Extensions of bounded holomorphic functions on the tridisk

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ABSTRACT: A set $\mathcal{V}$ in the tridisk $\mathbb{D}^3$ has the polynomial extension property if for every polynomial p there is a function ϕ on $\mathbb{D}^3$ so that $\|\phi\|_{\mathbb{D}^3} = \|p\|_{\mathcal{V}}$ and $\phi|_{\mathcal{V}} = p|_{\mathcal{V}}$. We study sets $\mathcal{V}$ that are relatively polynomially convex and have the polynomial extension property. If $\mathcal{V}$ is one-dimensional, and is either algebraic, or has polynomially convex projections, we show that it is a retract. If $\mathcal{V}$ is two-dimensional, we show that either it is a retract, or, for any choice of the coordinate functions, it is the graph of a function of two variables.

1 Introduction

A celebrated theorem of H. Cartan asserts that if Ω is a pseudoconvex domain in $\mathbb{C}^d$ and $\mathcal{V}$ is a holomorphic subvariety of Ω, then every holomorphic function on $\mathcal{V}$ extends to a holomorphic function on Ω [5]. It is not true, however, that every bounded holomorphic function on $\mathcal{V}$ necessarily extends to a bounded holomorphic function on Ω [13, 14]. It is even rarer for every bounded holomorphic function to extend to a bounded holomorphic function of the same norm, and when this does occur, there is a special relationship between $\mathcal{V}$ and Ω, which we seek to explore.

Let $\mathcal{V}$ be a subset of $\mathbb{C}^d$. By a *holomorphic function on* $\mathcal{V}$ we mean a function $f : \mathcal{V} \to \mathbb{C}$ with the property that for every point λ in $\mathcal{V}$, there is an open ball B in $\mathbb{C}^d$ that contains λ, and a holomorphic function $\phi : B \to \mathbb{C}$

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so that $\phi|_{B\cap\mathcal{V}} = f|_{B\cap\mathcal{V}}$. We shall denote the bounded holomorphic functions on $\mathcal{V}$ by $H^\infty(\mathcal{V})$, and equip this space with the supremum norm:

$$\|f\|_{H^\infty(\mathcal{V})} \ := \ \sup_{\lambda\in\mathcal{V}} |f(\lambda)|.$$

Definition 1.1. *Let Ω be a bounded domain in $\mathbb{C}^d$, and $\mathcal{V} \subseteq \Omega$ be non-empty. Let $\mathcal{A}$ be a subalgebra of $H^\infty(\mathcal{V})$. We say that $\mathcal{V}$ has the $\mathcal{A}$ extension property w.r.t. Ω if, for every $f \in \mathcal{A}$, there is a function ϕ in $H^\infty(\Omega)$ such that $\phi|_{\mathcal{V}} = f|_{\mathcal{V}}$ and*

$$\|\phi\|_{H^\infty(\Omega)} \ = \ \|f\|_{H^\infty(\mathcal{V})}.$$

When $\mathcal{A} = \mathbb{C}[z_1, \dots, z_d]$, the algebra of polynomials, we shall call this the polynomial extension property.

We say $\mathcal{V}$ is a *retract* of Ω if there is a holomorphic map $r : \Omega \to \mathcal{V}$ such that $r|_{\mathcal{V}} = \mathrm{id}|_{\mathcal{V}}$. Clearly any retract has the polynomial extension property, because $\phi := p \circ r$ gives a norm-preserving extension. The converse cannot be true without any regularity assumption on $\mathcal{V}$, because any set that is dense (or dense near the distinguished boundary of Ω) will trivially have the polynomial extension property. We shall restrict our attention, therefore, to sets that have some form of functional convexity. We shall say that $\mathcal{V}$ is a *relatively polynomial convex subset* of Ω if $\overline{\mathcal{V}}$ is polynomially convex and $\overline{\mathcal{V}} \cap \Omega = \mathcal{V}$. We shall say that $\mathcal{V}$ is $H^\infty(\Omega)$ convex if, for all $\lambda \in \Omega \setminus \mathcal{V}$, there exists a $\phi \in H^\infty(\Omega)$ such that

$$|\phi(\lambda)| \ > \ \sup_{z\in\mathcal{V}} |\phi(z)|.$$

Question 1.2. *If $\mathcal{V}$ is a relatively polynomial convex subset of Ω that has the polynomial extension property, must $\mathcal{V}$ be a retract of Ω?*

In [3] it was shown that the answer to Question 1.2 is yes if Ω is the bidisk $\mathbb{D}^2$. We give another proof of this in Section 3.

Theorem 1.3. *A relatively polynomially convex set $\mathcal{V} \subseteq \mathbb{D}^2$ has the polynomial extension property if and only if it is a retract.*

Not every retract is polynomially convex. Indeed, suppose B is a Blaschke product whose zeros are dense on the unit circle. Then $\mathcal{V} = (z, B(z))$ is a retract whose closure contains $\mathbb{T}\times\overline{\mathbb{D}}$, so its polynomial hull is the whole bidisk. Moreover, any superset of $\mathcal{V}$ has the polynomial extension property trivially (since polynomials attain their supremum), so *e.g.* $\mathcal{V} \cup \{(1/2, w) : w \in \mathbb{D}\}$ is a holomorphic variety with the polynomial extension property.

A more general version of Question 1.2 is the following. We do not know the answer even for Ω equal to the bidisk.

Question 1.4. *Is $\mathcal{V}$ a retract of Ω if and only if $\mathcal{V}$ is an $H^\infty(\Omega)$ convex subset of Ω that has the $H^\infty(\mathcal{V})$ extension property?*

Let ρ denote the pseudo-hyperbolic metric on the disk

$$\rho(z, w) \;=\; \left| \frac{z - w}{1 - \bar{w}z} \right|.$$

A *Kobayashi extremal* for a pair of points λ and μ in a domain Ω is a holomorphic function $f : \mathbb{D} \to \Omega$ such that λ and μ are in the range of f, and so that $\rho(f^{-1}(\lambda), f^{-1}(\mu))$ is minimized over all holomorphic functions $g : \mathbb{D} \to \Omega$ that have λ and μ in their range. A *Carathéodory extremal* is a map $\phi : \Omega \to \mathbb{D}$ that maximizes $\rho(\phi(\lambda), \phi(\mu))$.

If Ω is convex, there is a Kobayashi extremal for every pair of points, and by a theorem of L. Lempert [17], for every Kobayashi extremal $f : \mathbb{D} \to \Omega$ for the pair (λ, μ) there is a Carathéodory extremal $\phi : \Omega \to \mathbb{D}$ for the pair that is a left-inverse to f, *i.e.* $\phi \circ f = \mathrm{id}|_{\mathbb{D}}$. The range of a Kobayashi extremal is called a *geodesic*.

A pair of points $\lambda = (\lambda_1, \lambda_2)$ and $\mu = (\mu_1, \mu_2)$ in $\mathbb{D}^2$ is called *balanced* if $\rho(\lambda_1, \mu_1) = \rho(\lambda_2, \mu_2)$. The Kobayashi extremal for λ and μ is unique (up to precomposition with a Möbius map) if and only if λ and μ are balanced. A key part of the proof in [3] was to show that if a set with the polynomial extension property contained a balanced pair of points, then it contained the entire geodesic containing these points. We prove a generalization of this result to the polydisk in Theorem 2.5.

In [11], K. Guo, H. Huang and K. Wang proved that the answer to Question 1.2 is yes if Ω is the tridisk, $\mathcal{V}$ is the intersection of an algebraic set with $\mathbb{D}^3$, and in addition the polynomial extension operator is given by a linear operator L from $H^\infty(\mathcal{V})$ to $H^\infty(\mathbb{D}^3)$. (This is called the *strong extension property* in [20]). The principle focus of this paper is to examine what happens for the tridisk without the assumption that there is a linear extension operator.

