

Calcular algebras

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To the memory of Richard Timoney

Abstract

A calcular algebra is a subalgebra of $H^\infty(\Omega)$ with norm given by $\|\phi\| = \sup \|\phi(T)\|$ as T ranges over a given class of commutative d -tuples of operators with Taylor spectrum in Ω . We discuss what algebras arise this way, and how they can be represented.

1 Introduction

Let Ω be a bounded open set in \mathbb{C}^d . We say that a class \mathcal{C} is *subordinate to* Ω if:

- (i) Each element T of \mathcal{C} is a commuting d -tuple of bounded operators on a Hilbert space, with its Taylor spectrum¹ $\sigma(T)$ in Ω .
- (ii) For some non-zero Hilbert space \mathcal{H} , the set of scalars

$$\{(\lambda^1, \dots, \lambda^d) : \lambda \in \Omega\} \subseteq \mathcal{C},$$

where we think of λ as a d -tuple of scalar multiples of the identity acting on \mathcal{H} . (Note that we use superscripts to denote the coordinates.)

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¹For a definition of Taylor spectrum of a commuting tuple, see [12].

Given a class \mathcal{C} subordinate to Ω , we define $H^\infty(\mathcal{C})$ to be those holomorphic functions on Ω for which

$$\|\phi\|_{\mathcal{C}} = \sup\{\|\phi(T)\| : T \in \mathcal{C}\}$$

is finite. It can be shown (see Prop. 2.1 below) that this algebra is always complete, so it is a Banach algebra, which by Property (ii) is always contained contractively in the algebra $H^\infty(\Omega)$ of bounded holomorphic functions on Ω . (We are using H^∞ in two apparently different ways, but identifying Ω with the set of scalars makes the two usages agree). Any Banach algebra of holomorphic functions arising in this way we shall call a *calcular algebra over Ω* .

We shall call the closed unit ball of $H^\infty(\mathcal{C})$ the *Schur class of \mathcal{C}* , and denote it by $\mathcal{S}(\mathcal{C})$.

$$\mathcal{S}(\mathcal{C}) = \{\phi \in \text{Hol}(\Omega) : \|\phi(T)\| \leq 1, \forall T \in \mathcal{C}\}.$$

Let $\mathcal{S}(\Omega)$ denote the closed unit ball of $H^\infty(\Omega)$.

If \mathcal{H} is a Hilbert space (we shall always assume that Hilbert spaces are not zero-dimensional to avoid trivialities), let $CB(\mathcal{H})^d$ denote the set of commuting d -tuples of elements of $B(\mathcal{H})$, the bounded linear operators on \mathcal{H} . Given a set S of bounded holomorphic functions on Ω , and a Hilbert space \mathcal{H} , one can form the set

$$\mathcal{H}(S) = \{T \in CB(\mathcal{H})^d : \sigma(T) \subseteq \Omega \text{ and } \|\phi(T)\| \leq 1 \forall \phi \in S\}.$$

If \mathcal{H} is a Hilbert space, $\mathcal{C} \subseteq CB(\mathcal{H})^d$ and $S \subseteq \mathcal{S}(\Omega)$, then tautologically one has

$$\mathcal{H}(\mathcal{S}(\mathcal{C})) \supseteq \mathcal{C} \tag{1.1}$$

$$\mathcal{S}(\mathcal{H}(S)) \supseteq S. \tag{1.2}$$

Typically these inclusions will be strict. For example, let $d = 1$, and let Ω be the open unit disk \mathbb{D} . Let \mathcal{H} be any Hilbert space, and let \mathcal{C} be the set $\{\lambda I : \lambda \in \mathbb{D}\}$. Then $\mathcal{S}(\mathcal{C})$ will equal $\mathcal{S}(\mathbb{D})$, and, by von Neumann's inequality [14], $\mathcal{H}(\mathcal{S}(\mathbb{D}))$ will consist of all contractions on \mathcal{H} whose spectrum is in \mathbb{D} . Likewise if S just contains the function z , then $\mathcal{H}(S)$ will be the contractions on \mathcal{H} whose spectrum is in \mathbb{D} , and the Schur class of this set will be all of $\mathcal{S}(\mathbb{D})$.

