c \sum_{k=0}^{N} \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} \|R^k \varphi_j R^{N-k} h_i\|^2_{L^2_a(\tilde{\omega})}\right) \\
&\lesssim \sum_{k=0}^{N} \sum_{i=1}^{\infty} \|h_i\|^2_{B^N_\omega} \lesssim \sum_{i=1}^{\infty} \|h_i\|^2_{\mathcal{H}}
\end{aligned}$$

by Lemma 4.1. ■

4.2. **A bounded row, but unbounded column operator.** In this Section we will show that if $d \geq 2$ then there are sequences of multipliers $\Phi = \{\varphi_1, \varphi_2, ...\}$ of the Drury-Arveson space H^2_d such that Φ^R is bounded $\bigoplus_{j=1}^{\infty} H^2_d \to H^2_d$, but Φ^C is unbounded $H^2_d \to \bigoplus_{j=1}^{\infty} H^2_d$. The proof is patterned after an example of Trent for the Dirichlet space, [26]. The case at hand requires more work due to the fact that for the Drury-Arveson kernel k the expression $1 - 1/k_w(z)$ is a sum of only finitely many terms of the form $\varphi_i(z)\overline{\varphi_i(w)}$.

For $n \in \mathbb{N}$ let $S_n = \{\mu = (\mu_1, ..., \mu_n) : \mu_j \in \{1, 2, ..., d\}\}$ be the set of $n-$tuples of elements in $\{1, ..., d\}$. Set $S = \bigcup_{n=1}^{\infty} S_n$ and for $\mu \in S$ write $l(\mu) = n$ if $\mu \in S_n$, the length of μ.

Since S is countable it will suffice to exhibit a family of functions $\{\varphi_\mu\}_{\mu\in S}$ on $\mathbb{B}_d$ such that

$$\|\sum_{\mu\in S}\varphi_\mu h_\mu\|^2 \le C\sum_{\mu\in S}\|h_\mu\|^2 \text{ whenever } h_\mu\in\mathcal{H}_\gamma \text{ and } \sum_{\mu\in S}\|\varphi_\mu\|^2=\infty.$$

For $\mu\in S_n$ and $z\in\mathbb{C}^d$ set $z_\mu = z_{\mu_1}z_{\mu_2}\cdots z_{\mu_n}$.

Define $\varphi_\mu(z)=\frac{1}{l(\mu)}z_\mu$. Note that for each $\mu\in S_n$ there is a multi index $\alpha=(\alpha_1,...,\alpha_d)\in\mathbb{N}_0^d$ with $|\alpha|=n$ and such that $\varphi_\mu(z)=\frac{1}{n}z^\alpha$. Conversely, if $\alpha\in\mathbb{N}_0^d$ is a multi index with $|\alpha|=n$, then there are $\frac{|\alpha|!}{\alpha!}$ distinct $\mu\in S_n$ with $\varphi_\mu(z)=\frac{1}{n}z^\alpha$. Recall that in H^2_d we have $\|z^\alpha\|^2=\frac{\alpha!}{|\alpha|!}$.

Thus

$$\begin{aligned}\sum_{\mu\in S}\|\varphi_\mu\|^2 &= \sum_{n=1}^\infty\frac{1}{n^2}\sum_{\mu\in S_n}\|z_{\mu_1}z_{\mu_2}\cdots z_{\mu_n}\|^2\\ &= \sum_{n=1}^\infty\frac{1}{n^2}\sum_{|\alpha|=n}\frac{|\alpha|!}{\alpha!}\|z^\alpha\|^2\\ &= \sum_{n=1}^\infty\frac{1}{n^2}\sum_{|\alpha|=n}1\\ &\ge \sum_{n=1}^\infty\frac{1}{n}=\infty\end{aligned}$$

since for each $n\in\mathbb{N}$ there are n multi-indices α of the type $\alpha=(k,n-k,0,...,0)$. Here we use $d\ge 2$.

On the other hand consider

$$\|\sum_{\mu\in S}^\infty\varphi_\mu h_\mu\|^2 = \|\sum_{n=1}^\infty\frac{1}{n}\sum_{\mu\in S_n}z_\mu h_\mu\|^2 \le \left(\sum_{n=1}^\infty\frac{1}{n^2}\right)\left(\sum_{n=1}^\infty\|\sum_{\mu\in S_n}z_\mu h_\mu\|^2\right)$$

We now show by induction that for each $n\ge 1$ we have

$$\|\sum_{\mu\in S_n}z_\mu h_\mu\|^2\le\sum_{\mu\in S_n}\|h_\mu\|^2$$

and that will finish the proof.

The case $n=1$ follows since M_z is a d-contraction, i.e. $\|\sum_{k=1}^d z_kf_k\|^2\le\sum_{k=1}^d\|f_k\|^2$. This shows that $\|\sum_{\mu\in S_1}z_\mu h_\mu\|^2\le\sum_{\mu\in S_1}\|h_\mu\|^2$. Now suppose that the claim holds for some $n\ge 1$. We will show that it holds for $n+1$. Note that $S_{n+1}=S_1\times S_n$, hence as above

$$\begin{aligned}\|\sum_{\mu\in S_{n+1}} z_\mu h_\mu\|^2 &= \|\sum_{k=1}^{d} z_k \sum_{\mu'\in S_n} z_{\mu'} h_{(k,\mu')}\|^2\\ &\leq \sum_{k=1}^{d} \|\sum_{\mu'\in S_n} z_{\mu'} h_{(k,\mu')}\|^2\\ &\leq \sum_{k=1}^{d}\sum_{\mu'\in S_n} \|h_{(k,\mu')}\|^2 = \sum_{\mu\in S_{n+1}} \|h_\mu\|^2\end{aligned}$$

by the induction hypothesis.

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