For the polydisk, all retracts are described by the following theorem of L. Heath and T. Suffridge [12]:

Theorem 1.5. *The set $\mathcal{V}$ is a retract of $\mathbb{D}^d$ if and only if, after a permutation of coordinates, $\mathcal{V}$ is the graph of a map from $\mathbb{D}^n$ to $\mathbb{D}^{d-n}$ for some $0 \le n \le d$.*

Any relatively polynomially convex subset of $\mathbb{D}^3$ with the polynomial extension property is a holomorphic subvariety (Lemma 2.1), so is of dimension $0, 1, 2$, or 3. The only 3-dimensional holomorphic subvariety of $\mathbb{D}^3$ is $\mathbb{D}^3$. The only 0-dimensional sets with the extension property are singletons (Lemma 2.2). So we just have to consider the cases when $\mathcal{V}$ has dimension 1 and 2. In Theorems 4.1 and 5.1 we show:

Theorem 1.6. *Let $\mathcal{V}$ be a relatively polynomially convex subset of $\mathbb{D}^3$ that has the polynomial extension property and is one dimensional. If $\mathcal{V}$ is algebraic or has polynomially convex projections, then $\mathcal{V}$ is a retract of $\mathbb{D}^3$.*

In Section 6 in Theorem 6.1 we prove:

Theorem 1.7. *Let $\mathcal{V}$ be a relatively polynomially convex subset of $\mathbb{D}^3$ that has the polynomial extension property and is two dimensional. Then either $\mathcal{V}$ is a retract, or, for each $r = 1, 2, 3$, there is a domain $U_r \subseteq \mathbb{D}^2$ and a holomorphic function $h_r : U_r \to \mathbb{D}$ so that*

$$\begin{aligned} \mathcal{V} =& \{(z_1, z_2, h_3(z_1, z_2)) : (z_1, z_2) \in U_3\} \\ =& \{(z_1, h_2(z_1, z_3), z_3) : (z_1, z_3) \in U_2\} \\ =& \{(h_1(z_2, z_3), z_2, z_3) : (z_2, z_3) \in U_1\}. \end{aligned} \tag{1.8}$$

If we could show that one of the sets U_r were the whole bidisk, then $\mathcal{V}$ would be a retract. In Section 7, we show that the set

$$\{z \in \mathbb{D}^3 : z_1 + z_2 + z_3 = 0\}$$

does not have the polynomial extension property, although it does satisfy (1.8).

In Section 8 we look at the spectral theory connections, and show that a holomorphic subvariety $\mathcal{V} \subseteq \mathbb{D}^d$ has the $\mathcal{A}$-extension property if and only it has the $\mathcal{A}$ von Neumann property. Loosely speaking, the $\mathcal{A}$ von Neumann property means that any d-tuple of operators that "lives on" $\mathcal{V}$ has $\mathcal{V}$ as an $\mathcal{A}$ spectral set; we give a precise definition in Def. 8.1.

In [16] it was shown that the answer to Question 1.2 is yes if Ω is the ball in any dimension, or in dimension 2 if Ω is either strictly convex or strongly linearly convex.

There is a domain for which the answer to Question 1.2 is known to be no. This is the symmetrized bidisk, the set $G := \{(z+w, zw) : z, w \in \mathbb{D}\}$. In [1], J. Agler, Z. Lykova and N. Young proved

Theorem 1.9. *The set $\mathcal{V}$ is an algebraic subset of G having the $H^\infty(\mathcal{V})$ extension property if and only if either $\mathcal{V}$ is a retract of G, or $\mathcal{V} = \mathcal{R} \cup \mathcal{D}_\beta$, where $\mathcal{R} = \{(2z, z^2) : z \in \mathbb{D}\}$ and $\mathcal{D}_\beta = \{(\beta + \bar{\beta} z, z) : z \in \mathbb{D}\}$, where $\beta \in \mathbb{D}$.*

2 Preliminaries

2.1 General Domains

Note that if $\mathcal{V} \subseteq \Omega$ is relatively polynomially convex, it is automatically $H^\infty(\Omega)$ convex.

We shall use the following assumptions throughout this section:

(A1) *Ω is a bounded domain, and $\mathcal{V}$ is a relatively polynomially convex subset of Ω that has the polynomial extension property.*

(A2) *$\mathcal{V}$ is an $H^\infty(\Omega)$ convex subset of Ω that has the $H^\infty(\mathcal{V})$ extension property.*

The first lemma is straightforward.

Lemma 2.1. *If either (A1) or (A2) hold, then $\mathcal{V}$ is a holomorphic subvariety of Ω.*

Proof: Under (A1), for every point λ in $\Omega \setminus \mathcal{V}$, there is a polynomial p_λ such that $|p_\lambda(\lambda)| > \|p\|_{H^\infty(\mathcal{V})}$. Let ϕ_λ be the norm preserving extension of p_λ from $\mathcal{V}$ to Ω. Then

$$\mathcal{V} = \bigcap_{\lambda \in \Omega \setminus \mathcal{V}} Z_{\phi_\lambda - p_\lambda},$$

where we use Z_f to denote the zero set of a function f.

Locally, at any point a in $\mathcal{V}$, the ring of germs of holomorphic functions is Noetherian [10, Thm. B.10]. Therefore $\mathcal{V}$ is locally the intersection of finitely many zero zets of functions in $H^\infty(\Omega)$, and therefore is a holomorphic subvariety.

Under (A2) the same argument works, where now p_λ is in $H^\infty(\Omega)$ but not necessarily a polynomial. □

The following lemma is a modification of [3, Lemma 5.1].

Lemma 2.2. *If (A1) holds, then $\overline{\mathcal{V}}$ is connected. If (A2) holds, then $\mathcal{V}$ is connected.*

PROOF: In the first case, consider the Banach algebra $P(\overline{\mathcal{V}})$, the uniform closure of the polynomials in $C(\overline{\mathcal{V}})$. The maximal ideal space of $P(\overline{\mathcal{V}})$ is $\overline{\mathcal{V}}$ [8, Thm. III.1.2]. Assume E is a clopen proper subset of $\overline{\mathcal{V}}$. By the Shilov idempotent theorem [8, Thm. III.6.5], the characteristic function of E is in $P(\overline{\mathcal{V}})$. For each n, there exists a polynomial p_n such that $|p_n - 1| < 1/n$ on E, and $|p_n| < 1/n$ on $\overline{\mathcal{V}} \setminus E$. By the extension property, there are functions ϕ_n of norm at most $1 + 1/n$ in $H^\infty(\Omega)$ that extend p_n. By normal families, there is a subsequence of these functions that converge to a function ϕ of norm 1 in $H^\infty(\Omega)$ that is 1 on $E \cap \Omega$ and 0 on $(\overline{\mathcal{V}} \setminus E) \cap \Omega$. Since $E \cap \Omega$ is non-empty, by the maximum modulus theorem, ϕ must be constant, a contradiction to $\overline{\mathcal{V}} \setminus E$ being non-empty. Therefore $\overline{\mathcal{V}}$ is connected.

In the second case, if E is a clopen subset of $\mathcal{V}$, then the characteristic function of E is in $H^\infty(\mathcal{V})$, so has an extension to $H^\infty(\Omega)$, and the maximum modulus theorem yields that $\mathcal{V}$ is connected. □

An immediate consequence of Lemma 2.2 is that if $\mathcal{V}$ is 0-dimensional, then it is a single point.

In Section 1 we defined the Kobayashi and Carathéodory extremals for a pair of points λ, μ in a set $\Omega \subseteq \mathbb{C}^d$. There is also an infinitesimal version, where one chooses one point $\lambda \in \Omega$ and a non-zero vector v in the tangent space of Ω at λ. A Kobayashi extremal is then a holomorphic map $f : \mathbb{D} \to \Omega$ such that $f(0) = \lambda$ and $Df(0)$ points in the direction of v and has the largest magnitude possible (or any such f precomposed with a Möbius transformation of $\mathbb{D}$). A Carathéodory extremal is a holomorphic map $\phi : \Omega \to \mathbb{D}$ such that $\phi(\lambda) = 0$ and $D\phi(\lambda)[v]$ is maximal (or any such ϕ postcomposed with a Möbius transformation).

More generally, we shall say that Carathéodory-Pick data consists of distinct points $\lambda_1, \dots \lambda_N$ in Ω, and, for each $1 \leq j \leq N$, between 0 and d linearly independent vectors v_j^k (thought of as tangent vectors in $T_{\lambda_j}(\Omega)$), and correspondingly N points $w_1, \dots, w_N$ in $\mathbb{D}$ and complex numbers u_j^k (thought of as tangent vectors in $T_{w_j}(\mathbb{D})$). A Carathéodory-Pick solution to this data is a holomorphic function $\phi : \Omega \to \mathbb{D}$ such that

$$\begin{aligned} \phi(\lambda_j) &= w_j \\ D\phi(\lambda_j) v_j^k &= u_j^k \quad \forall j, k. \end{aligned}$$

We shall say that ϕ is a Carathéodory-Pick extremal for some data if ϕ is a Carathéodory-Pick solution, and no function of $H^\infty(\Omega)$ norm less than 1 is a solution. If $\mathcal{V} \subseteq \Omega$, we shall say that the data is contained in $\mathcal{V}$ if each λ_j is in $\mathcal{V}$ and for each v_j^k there is a sequence of points μ_n in $\mathcal{V}$ that converge to λ_j such that

$$v_j^k \;=\; \|v_j^k\| \lim_{n\to\infty} \frac{\lambda_j - \mu_n}{\|\lambda_j - \mu_n\|}.$$

The next theorem is based on an idea of P. Thomas [21]. Let $P(K)$ denote the uniform closure of the polynomials in $C(K)$.