Our first result is that the operations \mathcal{H} and \mathcal{S} stabilize after 3 steps, provided \mathcal{H} is infinite dimensional.

Notation: If T is a commuting d -tuple of bounded operators on a Hilbert space \mathcal{H} , we call \mathcal{H} the carrier of T , and write $\mathcal{H} = \text{car}(T)$.

Theorem 1.3. *Let Ω be a bounded open set in \mathbb{C}^d , and let \mathcal{C} be any class subordinate to Ω . Let S be a non-empty subset of $\mathcal{S}(\Omega)$. For any Hilbert space \mathcal{H} , we have*

$$\mathcal{H}(\mathcal{S}(\mathcal{H}(S))) = \mathcal{H}(S). \quad (1.4)$$

If the dimension of \mathcal{H} is either infinite, or greater than or equal to $\sup\{\dim(\text{car}(T)) : T \in \mathcal{C}\}$, then

$$\mathcal{S}(\mathcal{H}(\mathcal{S}(\mathcal{C}))) = \mathcal{S}(\mathcal{C}) \quad (1.5)$$

PROOF: By (1.2), we have

$$\mathcal{H}(\mathcal{S}(\mathcal{H}(S))) \subseteq \mathcal{H}(S). \quad (1.6)$$

Suppose now that $T \in \mathcal{H}(S)$, and ϕ is any function in $\mathcal{S}(\mathcal{H}(S))$. Then $\|\phi(T)\| \leq 1$, so T is in $\mathcal{H}(\mathcal{S}(\mathcal{H}(S)))$, proving (1.4).

By (1.2) again, with $S = \mathcal{S}(\mathcal{C})$, we get

$$\mathcal{S}(\mathcal{C}) \subseteq \mathcal{S}(\mathcal{H}(\mathcal{S}(\mathcal{C}))). \quad (1.7)$$

Now, assume the dimension of \mathcal{H} is as in the hypothesis. Let $\phi \in \mathcal{S}(\mathcal{H}(\mathcal{S}(\mathcal{C})))$, and let $T \in \mathcal{C}$, with $\text{car}(T) = \mathcal{K}$. We need to show $\|\phi(T)\| \leq 1$. If the dimension of \mathcal{H} is equal to the dimension of \mathcal{K} , then T is unitarily equivalent to a d -tuple R on \mathcal{H} , and $R \in \mathcal{H}(\mathcal{S}(\mathcal{C}))$ since $\|\psi(R)\| = \|\psi(T)\| \leq 1$ for every ψ in $\mathcal{S}(\mathcal{C})$. Therefore $\|\phi(T)\| = \|\phi(R)\| \leq 1$, and we are done.

If the dimension of \mathcal{H} is larger than the dimension of \mathcal{K} , write $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ where $\dim(\mathcal{H}_1) = \dim(\mathcal{K})$, and let R_1 on \mathcal{H}_1 be unitarily equivalent to T . Choose $\lambda = (\lambda^1, \dots, \lambda^d)$ in Ω , and let

$$R = (R_1^1 \oplus \lambda^1 I_{\mathcal{H}_2}, \dots, R_1^d \oplus \lambda^d I_{\mathcal{H}_2}),$$

Then for any $\psi \in O(\Omega)$, the set of holomorphic functions on Ω , we have $\psi(R) = \psi(R_1) \oplus \psi(\lambda) I_{\mathcal{H}_2}$, so if $\psi \in \mathcal{S}(\mathcal{C})$, we have $\|\psi(R)\| \leq 1$, and therefore $R \in \mathcal{H}(\mathcal{S}(\mathcal{C}))$. Now we get $\|\phi(T)\| \leq \|\phi(R)\| \leq 1$, and again we are done.