Theorem 2.3. *Let Ω be bounded, and assume that $\mathcal{V} \subseteq \Omega$ has the polynomial extension property. Let ϕ be a Carathéodory-Pick extremal for Ω for some data. If $\phi|_{\mathcal{V}}$ is in $P(\overline{\mathcal{V}})$, then $\overline{\phi(\mathcal{V})}$ contains the unit circle $\mathbb{T}$.*

Proof: Assume that $\overline{\phi(\mathcal{V})}$ omits some point on $\mathbb{T}$.

Then there is a simply connected star-shaped open set U such that

$$\overline{\phi(\mathcal{V})} \;\subseteq\; \overline{U} \subsetneq \overline{\mathbb{D}}.$$

(Take $U = \mathbb{D} \setminus \mathbb{D}((1+\varepsilon)e^{i\theta}, 2\varepsilon)$ for suitably chosen θ and ε).

Let $h : U \to \mathbb{D}$ be a Riemann map, and $f : \mathbb{D} \to U$ be its inverse.

Consider the Carathéodory-Pick problem on $\mathbb{D}$

$$\begin{aligned} g : w_j &\;\mapsto\; f(w_j) \\ g'(w_j) &\;=\; f'(w_j). \end{aligned}$$

This can clearly be solved by f, so has some solution. But it is well-known that the solution to every extremal Carathéodory-Pick problem on the disk is given by a unique finite Blaschke product. (See for instance [2, Thms. 5.34, 6.15] or [9, Sec. I.2]), and the range of f is contained in U. So there is also a solution g of norm strictly less than one, which can be taken to be a constant multiple r of a Blaschke product.

Then $g \circ h \circ \phi$ is a solution to the original Carathéodory-Pick problem on $\mathcal{V}$, and $\|g \circ h \circ \phi\|$ in $H^\infty(\mathcal{V})$ is less than or equal to r. Since $\phi \in P(\overline{\mathcal{V}})$, for each n, there is a polynomial p_n in d variables and a constant C depending on U so that $\|\phi - p_n\|_{H^\infty(\mathcal{V})} \leq \frac{C}{n}$ and $p_n(\mathcal{V}) \subseteq (1 - \frac{1}{n})U$. As $g \circ h$ can be uniformly approximated on $(1-\frac{1}{n})\overline{U}$ by a sequence of polynomials q_n, the sequence $q_n \circ p_n \in \mathbb{C}[z_1, \ldots, z_d]$ are polynomials that converge uniformly to $g \circ h \circ \phi$ on $\mathcal{V}$. Each such polynomial can be extended to a function ψ_n in

$H^\infty(\Omega)$ with $\|\psi_n\|_{H^\infty(\Omega)} = \|q_n \circ p_n\|_{H^\infty(\mathcal{V})}$. Finally, by normal families, a subsequence of ψ_n will converge to a function ψ in $H^\infty(\Omega)$ of norm at most r that solves the original Carathéodory-Pick problem. This contradicts the assumption that ϕ was an extremal. $\square$

2.2 Balanced Points in the polydisk

Let Ω now be the polydisk, $\mathbb{D}^d$. The automorphisms of $\mathbb{D}^d$ are precisely the maps

$$\lambda \ \mapsto \ (\psi_1(\lambda_{i_1}), \dots, \psi_d(\lambda_{i_d})),$$

where $(i_1, \dots, i_d)$ is some permutation of $(1, \dots, d)$ and each ψ_j is a Möbius map [20, p.167]. The properties of being a retract, being connected, being relatively polynomially convex, and having the polynomial extension property, are all invariant with respect to automorphisms of $\mathbb{D}^d$. The last assertion is because any polynomial composed with an automorphism is in $P(\overline{\mathbb{D}^d})$, the uniform closure of the polynomials. We shall often use this to move points to the origin for convenience.

Definition 2.4. *A pair of distinct points (λ, μ) in $\mathbb{D}^d$ is called n-balanced, for $1 \leq n \leq d$, if, for some permutation $(i_1, \dots, i_d)$ of $(1, \dots, d)$, we have*

$$\rho(\lambda_{i_1}, \mu_{i_1}) = \dots = \rho(\lambda_{i_n}, \mu_{i_n}) \geq \rho(\lambda_{i_{n+1}}, \mu_{i_{n+1}}) \geq \dots \geq \rho(\lambda_{i_d}, \mu_{i_d}).$$

We shall say the pair is n-balanced w.r.t. the first n coordinates if $(i_1, \dots, i_n) = (1, \dots, n)$.

If a pair is n-balanced, we can always permute the coordinates so that it is n-balanced w.r.t. the first n coordinates. Let $\pi_{(n)} : \mathbb{C}^d \to \mathbb{C}^n$ be projection onto the coordinates $z_1, \dots, z_n$.

A pair of points is d-balanced if and only if there is a unique Kobayashi geodesic passing through them. The Carathéodory extremal is unique (up to a Möbius transformation) if and only if the pair is not 2-balanced. Theorem 2.3 has the following important consequence.

Theorem 2.5. *Suppose $\mathcal{V}$ is a set that has the polynomial extension property with respect to $\mathbb{D}^d$. Suppose $\mathcal{V}$ contains a pair of points (λ, μ) that is n-balanced w.r.t. the first n coordinates. If $\pi_{(n)}(\mathcal{V})$ is relatively polynomially convex in $\mathbb{D}^n$, then $\pi_{(n)}(\mathcal{V})$ contains an n-balanced disk of the form*

$$\{(\psi_1(\zeta), \dots, \psi_n(\zeta)) : \zeta \in \mathbb{D}\}$$

for some Möbius transformations $\psi_1, \dots, \psi_n$.

Proof: By composing with an automorphism of $\mathbb{D}^d$ that fixes the first n coordinates, we can assume that

$$\rho(\lambda_1, \mu_1) = \cdots = \rho(\lambda_n, \mu_n) \geq \rho(\lambda_{n+1}, \mu_{n+1}) \geq \cdots \geq \rho(\lambda_d, \mu_d),$$

that $\mu = 0$, and that $\lambda_j \geq 0$ for each j. Let $\phi(z) = \frac{1}{n}(z_1 + \ldots z_n)$. By the Schwarz lemma, ϕ is a Carathéodory extremal for the pair (λ, μ), so by Theorem 2.3, $\overline{\pi_{(n)}(\mathcal{V})} = \pi_{(n)}(\overline{\mathcal{V}})$ contains the unit circle $\{(\tau, \ldots, \tau) : |\tau| = 1\}$. Since $\pi_{(n)}(\overline{\mathcal{V}})$ is polynomially convex, $\pi_{(n)}(\mathcal{V})$ contains $\{(\zeta, \ldots, \zeta) : \zeta \in \mathbb{D}\}$. $\Box$

There is an infinitesimal version of Theorem 2.5, most conveniently expressed when we use an automorphism to move the point of interest to the origin. It is proved from Theorem 2.3 in the same way.

Theorem 2.6. *Suppose $\mathcal{V}$ has the polynomial extension property with respect to $\mathbb{D}^d$, and $0 \in \mathcal{V}$. Suppose there is a non-zero tangent vector $v \in T_0(\mathcal{V})$ such that*

$$|v_1| = \cdots = |v_n| \geq |v_{n+1}| \geq \cdots \geq |v_d|.$$

If $\pi_{(n)}(\overline{\mathcal{V}})$ is relatively polynomially convex, then it contains the disk

$$\{(\zeta, \omega_2\zeta, \ldots, \omega_n\zeta) : \zeta \in \mathbb{D}\}$$

for some unimodular $\omega_2, \ldots, \omega_n$.

3 The bidisk

In this section we will take our domain Ω to be the bidisk $\mathbb{D}^2$, and make the following assumption about $\mathcal{V} \subseteq \mathbb{D}^2$:

(A3) $\mathcal{V}$ is relatively polynomially convex and has the polynomial extension property w.r.t. $\mathbb{D}^2$.

We shall let

$$m_a(z) \;=\; \frac{a - z}{1 - \overline{a}z}$$

be the Möbius map that interchanges a and 0. A subset of $\mathbb{D}^2$ is called balanced if, whenever it contains a 2-balanced pair of points, it contains the entire geodesic through these points.