Finally we consider the case where \mathcal{H} is infinite dimensional, but the carriers of the elements of \mathcal{C} may have larger dimension. We can assume

without loss of generality that \mathcal{H} is separable. We need to find $R \in \mathcal{H}(\mathcal{S}(\mathcal{C}))$ with $\|\phi(R)\| = \|\phi(T)\|$. To do this, it is sufficient to show that there is a separable subspace \mathcal{K}_1 of \mathcal{K} that is reducing for $f(T)$ for every $f \in O(\Omega)$ and such that $\|\phi(T)|_{\mathcal{K}_1}\| = \|\phi(T)\|$; for then we can choose R_1 on \mathcal{H} unitarily equivalent to $P_{\mathcal{K}_1}T|_{\mathcal{K}_1}$, where $P_{\mathcal{K}_1}$ is projection onto \mathcal{K}_1 ; the fact that \mathcal{K}_1 is reducing means that $\phi(P_{\mathcal{K}_1}T|_{\mathcal{K}_1}) = P_{\mathcal{K}_1}\phi(T)|_{\mathcal{K}_1}$.

Observe that $\{f(T) : f \in O(\Omega)\}$ has a countable dense subset \mathcal{D} in the norm topology of $CB(\mathcal{K})^d$, since $O(\Omega)$ is separable. Let u_j be a sequence of unit vectors in \mathcal{K} such that $\|\phi(T)u_j\| \rightarrow \|\phi(T)\|$. Let \mathcal{K}_1 be the closed linear span of finite products of elements of $\mathcal{D} \cup \mathcal{D}^*$ applied to finite linear combinations of the vectors u_j . By \mathcal{D}^* we mean

$$\mathcal{D}^* = \{((T^1)^*, \dots, (T^d)^*) : (T^1, \dots, T^d) \in \mathcal{D}\}.$$

Then \mathcal{K}_1 is a separable subspace of \mathcal{K} on which $\phi(T)$ achieves its norm and that is reducing for every $f(T)$. \square

For a given class \mathcal{C} , it is of interest to know the smallest dimension of \mathcal{H} that gives equality in (1.5).

Example 1.8 Let $\Omega = \mathbb{D}$, and let \mathcal{C} be all contractions with spectrum in \mathbb{D} . Then we can take \mathcal{H} to be one dimensional. Similarly, if $\Omega = \mathbb{D}^2$, and \mathcal{C} is all pairs of commuting contractions with spectrum in \mathbb{D}^2 , Andô's inequality [4] yields that we can take \mathcal{H} to be one dimensional again.

However, if $d \geq 3$, we let $\Omega = \mathbb{D}^d$, and \mathcal{C} be the class of all d -tuples of commuting contractions with spectrum contained in \mathbb{D}^d , then $\mathcal{S}(\mathcal{C})$ is the Schur-Agler class, a proper subset of $\mathcal{S}(\mathbb{D}^d)$ [6, 13]. If $\mathcal{H} = \mathbb{C}^n$, then $\mathcal{H}(\mathcal{S}(\mathcal{C}))$ will be all d -tuples of commuting contractive n -by- n matrices with spectrum in \mathbb{D}^d . In [9], it is shown that if $n = 3$, then

$$\mathcal{S}(\mathbb{C}^3(\mathcal{S}(\mathcal{C}))) = \mathcal{S}(\mathbb{D}^d).$$

It is unknown what the minimal dimension of \mathcal{H} must be in this case to get equality in (1.5), or even whether it must be infinite.

Example 1.9 Let \mathcal{K} be a Hilbert function space on Ω with reproducing kernel k . The multiplier algebra $\text{Mult}(\mathcal{K})$ is always a calcular algebra. Indeed, for each finite set $F = \{\lambda_1, \dots, \lambda_n\} \subset \Omega$, let T_F be the commuting d -tuple (T_F^1, \dots, T_F^d) acting on the n -dimensional subspace of \mathcal{K} spanned by the kernel

functions $\{k_{\lambda_j} : 1 \leq j \leq n\}$ defined by

$$T_F^r k_{\lambda_j} = \overline{\lambda_j^r} k_{\lambda_j} \quad 1 \leq r \leq d, 1 \leq j \leq n.$$

Define

$$\mathcal{C} = \{T_F^* : F \text{ a finite subset of } \Omega\}.$$

It is straightforward to show that $H^\infty(\mathcal{C}) = \text{Mult}(\mathcal{K})$. Many other examples of calcular algebras are given in [2, Chapter 9].