Let

$$
\begin{aligned}
R_1 &= \{(z_1, z_2) \in \mathbb{D}^2 \ : \ |z_2| < |z_1|\} \\
R_2 &= \{(z_1, z_2) \in \mathbb{D}^2 \ : \ |z_1| < |z_2|\} \\
D_\omega &= \{(\zeta, \omega\zeta) \ : \ \zeta \in \mathbb{D}\}.
\end{aligned}
$$

Theorem 3.1. *Assume that $\mathcal{V}$ is relatively polynomially convex and has the polynomial extension property w.r.t. $\mathbb{D}^2$. Then $\mathcal{V}$ is a retract of $\mathbb{D}^2$.*

Proof: Without loss of generality, we can assume $0 \in \mathcal{V}$. By Lemma 2.1 we know that $\mathcal{V}$ is a holomorphic subvariety, and by Lemma 2.2, if it is 0-dimensional, it is a point, so we shall assume it is not 0-dimensional.

Step 1: If $\mathcal{V}_0$ is a connected component of $\mathcal{V}$, and if $\mathcal{V}_0 \cap R_1$ and $\mathcal{V}_0 \cap R_2$ are both non-empty, then $\mathcal{V}_0$ contains D_ω for some unimodular ω.

A curve in $\mathcal{V}_0$ that goes from R_1 to R_2 crosses the set $\{(z_1, z_2) \in \mathbb{D}^2 : |z_1| = |z_2|\}$ either through a nonzero point or 0. If there is a curve for which the first possibility holds, that is if $\mathcal{V}_0$ contains a non-zero point λ with $|\lambda_1| = |\lambda_2|$, then the pair $(0, \lambda)$ is 2-balanced, so by Theorem 2.5 we get some D_ω in $\mathcal{V}$ and therefore in $\mathcal{V}_0$. Otherwise, there are sequences (z_n) and (w_n) tending to 0 in $\mathbb{C}$, and numbers a_n and b_n in $\overline{\mathbb{D}}$, so that $(z_n, a_n z_n)$ and $(b_n w_n, w_n)$ are in $\mathcal{V}_0$. Passing to a subsequence, we can assume that a_n converges to a and b_n converges to b. If $ab = 1$, choose non-zero α and β so that $|\alpha| + |\beta| = 1$ and $\alpha + a\beta = 1$. Otherwise, let α and β be any non-zero numbers such that $|\alpha| + |\beta| = 1$. Let $\phi(\lambda) = \alpha\lambda_1 + \beta\lambda_2$. We will show that $\overline{\phi(\mathcal{V})} \supseteq \mathbb{T}$ by Theorem 2.3.

Let $v^1 = (1, a)^t$ and $v^2 = (b, 1)^t$ be tangent vectors at 0. We claim that ϕ is extremal for the Carathéodory-Pick problem on $\mathbb{D}^2$

$$
\begin{aligned}
\psi(0) &= 0 \\
D\psi(0)v^1 &= \alpha + \beta a \\
D\psi(0)v^2 &= \alpha b + \beta
\end{aligned}
\tag{3.2}
$$

Indeed, if v^1 and v^2 are linearly independent, then (3.2) determines that $D\psi(0) = (\alpha \ \beta)$, so ϕ is extremal by the Schwarz lemma. If they are not, which occurs when $ab = 1$, then our choice that $\alpha + a\beta = 1$ still yields ϕ is extremal (though no longer the unique solution).

So by Theorem 2.3, we get $\overline{\phi(\mathcal{V})} \supseteq \mathbb{T}$, and since $\mathcal{V}$ is relatively polynomially convex, with

$$
\omega = \frac{\alpha|\beta|}{\beta|\alpha|}
$$

we get $D_\omega \subseteq \mathcal{V}_0$.

Step 2: If $D_\omega \subsetneq \mathcal{V}$, then $\mathcal{V} = \mathbb{D}^2$.

Let $\lambda \in \mathcal{V} \setminus D_\omega$. There exists some point μ in D_ω so that (λ, μ) is 2-balanced, so by Theorem 2.5, $\mathcal{V}$ contains two intersecting balanced geodesics. Composing with an automorphism, we can assume that they intersect at 0, so that $\mathcal{V}$ contains D_ω and D_η for two different unimodular numbers ω and η. Now we repeat the argument in the proof of Step 1 with $a = \omega$ and $b = \overline{\eta}$. Since $ab \neq 1$, we can choose any α and β whose moduli sum to 1, so we get that $\mathcal{V}$ contains D_τ for every unimodular τ. As $\mathcal{V}$ is a holomorphic variety, it must be all of $\mathbb{D}^2$.

After a permutation of coordinates, we can now assume that $\mathcal{V} \setminus \{0\} \subseteq R_1$. Let π_1 be projection onto the first coordinate.

Step 3: If $\mathcal{V}_0$ is a connected component of $\mathcal{V}$, then for every $z \in \mathbb{D}$, the set $\pi_1^{-1}(z) \cap \mathcal{V}_0$ contains at most one element.

Otherwise (z, w_1) and (z, w_2) are distinct points in $\mathcal{V}_0$. Composing with the automorphism of $\mathbb{D}^2$ that sends

$$(0,0), (z, w_1), (z, w_2) \ \mapsto \ (z, w_1), (0,0), (0, m_{w_1}(w_2))$$

respectively, we are in the situation of Step 1, and hence by Step 1 and Step 2, $\mathcal{V} = \mathbb{D}^2$.

Step 4: If $\mathcal{V}$ is connected, $\mathcal{V}$ is a retract.

By Step 3, the only remaining case is when $\mathcal{V} = \{(z, f(z)) : z \in U\}$ where $U \subseteq \mathbb{D}$ and $f : U \to \mathbb{D}$ satisfies $|f(z)| < |z|$ if $z \neq 0$. Since $\mathcal{V}$ is a holomorphic subvariety of $\mathbb{D}^2$, we must have $U = \mathbb{D}$ and f is holomorphic.

Step 5: The set $\mathcal{V}$ has to be connected.

It is sufficient to consider the case when it is one-dimensional. By Steps 1 and 3, $\mathcal{V}$ cannot have any branch points, so must be a disjoint union of single sheets. It cannot contain any 2-balanced pairs, or we are done by Step 2. Assuming $0 \in \mathcal{V}$, this means that, after a permutation of coordinates if necessary, there is some sheet $\mathcal{S} = \{(z, f(z)) : z \in \mathbb{D}\}$ in $\mathcal{V}$, where $|f(z)| < |z|$ if $z \neq 0$, and no point on $D_\omega \setminus \{0\}$ for any unimodular ω. By Lemma 3.3, $\mathcal{S}$ must be all of $\mathcal{V}$, for otherwise $\mathcal{V}$ would contain a 2-balanced pair. □

Lemma 3.3. *Suppose $X \subseteq \mathbb{D}^2$ contains the set $\mathcal{S} = \{(z, f(z)) : z \in \mathbb{D}\}$ where $f : \mathbb{D} \to \mathbb{D}$ is a holomorphic function satisfying $f(0) = 0$, and $\mathcal{S}$ is not all of X. Then X contains a 2-balanced pair.*

Proof: Let (z_1, w_1) be any point in $X \setminus \mathcal{S}$. Composing with the automorphism $(m_{z_1}, m_{f(z_1)})$ we can assume that $(0, w_1) \in X \setminus \mathcal{S}$, and $\mathcal{S} = \{(z, g(z)) : z \in \mathbb{D}\}$, where

$$g(z) \ = \ m_{f(z_1)} \circ f \circ m_{z_1}(z). \tag{3.4}$$

If X has no 2-balanced pairs, we must have that for all z in $\mathbb{D}$,

$$\rho(g(z), w_1) \ > \ \rho(z, 0). \tag{3.5}$$

Let $1 > r > |w_1|$, and consider $\{g(re^{i\theta}) : 0 \leq \theta \leq 2\pi\}$. By (3.5), this set must lie outside the pseudohyperbolic disk centered at w_1 of radius r, and inside the disk centered at 0 of radius r by the Schwarz Lemma (since $g(0) = 0$). By the argument principle, this would mean that g has no zero in $\mathbb{D}(0, r)$, a contradiction. □

4 $\mathcal{V}$ is one-dimensional with polynomially convex projections

In this section we take $\Omega = \mathbb{D}^3$. We make the following assumption about $\mathcal{V} \subseteq \mathbb{D}^3$:

(A4) The set $\mathcal{V}$ has the polynomial extension property with respect to $\mathbb{D}^3$, is one-dimensional, and both $\mathcal{V}$ and $\pi(\mathcal{V})$ are relatively polynomially convex for every projection π onto two of the coordinate functions.

Theorem 4.1. *If $\mathcal{V}$ satisfies (A4), then it is a retract of $\mathbb{D}^3$.*

We shall prove Theorem 4.1 in a series of three Lemmas, 4.2, 4.3 and 4.4. Composing with automorphisms of $\mathbb{D}^3$, we can assume without loss of generality that $0 \in \mathcal{V}$, and since $\mathcal{V}$ is a holomorphic subvariety, we can also assume that 0 is a regular point. Thus, there are germs f_2, f_3 such that $\mathcal{V}$ coincides with $\{(\zeta, \zeta f_2(\zeta), \zeta f_3(\zeta))\}$ in a neighborhood of 0. Permuting the coordinates we may also assume that $|f_j'(0)|$ are less than or equal to 1 for each j. Let $\mathcal{V}_0$ be the component of $\mathcal{V}$ containing 0.