2 When is a Banach algebra a calcular algebra?

Proposition 2.1. *Let \mathcal{C} be subordinate to Ω . Then $H^\infty(\mathcal{C})$ is a Banach algebra.*

PROOF: We need to prove completeness. Consider a Cauchy sequence $\{\phi_n\}$ in $H^\infty(\mathcal{C})$. Since \mathcal{C} is subordinate to Ω , $\{\phi_n\}$ is a Cauchy sequence in $H^\infty(\Omega)$. Therefore, as $H^\infty(\Omega)$ is complete, there exists $\phi \in H^\infty(\Omega)$ such that

$$\sup_{\lambda \in \Omega} |\phi_n(\lambda) - \phi(\lambda)| \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (2.2)$$

We claim that

$$\phi \in H^\infty(\mathcal{C}) \quad (2.3)$$

and

$$\phi_n \rightarrow \phi \text{ in } H^\infty(\mathcal{C}). \quad (2.4)$$

To prove statement (2.3), note that for each $T \in \mathcal{C}$, we have Ω is a neighborhood of $\sigma(T)$. Consequently, continuity of the functional calculus implies that

$$\phi_n(T) \rightarrow \phi(T) \quad \forall T \in \mathcal{C}. \quad (2.5)$$

Also, as $\{\phi_n\}$ is a Cauchy sequence in $H^\infty(\mathcal{C})$, there exists a constant M such that

$$\|\phi_n\|_{\mathcal{C}} \leq M \quad \forall n.$$

Therefore, if $T \in \mathcal{C}$,

$$\|\phi(T)\|_{\mathcal{C}} = \lim_{n \rightarrow \infty} \|\phi_n(T)\| \leq \limsup_{n \rightarrow \infty} \|\phi_n\|_{\mathcal{C}} \leq M.$$

Hence,

$$\|\phi\|_{\mathcal{C}} = \sup_{T \in \mathcal{C}} \|\phi(T)\|_{\mathcal{C}} \leq M,$$

which proves the membership (2.3).

To prove the limiting relation (2.4), let $\varepsilon > 0$. Choose N such that

$$m, n \geq N \implies \|\phi_n - \phi_m\|_{\mathcal{C}} < \varepsilon.$$

By definition of the norm, this means

$$m, n \geq N \implies \|\phi_n(T) - \phi_m(T)\| < \varepsilon \quad \forall T \in \mathcal{C}.$$

Letting $m \rightarrow \infty$ and using statement (2.5) we deduce that

$$n \geq N \implies \|\phi_n(T) - \phi(T)\| < \varepsilon \quad \forall T \in \mathcal{C}.$$

Hence, since $\|\phi_n - \phi\|_{\mathcal{C}} = \sup_{T \in \mathcal{C}} \|\phi_n(T) - \phi(T)\|$,

$$n \geq N \implies \|\phi_n - \phi\|_{\mathcal{C}} \leq \varepsilon.$$

□

Let \mathcal{A} be a unital Banach algebra contractively contained in $H^\infty(\Omega)$. When can it be realized as a calcular algebra? Let S be its unit ball. By Theorem 1.3, \mathcal{A} is a calcular algebra if and only if $\mathcal{S}(\mathcal{H}(S)) = S$, where \mathcal{H} is an infinite dimensional Hilbert space.

This imposes a constraint on \mathcal{A} . In particular, there must be an isometric homomorphism from \mathcal{A} into $B(\mathcal{K})$ for some Hilbert space \mathcal{K} . There is another constraint which stems from the requirement that all the operators in the class have spectrum in the open set Ω .

Proposition 2.6. *If \mathcal{A} is a calcular algebra, then:*

- (i) *There is an isometric homomorphism into $B(\mathcal{K})$ for some Hilbert space \mathcal{K} .*
- (ii) *If ϕ_n is a bounded sequence in \mathcal{A} that converges uniformly on compact subsets of Ω to a function ψ , then $\psi \in \mathcal{A}$, and $\|\psi\| \leq \liminf \|\phi_n\|$.*

PROOF: (i) Suppose \mathcal{A} is $H^\infty(\mathcal{C})$ for some class \mathcal{C} subordinate to an open set Ω . Let \mathcal{H} be any infinite dimensional Hilbert space, and let $\mathcal{C}_1 = \mathcal{H}(\mathcal{S}(\mathcal{C}))$. By Theorem 1.3, we have

$$\mathcal{A} = H^\infty(\mathcal{C}_1). \tag{2.7}$$

Let \mathcal{K} be the direct sum of $\text{cardinality}(\mathcal{C}_1)$ copies of \mathcal{H} , with the sum indexed by \mathcal{C}_1 . Define a map $\pi : \mathcal{A} \rightarrow B(\mathcal{K})$ by

$$\pi(\phi) = \oplus_{T \in \mathcal{C}_1} \phi(T).$$

Then π is a homomorphism, and by (2.7) it is isometric.