Recall that $\pi_{(2)} : \mathbb{C}^3 \to \mathbb{C}^2$ is projection onto the first two coordinates.

Lemma 4.2. *Either $\mathcal{V}_0$ is a retract of $\mathbb{D}^3$, or, up to a composition with an automorphism of $\mathbb{D}^3$, the set $\{(\zeta, \zeta) : \zeta \in \mathbb{D}\}$ is contained in $\pi_{(2)}(\mathcal{V}_0)$.*

Proof: If $|f_j'(0)| = 1$ for $j = 2$ or 3, we are done by Theorem 2.6. So we shall assume that they are both less than 1. Let

$$R := \{\lambda \in \mathbb{D}^3 : |\lambda_j| < |\lambda_1| \text{ for } j = 2, 3\}.$$

If $\mathcal{V}_0 \setminus \{0\}$ is not contained in R, then there is a point in $\mathcal{V}_0 \cap \partial R$ that is 2-balanced with respect to 0, so we are finished by Theorem 2.5.

Assume that $\mathcal{V}_0 \setminus \{0\}$ is contained in R, and that $\mathcal{V}_0$ is not single-sheeted over the first coordinate, so it contains $z = (z_1, z_2, z_3)$ and (z_1, w_2, w_3) where $(z_2, z_3) \neq (w_2, w_3)$. Moreover, we can assume that $\mathcal{V}_0$ is regular at z, since the singular points are of dimension 0.

Composing with an automorphism that interchanges 0 and (z_1, z_2, z_3), we see that $\mathcal{V}_0$ contains $0, z$ and a point $\mu = (0, \mu_2, \mu_3)$ and that $\mathcal{V}_0$ is regular at 0. Therefore the image of $\mathcal{V}_0$ under the chosen automorphism must contain a path connecting μ and z that does not go through 0. Therefore this path must intersect ∂R, so we are in the position considered at the beginning of the proof, which was covered by Theorem 2.5.

Consequently, we are left with the case when $\mathcal{V}_0$ is single-sheeted over the first coordinate. Then there is a sheet $\{(\zeta, \zeta g_2(\zeta), \zeta g_3(\zeta))\}$ passing through 0 and z that stays inside R. Since $\mathcal{V}_0$ is single sheeted, this means it is a retract. □

Lemma 4.3. *Suppose* $\{(\zeta, \zeta) : \zeta \in \mathbb{D}\}$ *is contained in* $\pi_{(2)}(\mathcal{V}_0)$. *Then there is a holomorphic* $f : \mathbb{D} \to \mathbb{D}$ *such that* $\{(\zeta, \zeta, f(\zeta)) : \zeta \in \mathbb{D}\} \subseteq \mathcal{V}_0$.

Proof: Let $\mathcal{W} = \{(\zeta, w) : (\zeta, \zeta, w) \in \mathcal{V}_0\}$. This is a one-dimensional variety. Let $\mathcal{W}_0$ be the connected component of 0. If $\mathcal{W}_0$ contains a point in $\{|\zeta| = |w|\}$ then $\mathcal{V}_0$ contains a 3-balanced point, and we are done. We assumed $|f_3'(0)| \leq 1$; if equality obtains, then for some unimodular ω we would have $\phi(\lambda) = \frac{1}{3}(\lambda_1 + \lambda_2 + \omega\lambda_3)$ would satisfy the hypotheses of Theorem 2.3, and polynomial convexity would again give a 3-balanced disk in $\mathcal{V}_0$.

So we can assume $\mathcal{W}_0 \subseteq \{|w| < |\zeta|\}$. Either $\mathcal{W}_0$ is single sheeted over ζ, and we are done, or as in Lemma 4.2 we find two distinct regular points (ζ, w_1) and (ζ, w_2). Composing with the automorphism $\alpha = (m_\zeta, m_{w_1})$, we get the points $(0,0), (\zeta, w_1)$ and $(0, m_{w_1}(w_2))$ all in $\alpha(\mathcal{W}_0)$, and by Lemma 3.3 we get $\mathcal{W}_0$ contains a 3-balanced pair. □

Lemma 4.4. *Suppose* $\{(\zeta, \zeta) : \zeta \in \mathbb{D}\}$ *is contained in* $\pi_{(2)}(\mathcal{V}_0)$. *Then there is a holomorphic* $f : \mathbb{D} \to \mathbb{D}$ *such that* $\{(\zeta, \zeta, f(\zeta)) : \zeta \in \mathbb{D}\} = \mathcal{V}$.

PROOF: By Lemma 4.3, we have $\mathcal{S} := \{(\zeta, \zeta, f(\zeta)) : \zeta \in \mathbb{D}\}$ is a subset of $\mathcal{V}$. Suppose the containment is proper, and there exists $(z_1, z_2, z_3) \in \mathcal{V} \setminus \mathcal{S}$. If $z_2 = z_1$, let $\mathcal{W} = \{(\zeta, w) : (\zeta, \zeta, w) \in \mathcal{V}\}$. This contains the set $\{(\zeta, f(\zeta)) : \zeta \in \mathbb{D}\}$ and a point (z_1, z_3) with $z_3 \neq f(z_1)$. By Lemma 3.3, this means $\mathcal{W}$ contains a 2-balanced pair, which means $\mathcal{V}$ contains a 3-balanced pair. Since $\mathcal{V}$ is relatively polynomially convex, by Theorem 2.5 this means $\mathcal{V}$ contains a 3-balanced disk. Since we are assuming that $\mathcal{V}$ is larger than a disk, there must be another point, and hence a 2-balanced disk through this point and the 3-balanced disk. So, after an automorphism, we can assume that $\mathcal{V}$ contains two different discs $\{(\zeta, \zeta, \zeta) : \zeta \in \mathbb{D}\}$ and $\{(\eta, \omega\eta, g(\eta)) : \eta \in \mathbb{D}\}$ for some unimodular ω. If $\omega \neq 1$, then for any α, β with $|\alpha| + |\beta| = 1$, the function $\phi(z) = \alpha z_1 + \beta z_2$ will be a Carathéodory-Pick extremal for the Carathéodory-Pick data

$$\begin{aligned} \psi(0) &= 0 \\ D\psi(0)(1\ 1\ 1)^t &= \alpha + \beta \\ D\psi(0)(1\ \omega\ g'(0))^t &= \alpha + \beta\omega \\ D\psi(0)(0\ 0\ 1)^t &= 0. \end{aligned} \tag{4.5}$$

So by Theorem 2.3, $\phi(\overline{\mathcal{V}})$ will contain $\mathbb{T}$, that is

$$\left(\tau\bar{\alpha}/|\alpha|, \tau\bar{\beta}/|\beta|\right) \in \pi_{(2)}(\overline{\mathcal{V}}) \tag{4.6}$$

for each $\tau \in \mathbb{T}$. By polynomial convexity (4.6) holds for every $\tau \in \mathbb{D}$ which means that, as α and β can vary independently over the circle of radius $\frac{1}{2}$, $\pi_{(2)}\mathcal{V} \supset \mathbb{D}^2$.

Hence $\mathcal{V}$ could not have been one dimensional. (Indeed, since $\pi_{(2)}$ is Lipschitz, it cannot increase Hausdorff dimension). If $\omega = 1$, then $g'(0) \neq 1$ (otherwise $g(\eta) = \eta$ for $\eta \in \mathbb{D}$), so we interchange the second and third coordinates and repeat the argument.

If $z_2 \neq z_1$, then there will be some point in $\mathcal{S}$ that is 2-balanced with respect to z, so by Theorem 2.5 $\mathcal{V}$ contains a 2-balanced disk in addition to $\mathcal{S}$. Repeating the previous argument again shows that $\mathcal{V}$ cannot be one-dimensional. □

5 $\mathcal{V}$ is one dimensional and algebraic

We shall say that $\mathcal{V} \subseteq \mathbb{D}^d$ is algebraic if there is a set of polynomials such that $\mathcal{V}$ is the intersection of $\mathbb{D}^d$ with their common zero set. (The set can always be chosen to be finite by the Hilbert basis theorem.) Let $\mathcal{W}$ be the common zero set of the polynomials in $\mathbb{C}^d$ (so $\mathcal{V} = \mathcal{W} \cap \mathbb{D}^d$).

Theorem 5.1. *Suppose: (A5) The set $\mathcal{V}$ is a one-dimensional algebraic subset of $\mathbb{D}^3$ that has the polynomial extension property.*

Then it is a retract.