(ii) Let ϕ_n be a bounded sequence in $H^\infty(\mathcal{C})$ converging to ψ locally uniformly on Ω . Without loss of generality, we may assume that each ϕ_n is in $\mathcal{S}(\mathcal{C})$. For each T in \mathcal{C} , since $\sigma(T) \subseteq \Omega$, it follows from the continuity of the functional calculus that $\psi(T)$ is the limit in norm of $\phi_n(T)$, so ψ is in $\mathcal{S}(\mathcal{C})$. Replacing ϕ_n by a subsequence whose norms converge to $\liminf \|\phi_n\|$ gives the last inequality. \square

Remark: If one defines $S = \oplus_{T \in \mathcal{C}_1} (T)$, then one can interpret $\phi(S)$ as $\pi(\phi)$. However, the spectrum of S will be $\overline{\Omega}$, so S is not contained in any class subordinate to Ω . The Taylor functional calculus is defined only for functions holomorphic on a neighborhood of the Taylor spectrum of the d -tuple.

A necessary condition for a Banach algebra to be isometrically isomorphic to an algebra of operators on a Hilbert space is that it satisfies von Neumann's inequality: $\|p(x)\| \leq \|p\|_{H^\infty(\mathbb{D})}$ for any x in the unit ball of the Banach algebra, and any polynomial p . It is not known whether this condition is sufficient.

Calcular algebras come with a sequence of matrix norms. If $[\phi_{ij}]$ is an n -by- n matrix of elements of $H^\infty(\mathcal{C})$, one can define

$$\|[\phi_{ij}]\|_n = \sup\{\|[\phi_{ij}(T)]\| : T \in \mathcal{C}\},$$

where the norm on the right-hand side is the operator norm on $\text{car}(T) \otimes \mathbb{C}^n$. By a similar argument to Proposition 2.6, one can show that calcular algebras have completely isometric homomorphic embeddings into some $B(\mathcal{K})$.

Algebras that can be completely isometrically realized in this way are characterized by the Blecher-Ruan-Sinclair theorem [5], [11, Cor. 16.7]. This says that the algebra \mathcal{A} must satisfy the Ruan axioms:

$$\begin{aligned} \forall n \in \mathbb{N}, \forall a \in M_n(\mathcal{A}), \forall X, Y \in M_n(\mathbb{C}), \quad \|XaY\|_n &\leq \|X\| \|a\|_n \|Y\| \\ \forall m, n \in \mathbb{N}, \forall a \in M_m(\mathcal{A}), b \in M_n(\mathcal{A}), \quad \|a \oplus b\|_{m+n} &= \max \|a\|_m, \|b\|_n, \end{aligned}$$

and hence be isometrically realizable as an operator space; and the matrix multiplication at each level n must be contractive, *i.e.* if $a = [a_{ij}]$ and $b = [b_{ij}]$

are in $M_n(\mathcal{A})$, then

$$\left\| \left[\sum_{k=1}^n a_{ik} b_{kj} \right] \right\|_n \leq \| [a_{ij}] \|_n \| [b_{ij}] \|_n.$$

It is straightforward to check that a calcular algebra satisfies the hypotheses of the Blecher-Ruan-Sinclair theorem.

We do not know in general what intrinsic necessary and sufficient conditions on a sub-algebra of $H^\infty(\Omega)$ make it a calcular algebra; we can say something with strong convexity assumptions. Let $\mathcal{P} = \mathbb{C}[z_1, \dots, z_d]$ denote the polynomials. If f is a function and $r > 0$, define f_r by $f_r(z) = f(rz)$.