First, we prove that it is smooth. We need to use the following four results. The first one is [23, Thm. 5.4A].

Proposition 5.2. *Let V, W be analytic spaces and $F : V \to W$ be proper and extend to be continuous and holomorphic on $\overline{V}$ (i.e. F is c-holomorphic). Then $F(V)$ is analytic in W.*

The next result is from [6, p. 122].

Proposition 5.3. *Let A be an analytic set in $\mathbb{C}^n$, $a \in A$, $dim_a A = p$. Assume that there is a connected neighborhood $U = U' \times U''$ of a such that $\pi : U \cap A \to U' \subseteq \mathbb{C}^p$ is proper.*

Then there exist an analytic set $W \subset U'$, $\dim W < p$, and $k \in \mathbb{N}$ such that

- *$\pi : U \setminus \pi^{-1}(W) \to U' \setminus W$ is a local k-sheeted covering;*
- *$\pi^{-1}(W)$ is nowhere dense in $A_p \cap U$, where $A_p = \{z \in A : \dim_z A = p\}$.*

The following proposition essentially can be found in proofs that are scattered over Sections 3.1 and 3.2 in [6]. For the convenience of the reader we recall its (elementary) proof.

Proposition 5.4. *Let A be an analytic set in a domain $\Omega \subset \mathbb{C}^p \times \mathbb{C}^m$ and $\pi : (z', z'') \mapsto z'$ be projection onto $\mathbb{C}^p$. If $a = (a', a'')$ is an isolated point of $\pi^{-1}(a') \cap A$, then there is a polydisc $U = U' \times U''$ with the center equal to a such that $\pi : U \cap A \to U'$ is proper.*

PROOF: One can find a polydisc U'' such that a is the only point of the intersection of $\{a'\} \times \overline{U''}$ and $\pi|_A^{-1}(a')$. Since A is closed, there is a polydisc U' such that A does not have limits points on $U' \times \partial U''$, which means that $\pi : U \cap A \to U'$ is proper. $\square$

The next tool that will be exploited in the present section is taken from [18, Chap. V.1].

Proposition 5.5 (The analytic graph theorem)**.** *Let U, V be complex manifolds and $f : U \to V$ locally bounded. Then f is holomorphic if and only if its graph $\{(x, f(x)) : \ x \in U\}$ is analytic in $U \times V$.*

We are now in position to start the proof of the theorem.

Lemma 5.6. *If (A5) holds, and there is an automorphism Φ of $\mathbb{D}^3$ so that $\Phi(\mathcal{V}) \subseteq \mathbb{D}^2 \times \{\eta\}$ for some $\eta \in \mathbb{D}$, then $\mathcal{V}$ is a retract.*

Proof: Under the hypotheses, $\pi_{(2)}(\Phi(\mathcal{V}))$ is a polynomially convex subset of $\mathbb{D}^2$ that has the polynomial extension property so by Theorem 1.3 it is a retract of the bidisk. It follows that $\Phi(\mathcal{V})$, and hence $\mathcal{V}$, are retracts of the tridisk. □

Lemma 5.7. *Assume (A5) holds, and $\mathcal{V}$ contains a 2-balanced pair (w', w'') that is not 3-balanced. Assume also that w'' is a regular point of $\mathcal{V}$. Then there is an automorphism Φ of $\mathbb{D}^3$ that takes w' to 0 and w'' to $w = (w_1, w_1, w_3)$ and an irreducible component $\mathcal{V}'$ of $\mathcal{V}$ such that $\Phi(\mathcal{V}')$ contains 0 and w and such that $\Phi(\mathcal{V}') \subseteq \{z_1 = z_2\}$.*

Proof: After an automorphism, we can assume that $\mathcal{V}$ contains 0 and a point $w = (w_1, w_1, w_3)$ with $|w_3| < |w_1|$. By Lemma 2.2, the point w is not isolated in $\mathcal{V}$. Consider the function

$$\begin{aligned} \psi : \mathcal{V} \times \mathbb{C} &\to \mathbb{C} \\ (z, \zeta) &\mapsto z_1 - \zeta z_2. \end{aligned}$$

This function vanishes at $(w, 1)$, and therefore on a one-dimensional subvariety of $\mathcal{V} \times \mathbb{C}$ in a neighborhood of that point. If it vanishes on the set $\zeta = 1$, then $z_2 - z_1$ vanishes on all of some component $\mathcal{V}'$ of $\mathcal{V}$.

Otherwise, by the Weierstrass preparation theorem, there is a non-empty open set E of the circle $\mathbb{T}$ so that $\mathcal{V}$ intersects

$$W_\omega \ := \ \{|z_3| < |z_2| = |z_1|\} \cap \{z_2 = \omega z_1\}$$

for every $\omega \in E$. Let F_ω be any left inverse to the map $\zeta \mapsto (\zeta, \omega\zeta)$, and define

$$\begin{aligned} G_\omega : \mathbb{D}^3 &\to \mathbb{D} \\ z &\mapsto F_\omega(z_1, z_2). \end{aligned}$$

For each $\omega \in E$, there is a point $w = (w_1, \omega w_1, w_3)$ in W_ω, and there is a geodesic in $\mathbb{D}^3$ that contains 0 and w. The function G_ω is a left inverse to the Kobayashi extremal through these points. By Theorem 2.3, $\overline{G_\omega(\mathcal{V})}$ contains the unit circle. Therefore $\pi_{(2)}(\overline{\mathcal{V}})$ contains $\mathbb{T} \times E$. But since $\mathcal{V}$ is algebraic, there can only be finitely many points lying over any point in $\mathbb{T}$, except perhaps for a zero-dimensional singular set. □

Lemma 5.8. *Let f_i, g_i, $i = 1, 2$, be holomorphic functions in the closed unit ball in $H^\infty(\mathbb{D}(t))$. Let $\mathcal{V}$ be an analytic variety in $\mathbb{D}^3$ that contains two discs $\mathcal{S} = \{(\lambda, \lambda f_1(\lambda), \lambda f_2(\lambda)) : \lambda \in \mathbb{D}(t)\}$ and $\mathcal{S}' = \{(\lambda g_1(\lambda), \lambda, \lambda g_2(\lambda)) : \lambda \in \mathbb{D}(t)\}$. If the germs of these discs at 0 are not equal, then one can find two points, one in $\mathcal{S} \setminus \{0\}$ and the second in $\mathcal{S}' \setminus \{0\}$, that are arbitrarily close to 0 and form a 2-balanced pair.*

Proof: Let us consider the values

$$\rho(\lambda, \mu g_1(\mu)),\ \rho(\lambda f_1(\lambda), \mu),\ \rho(\lambda f_2(\lambda), \mu g_2(\mu)). \tag{5.1}$$

If the inequality $\rho(\lambda, \mu g_1(\mu)) \le \rho(\lambda f_1(\lambda), \mu)$ is satisfied for $\lambda, \mu \in \mathbb{D}(s)$, where $s > 0$ is small, then f_1 is a unimodular constant (to see it take $\mu = 0$) and $g_1 f_1 = 1$ (put $\lambda f_1 = \mu$). If additionally $\rho(\lambda f_2(\lambda), \mu g_2(\mu)) \le \rho(\lambda f_1(\lambda), \mu)$, $\lambda, \mu \in \mathbb{D}(s)$, putting $\mu = \lambda f_1$ we find that $\mathcal{S}$ and $\mathcal{S}'$ coincide near 0. This shows that for λ and μ ranging within $\mathbb{D}(s)$ the maximum of the values in (5.1) cannot be attained by the second term listed there. By symmetry, the same is true for the first term and a similar argument shows that values in (5.1) cannot be dominated by the third term, as well.

Consequently, allowing λ and μ to range within $\mathbb{D}(s) \setminus \{0\}$ we see that the maximum of three hyperbolic distances in (5.1) is attained by at least two of them simultaneously. For such a choice of λ and μ the points $(\lambda, \lambda f_1(\lambda), \lambda f_2(\lambda))$ and $(\mu g_1(\mu), \mu, \mu g_2(\mu))$ form the 2-balanced pair we are looking for. □

Let $\mathbb{B}(t)$ be the polydisc $\mathbb{D}^3(t)$ of radius t centered at the origin.

Lemma 5.9. *If (A5) holds, then $\mathcal{V}$ is locally a graph of a holomorphic function.*

Proof: Since the property is local it suffices to show that $\mathcal{V}$ is smooth at $0 \in \mathcal{V}$.

Any analytic set is a locally finite union of its irreducible components. Therefore we can choose $t > 0$ so that any irreducible component of $\mathcal{V}$ that intersects $\mathbb{B}(t)$ contains 0.

For each $j = 1, 2, 3$, write $\mathcal{V}$ as the union of two analytic sets $\mathcal{W}_j \cup \mathcal{V}_j$ such that $\mathcal{W}_j$ is contained in $\{z_j = 0\}$ while 0 is an isolated point of $\mathcal{V}_j \cap \{z_j = 0\}$.

Let $\pi_j : \mathbb{C}^3 \to \mathbb{C}$ denote the projection on the j-th variable, $z \mapsto z_j$, $j = 1, 2, 3$. Decreasing t we can assume that $\pi_j|_{U_j \cap \mathcal{V}_j} \to U_j'$ is proper for some polydisc $U_j = U_j' \times U_j''$ containing $\mathbb{B}(t)$ (Proposition 5.4) and that any point of $\mathcal{V}_j \cap U_j$, possibly without 0, is a regular point of $\mathcal{V}$. Let $W_j \subset U_j'$ be as in Proposition 5.3. Since it is a discrete set, decreasing t we can also assume that W_j and $\mathbb{D}(t)$ have at most one common point and that the common point is 0, if it exists, $j = 1, 2, 3$.

Claim 1. Assume that there is a point x in $\mathbb{B}(t) \cap \mathcal{V}$ such that $|x_1| > |x_2|, |x_3|$. Then, near 0 the variety $\mathcal{V}_1$ is a graph $\{(\lambda, \lambda f(\lambda), \lambda g(\lambda)) : \ \lambda \in \mathbb{D}(t)\}$ for some f, g in the open unit ball of $H^\infty(\mathbb{D}(t))$.

Proof of Claim 1. To prove the assertion we need to show that $\mathcal{V}_1$ is single sheeted near 0, that is the multiplicity of the projection

$$\pi_1 : U_1 \cap \mathcal{V}_1 \to U_1'$$

is equal to 1. Actually, this would mean that in a neighborhood of 0 the variety is $\mathcal{V}_1$ is of the form $\{(\lambda, \lambda f(\lambda), \lambda g(\lambda)) : \ \lambda \in \mathbb{D}(t)\}$ for some functions f, g that are locally bounded, and thus holomorphic, according to Proposition 5.5. For any $z \in \mathcal{V}_1 \cap U_1$ the pair $(0, z)$ is not 2-balanced. If it were not the case, we could use Lemma 5.7 to find that f and g are unimodular constants, which contradicts the assumption that $|x_j| < |x_1|$, $j = 1, 2$. Consequently $|f(\lambda)|, |g(\lambda)| < 1$, $\lambda \in \mathbb{D}(t)$, proving the assertion.

Since $x_1 \notin W_1$ it is enough to show that $\mathcal{V}_1 \cap U_1$ is single sheeted over x_1. Suppose the contrary, that is we can find another point $x' = (x_1, y_2, y_3)$ in $\mathcal{V} \cap U_1$. Let γ be a curve in $\mathcal{V} \cap U$ joining 0 and x' and such that $\gamma(t) \neq x$ for any $0 \leq t \leq 1$ and $\gamma(t)$ is a regular point of $\mathcal{V}$ for any $t \in (0, 1]$. The automorphism

$$\Phi(z) = (m_{x_1}(z_1), m_{x_2}(z_2), m_{x_3}(z_3))$$

switches 0 with x and the curve $\Phi \circ \gamma$ joins (x_1, x_2, x_3) with $(0, m_{x_2}(y_2), m_{x_3}(y_3))$. Therefore the curve must meet one of sets $\{|z_3| \leq |z_1| = |z_2|\}$ or $\{|z_2| \leq |z_1| = |z_3|\}$. We lose no generality assuming that the first possibility holds. If $\Phi \circ \gamma$ meets a point $w \neq 0$ such that $|w_1| = |w_2| = |w_3|$, then by Theorem 2.5 the set $\Phi(\mathcal{V})$ contains a 3-balanced disk through 0, in addition to a curve joining 0 to x. This would make 0 a multiple point of $\Phi(\mathcal{V})$, so x would be a multiple point of $\mathcal{V}$, contradicting the assumption that it was smooth. If $\Phi \circ \gamma$ meets the set

$$\Sigma := \{z \in \mathbb{C}^3 : \ |z_3| < |z_1| = |z_2|\}$$

we get a contradiction using Lemma 5.7, since at the first point of intersection of $\Phi \circ \gamma$ with Σ, say at $w = \Phi \circ \gamma(t_0)$, a neighborhood of w in $\mathcal{V}$ contains an analytic disk inside the image of Σ under Φ. This means that $\mathcal{V}$ is not smooth at w. □

Claim 2. Assume that $x = (x_1, x_1, x_3) \in \mathcal{V} \cap \mathbb{B}(t)$ is such that $0 < |x_3| \leq |x_1|$. Then $\mathcal{V}_1 \cap \{z \in \mathbb{B}(t) : \ z_1 = z_2\} = \{(\lambda, \lambda, \lambda f(\lambda)) : \ \lambda \in \mathbb{D}(t)\}$ for some f in the closed unit ball of $H^\infty(\mathbb{D}(t))$.

Proof of Claim 2. Since $(0, x)$ is either a two- or three-balanced pair, the variety $\mathcal{W} := \mathcal{V}_1 \cap \{z_1 = z_2\}$ is one dimensional at 0. Repeating the argument used in Claim 1 it is enough to show that $\mathcal{W}$ is single sheeted over $\{z \in \mathbb{D}(t)^2 : z_1 = z_2\}$. To see it, take two points $\mu = (\mu_1, \mu_1, \mu_3)$ and $\nu = (\mu_1, \mu_1, \nu_3)$ in $\mathcal{V}_1 \cap \mathbb{B}(t)$.

If either $|\mu_3| > |\mu_1|$ or $|\nu_3| > |\mu_1|$ then, after a proper permutation of coordinates, we find from Claim 1 that $\mathcal{V} \setminus \{z \in \mathbb{C}^3_* : \ z_3 = 0\}$ is a graph of a function over the third variable:

$$\{(\lambda f(\lambda), \lambda g(\lambda), \lambda) : \ \lambda \in \mathbb{D}(t)\},$$

where $|f|, |g| < 1$ on $\mathbb{D}(t)$, which is impossible, as x belongs to it.

If in turn $|\mu_3| = |\mu_1|$, then for some unimodular ω, the variety $\mathcal{V}$ contains the disc $\{(\lambda, \lambda, \omega\lambda) : \ \lambda \in \mathbb{D}\}$. One can find $\lambda \in \mathbb{D}(t)$ such that

$$\rho(\mu_1, \lambda) = \rho(\mu_3, \omega\lambda),$$

which entails that there is a 3-balanced disc in $\mathcal{V}$ passing through μ and $(\lambda, \lambda, \omega\lambda)$. Consequently, $\mathcal{V}$ is not smooth at μ. Of course, the same holds if $|\nu_3| = |\mu_1|$.

Finally consider the case when $|\mu_3| < |\mu_1|$ and $|\nu_3| < |\mu_1|$. Let γ be a curve in $\mathcal{W} \cap \mathbb{B}(t)$ joining 0 and μ. A continuity argument proves that there is $s > 0$ that satisfies the equality

$$\rho(\mu_1, \gamma_1(s)) = \rho(\nu_3, \gamma_3(s)).$$

Again, we can obtain a contradiction with the smoothness constructing a balanced disc passing through $\gamma(s)$ and μ. □

Claim 3. Suppose that there is a point $x \in \mathcal{V}_1 \cap \mathbb{B}(t) \setminus \{0\}$ such that $|x_1| \geq |x_2| \geq |x_3|$. Then $\mathcal{V}_1 \cap \mathbb{B}(t)$ is the graph of a holomorphic function.

Proof of Claim 3. Note first that the assertion is an immediate consequence of Claim 1 provided that there is a point in $y \in \mathcal{V}_1 \cap \mathbb{B}(t)$ such that $|y_1| > \max(|y_2|, |y_3|)$. On the other hand, if there is a point y in $\mathcal{V}_1 \cap \mathbb{B}(t)$ that satisfies $|y_2| > \max(|y_1|, |y_3|)$ or $|y_3| > \max(|y_1|, |y_2|)$, then Claim 1 gives a contradiction.