Theorem 2.8. *Let \mathcal{A} be a unital Banach algebra that is contractively contained in $H^\infty(\Omega)$, for some bounded open convex set Ω in \mathbb{C}^d that contains 0. Suppose that \mathcal{P} is contained in \mathcal{A} and that for every function $\phi \in \mathcal{A}$, there is a sequence in \mathcal{P} that is bounded in norm by $\|\phi\|$ and converges to ϕ locally uniformly on Ω . Suppose moreover that for every polynomial $p \in \mathcal{P}$, we have $\|p_r\| \leq \|p\|$ for $0 < r < 1$.*

Then \mathcal{A} is a calcular algebra over Ω if and only if the necessary conditions of Proposition 2.6 hold.

PROOF: Suppose both conditions hold, and π embeds \mathcal{A} isometrically in $B(\mathcal{K})$. For each of the coordinate functions z^j , $1 \leq j \leq d$, define $T^j = \pi(z^j)$. Let $T \in CB(\mathcal{K})^d$ be the tuple (T^1, \dots, T^d) . Then for any polynomial $p \in \mathcal{P}$ we have $\pi(p) = p(T)$. Moreover, if p has no zeroes on $\overline{\Omega}$, then

$$\pi\left(p \frac{1}{p}\right) = 1_{\mathcal{K}} = p(T)\pi\left(\frac{1}{p}\right),$$

so

$$\pi\left(\frac{1}{p}\right) = p(T)^{-1}.$$

As p ranges over affine functions whose zero sets are hyperplanes not intersecting $\overline{\Omega}$, we see that $\sigma(T)$ must be contained in $\overline{\Omega}$.

We want the elements of \mathcal{C} to have spectrum in Ω . Let $\mathcal{C} = \{rT : 0 \leq r < 1\}$.

For any polynomial p and any sequence $r_n \uparrow 1$ we have

$$\begin{aligned}
\|p\|_{\mathcal{A}} &= \|\pi(p)\| \\
&= \|p(T)\| \\
&= \lim_{n \rightarrow \infty} \|p(r_n T)\| \\
&\leq \|p\|_c \\
&= \sup_{0 < r < 1} \|p_r(T)\| \\
&= \sup_{0 < r < 1} \|\pi(p_r)\| \\
&\leq \|p\|_{\mathcal{A}}.
\end{aligned}$$

So \mathcal{A} and $H^\infty(C)$ assign the same norm to polynomials.

Let ψ be in \mathcal{A} of norm 1. By hypothesis, there is a sequence of polynomials q_n that converges locally uniformly to ψ , with $\|q_n\|_{\mathcal{A}} \leq 1$. Therefore for each $0 \leq r < 1$,

$$\|\psi(rT)\| = \lim_{n \rightarrow \infty} \|q_n(rT)\| \leq 1.$$

Therefore ψ is in the unit ball of $H^\infty(C)$, and hence \mathcal{A} is contractively contained in $H^\infty(C)$.

Conversely, let $\phi \in \mathcal{S}(C)$. Since Ω is convex, ϕ_r will converge to ϕ locally uniformly on Ω as $r \uparrow 1$. Fix $r < 1$. There is a sequence q_n of polynomials that converges uniformly to ϕ_r on a neighborhood of $\bar{\Omega}$. Therefore $\lim_{n \rightarrow \infty} q_n(T) = \phi_r(T)$ is a contraction, and so by Property (ii) we have

$$\|\phi_r\|_{\mathcal{A}} \leq 1.$$

By a diagonalization argument, we can modify this construction to find polynomials q_n in the unit ball of \mathcal{A} that converge locally uniformly to ϕ , and hence

$$\|\phi\|_{\mathcal{A}} \leq 1.$$

So $H^\infty(C)$ is contractively contained in \mathcal{A} , and hence the two algebras are isometrically isomorphic. \square

Example 2.9 The disk algebra $A(\mathbb{D})$ cannot be a calcular algebra, since it fails (ii). However, there are subalgebras of the disk algebra that are the multiplier algebra of some Hilbert function spaces on the disk, *e.g.* the space

with reproducing kernel

$$k(w, z) = \sum_{n=0}^{\infty} (n+1)^2 z^n \bar{w}^n.$$

Multiplier algebras for spaces of holomorphic functions are always calcular, as shown in Example 1.9.

Problem 2.10 Find necessary and sufficient conditions for a subalgebra of $H^\infty(\Omega)$ to be a calcular algebra.

One answer to this problem is given in [10, Thm. 2.4]—the algebra must be a multiplier algebra for some Hilbert function space, and be closed under pointwise limits of bounded nets.

3 Realization formulas

In [7] and [8], Dritschel, Marcantognini, and McCullough proved a very general realization formula, building on work of Ambrozie and Timotin in [3], which can be adapted to our current setting.

Let S be a set of functions from a set X to the unit disk \mathbb{D} . In this section, we shall make the standing assumption that S restricted to any finite set F generates, as an algebra, all the complex-valued functions on F .

We define K_S to be the set of kernels on X that satisfy

$$K_S = \{k \mid (1 - \psi(z)\bar{\psi}(w))k(z, w) \geq 0 \quad \forall \psi \in S\}.$$

We define $A^\infty(K_S)$ to be

$$A^\infty(K_S) = \{\phi : X \rightarrow \mathbb{C} \mid \exists M \geq 0 \text{ s.t. } (M^2 - \phi(z)\bar{\phi}(w))k(z, w) \geq 0 \quad \forall k \in K_S\},$$

and define $\|\phi\|$ to be the smallest M that works.

Endow S with the topology of pointwise convergence. Let $C_b(S)$ denote the continuous bounded functions on S , which we think of as a C^* -algebra. Let $E : X \rightarrow C_b(S)$ be the evaluation map $E(z)(\psi) = \psi(z)$, and let $E(w)^*$ mean the complex conjugate of this, the adjoint in the C^* -algebra, $E(w)^*(\psi) = \overline{\psi(w)}$.

If ψ is a function from X to \mathbb{C} , we say it has a *network realization formula* if there exists a Hilbert space \mathcal{M} , a unital $*$ -representation $\rho : C_b(S) \rightarrow$

$B(\mathcal{M})$, and a unitary $U : \mathbb{C} \oplus \mathcal{M} \rightarrow \mathbb{C} \oplus \mathcal{M}$ that in block matrix form is

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

so that

$$\psi(z) = A + B\rho(E(z))(I - D\rho(E(z)))^{-1}C. \quad (3.1)$$

If \mathcal{B} is a C^* -algebra, a positive kernel on a set X with values in \mathcal{B}^* , the dual of \mathcal{B} , is a function $\Gamma : X \times X \rightarrow \mathcal{B}^*$ such that for every finite set $F \subset X$, and every $f : F \rightarrow \mathcal{B}$ we have

$$\sum_{z,w \in F} \Gamma(z,w)(f(w)^*f(z)) \geq 0.$$

Here is the Dritschel, Marcantognini, and McCullough theorem.

Theorem 3.2. *Let S be a set of functions from X to \mathbb{D} , and let $\phi : X \rightarrow \overline{\mathbb{D}}$. The following are equivalent:*

- (i) $\phi \in A^\infty(K_S)$ and $\|\phi\|_{A^\infty(K_S)} \leq 1$.
- (ii) For each finite set $F \subseteq X$ there exists a positive kernel $\Gamma : F \times F \rightarrow C_b(S)^*$ so that, for all $z, w \in F$,

$$1 - \phi(z)\overline{\phi(w)} = \Gamma(z,w)(1 - E(z)E(w)^*). \quad (3.3)$$

- (iii) ϕ has a network realization formula.

Now let us assume that the functions in S are all holomorphic functions on the open set Ω in \mathbb{C}^d . By definition, we always have S is contained in the unit ball of $A^\infty(K_S)$, so when \mathcal{H} is infinite dimensional we have $H^\infty(\mathcal{H}(S))$ is contractively contained in $A^\infty(K_S)$ by Theorem 1.3. We shall show in Theorem 3.7 and Proposition 3.5 that the converse holds if S is finite, or if a certain generic assumption holds.

We shall say that T is a *generic matrix d -tuple on Ω* if, for some $n \in \mathbb{N}$, we have that T is a d -tuple of commuting n -by- n matrices that have a common set of n linearly independent eigenvectors with distinct joint eigenvalues, which means there are n linearly independent eigenvectors v_j in \mathbb{C}^n so that

$$T^r v_j = \lambda_j^r v_j, \quad 1 \leq r \leq d, \quad 1 \leq j \leq n, \quad (3.4)$$

and the n points $\lambda_j = (\lambda_j^1, \dots, \lambda_j^d)$ are distinct points in Ω . The advantages of working with generic d -tuples were pointed out in [1].

We shall define an algebra $H_{\text{gen}}^\infty(S)$ to be the holomorphic functions on Ω for which the norm

$$\|\phi\|_{H_{\text{gen}}^\infty(S)} := \sup\{\|\phi(T)\| : T \text{ is a generic matrix } d\text{-tuple on } \Omega, \\ \text{and } \|\psi(T)\| \leq 1 \forall \psi \in S\}.$$

Proposition 3.5. *Let S be a set of holomorphic functions from Ω to \mathbb{D} . Then $H_{\text{gen}}^\infty(S) = A^\infty(K_S)$ isometrically.*

PROOF: Let ϕ be in the closed unit ball of $A^\infty(K_S)$. Let T be a generic matrix tuple on Ω , with eigenvectors as in (3.4), and assume that $\|\psi(T)\| \leq 1$ for all ψ in S . Let $F = \{\lambda_1, \dots, \lambda_n\}$. Define a kernel $k(z, w)$ on Ω by setting it to zero unless both z and w are in F , and on F define

$$k(\lambda_i, \lambda_j) = \langle v_i, v_j \rangle.$$

Then $k \in K_S$, so

$$(1 - \phi(\lambda_i)\overline{\phi(\lambda_j)})\langle v_i, v_j \rangle \geq 0. \quad (3.6)$$

Then (3.6) says that $\|\phi(T)\| \leq 1$, so ϕ is in the closed unit ball of $H_{\text{gen}}^\infty(S)$.

Conversely, if ϕ is in the closed unit ball of $H_{\text{gen}}^\infty(S)$, then for every finite set $F \subset \Omega$, by Theorem 3.2 applied to F , we have that (3.3) holds on F . Hence by the Theorem again, we have ϕ is in the closed unit ball of $A^\infty(K_S)$.

□

Theorem 3.7. *Let S be a set of holomorphic functions from Ω to \mathbb{D} . Let \mathcal{H} be an infinite dimensional Hilbert space. If S is finite, then $H^\infty(\mathcal{H}(S)) = A^\infty(K_S)$.*

PROOF: By Theorem 1.3, we have $H^\infty(\mathcal{H}(S))$ is contractively contained in $A^\infty(K_S)$. For the converse, let ϕ be in the closed unit ball of $A^\infty(K_S)$, with a network realization formula as above. Let $S = \{\psi_1, \dots, \psi_n\}$.

Let Λ_j be the elements of $C_b(S)$ defined by $\Lambda_j(\psi_i) = \delta_{ij}$. Since each Λ_j is a projection, we get that $\rho(\Lambda_j) = P_j$ gives n mutually orthogonal projections that sum to the identity on \mathcal{M} . Then $\rho(E(z)) = \sum_{j=1}^n \psi_j(z)P_j$.

Expanding (3.1) as a Neumann series in $D\rho(E(z))$, the partial sums ϕ_n will converge locally uniformly on Ω . Therefore if T is in $\mathcal{H}(S)$, since its

spectrum is a compact subset of Ω , we get that $\phi(T) = \lim_n \phi_n(T)$. We have $\rho(E(T)) = \sum_{j=1}^n \psi_j(T) \otimes P_j$, and (3.1) extends to

$$\phi(T) = I_{\mathcal{H}} \otimes A + (I_{\mathcal{H}} \otimes B)\rho(E(T))(I - (I_{\mathcal{H}} \otimes D)\rho(E(T)))^{-1}I_{\mathcal{H}} \otimes C. \quad (3.8)$$

A calculation with (3.8) shows that $I_{\mathcal{H}} - \phi(T)^*\phi(T) \geq 0$, so we conclude $\phi \in \mathcal{S}(\mathcal{H}(S))$. \square

Problem 3.9 Let \mathcal{H} be an infinite dimensional Hilbert space. Do $H^\infty(\mathcal{H}(S))$ and $A^\infty(K_S)$ coincide for all non-empty sets S of holomorphic functions from Ω to \mathbb{D} ?

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