Therefore, we need to focus on the case when any $y \in \mathcal{V}_1 \cap \mathbb{B}(t)$ satisfies $|y_1| = |y_2| \geq |y_3|$ or $|y_1| = |y_3| \geq |y_2|$. If $\mathcal{V}_1 \cap \{z_3 = 0\}$ is not discrete, then it is the union of discs of the form $\{(\lambda, \omega\lambda, 0) : \ \lambda \in \mathbb{D}\}$ with unimodular ω, and the analogous statement holds if $\mathcal{V}_1 \cap \{z_2 = 0\}$ is not discrete. Otherwise we can use Claim 2, which provides us with a description of the intersections $\mathcal{V}_1 \setminus \{z_2 z_3 = 0\}$ with the hyperplanes

$$l_1 = \{z_1 = \omega z_2\}, \quad l_2 = \{z_1 = \omega z_3\}$$

for unimodular constants ω. In particular, if $\mathcal{V}_1$ lies entirely in one of these hyperplanes, we are done. Otherwise, there are at least two points in $\mathcal{V}_1 \cap \mathbb{B}(t) \setminus \{0\}$ that lie in two different hyperplanes. Applying Claim 2 (after an appropriate permutation of coordinates and multiplication of them by unimodular constants) we find that $\mathcal{V}_1$ contains two different analytic discs. The possibilities that may occur here are listed below. The first one describes the case when both points lie in hyperplanes of type l_1 (or type l_2, after a change of coordinates) while the second one refers to the case when one of the points is in l_1 and the second in l_2:

i) $(\lambda, \lambda, \lambda f(\lambda)), (\lambda, \omega\lambda, \lambda g(\lambda)) \in \mathcal{V}$ for $\lambda \in \mathbb{D}(t)$,

ii) $(\lambda, \lambda, \lambda f(\lambda)), (\lambda, \lambda g(\lambda), \lambda) \in \mathcal{V}$ for $\lambda \in \mathbb{D}(t)$,

where f and g are in the closed unit ball of $H^\infty(\mathbb{D}(t))$ and $\omega \in \mathbb{T}$. Making use of Lemmas 5.8 and 5.7 we see that both cases contradict the smoothness of $\mathcal{V}$ outside the origin. □

Claim 4 If $\mathcal{W}_1$ is not discrete, then it is the graph of a holomorphic function.

Proof of Claim 4. If there is a point $(0, y_2, y_3)$ in $\mathcal{W}_1$ such that $|y_2| \neq |y_3|$ we are done, as after an appropriate change of coordinates Claim 1 can be applied here. Otherwise, for some unimodular ω there is a disc of the form $\{(0,\lambda,\omega\lambda) : \lambda\in\mathbb{D}\}$ that is entirely contained in $\mathcal{V}$. If there are two different discs in $\mathcal{V}$ we end up with a particular case of possibility i) that occurred in Claim 3 (take f and g equal to 0 and and multiply the coordinates by unimodular constants). $\square$

We come back to the proof of Lemma 5.9. If there is a point $x\in\mathcal{V}\cap\mathbb{B}(t)\setminus\{0\}$ all of whose coordinates do not vanish, then $\mathcal{V}$ is a union of at most two graphs of holomorphic functions, due to Claims 3 and 4. If there is no such point, then $\mathcal{V}\cap\mathbb{D}(t)$ can also be expressed as at most three graphs, according to Claim 4.

If $\mathcal{V}$ is not a graph of one function, then permuting coordinates we see that it contains two discs

$$\{(\lambda,\lambda f(\lambda),\lambda g(\lambda)) :\ \lambda\in\mathbb{D}(t)\},\quad \{(0,\lambda,\lambda h(\lambda)) :\ \lambda\in\mathbb{D}(t)\},$$

where f,g,h are in the closed unit ball of $H^\infty(\mathbb{D}(t))$. Here, again, Lemmas 5.7 and 5.8 give a contradiction. $\square$

Proof of Theorem 5.1. We have shown that $\mathcal{V}$ is smooth. If $\mathcal{V}$ contains a 3-balanced pair, then it contains a 3-balanced disk by Theorem 2.5. If this is all, then it is a retract. If it contains any other point, then that point and some point in the 3-balanced disk would form a 2-balanced pair that is not 3-balanced, and we get a contradiction from Lemma 5.7.

So we can assume that $\mathcal{V}$ contains no 3-balanced pairs, and, by Lemma 5.7 again, no 2-balanced pairs either, or else it would be a retract. After an automorphism, we can assume that $0\in\mathcal{V}$ and and

$$\mathcal{V}\setminus\{0\}\ \subseteq\ \{\max(|z_2|,|z_3|)<|z_1|\}. \tag{5.10}$$

In a neighborhood of 0, we can write $\mathcal{V}$ as $\{(\zeta,f_2(\zeta),f_3(\zeta))\}$ for some holomorphic functions f_2 and f_3 that vanish at 0. Apply Proposition 5.3 with $U=\mathbb{D}^2\times\mathbb{D}$ and $A=\mathcal{V}$. By (5.10), the projection so that $\pi_1:\mathcal{V}\to\mathbb{D}$ is proper. Thus we get that $\mathcal{V}$ is locally k-sheeted over $\mathbb{D}$, except over possibly finitely discrete set of points. But since $\mathcal{V}$ is smooth, and squeezed by (5.10), we must have $k=1$. Therefore f_2 and f_3 extend to be holomorphic from $\mathbb{D}$ to $\mathbb{D}$, and $\mathcal{V}$ is a retract. $\square$

6 $\mathcal{V}$ is two-dimensional

Theorem 6.1. *Let $\mathcal{V}$ be a polynomially convex 2-dimensional analytic variety that has the polynomial extension property in $\mathbb{D}^3$.*

Then, either $\mathcal{V}$ is a holomorphic retract or, for any permutation of the coordinates, $\mathcal{V}$ is of the form $\{(z_1, z_2, f(z_1, z_2)) : (z_1, z_2) \in D\}$, where $D \subset \mathbb{D}^2$ and $f \in \mathcal{O}(D)$.

Throughout this section let $\pi_{ij} : \mathbb{C}^3 \to \mathbb{C}^2$ denote the projection onto (z_i, z_j) variables.

The following lemma may be seen as the infinitesimal version of Theorem 2.3.

Lemma 6.2. *(1). Suppose that there there is a sequence $\{(t_n, \gamma_n t_n, \delta_n t_n)\}$ in $\mathcal{V}$ converging to 0 such that $\gamma_n \to \gamma_0 \in \mathbb{T}$ and $\delta_n \to \delta_0 \in \mathbb{T}$. Then $\{(\zeta, \gamma_0\zeta, \delta_0\zeta) : \zeta \in \mathbb{D}\} \subset \mathcal{V}$.*

(2). Suppose that there are two sequences $\{(t_n, t_n, \delta_n^j t_n)\}$ in $\mathcal{V}$ converging to 0 such that $\delta_n^j \to \delta_0^j$, $j = 1, 2$, and $\delta_0^1 \neq \delta_0^2$. Then $\{(\zeta, \zeta, \eta) : \zeta \in \mathbb{D}, \eta \in \mathbb{D}\} \subset \mathcal{V}$.

Proof. 1. It is enough to prove the lemma for $\gamma_0 = \delta_0 = 1$. Let $F(z) = (z_1 + z_2 + z_3)/3$. We then need to show that $\overline{F(\mathcal{V})}$ contains the unit circle.

Assume that the assertion is not true. Then we can find a triangle Γ in $\mathbb{D}$ with one vertex on $\mathbb{T}$ such that $F(\mathcal{V}) \subset D := \mathbb{D} \setminus \bar{\Gamma}$. Let $\Phi_D : D \to \mathbb{D}$ be a mapping fixing the origin such that $\Phi_D'(0) > 1$. Let $G : \mathbb{D}^3 \to \mathbb{D}$ be a holomorphic extension of $\Phi_D \circ F$. It is clear that $G(0) = 0$, so it follows from the Schwarz lemma that $|G_{z_1}'(0)| + |G_{z_2}'(0)| + |G_{z_3}'(0)| \leq 1$. Now dividing the equality $\Phi_D(F(t_n, \gamma_n t_n, \delta_n t_n)) = G(t_n, \gamma_n t_n, \delta_n t_n)$ by t_n and letting $n \to \infty$ we find that $\Phi_D'(0) = G_{z_1}'(0) + G_{z_2}'(0) + G_{z_3}'(0)$; a contradiction.

2. We proceed as in the previous part, with the exception that we take $F(z) = \alpha z_1 + \alpha z_2 + \beta z_3$, where α and β are any complex numbers satisfying $2|\alpha| + |\beta| = 1$. Again, what we need is to show that $\overline{F(\mathcal{V})}$ contains the unit circle.

Suppose that the assertion is not true, *i.e.* $F(\mathcal{V}) \subset D := \mathbb{D} \setminus \Gamma$, where Γ is a triangle chosen analogously to before. With $t = \Phi_D'(0) > 1$ and G a norm-preserving extension of $\Phi_D \circ F$, we get for both $j = 1$ and $j = 2$

$$G_{z_1}'(0) + G_{z_2}'(0) + \delta_0^j G_{z_3}'(0) = t(F_{z_1}'(0) + F_{z_2}'(0) + \delta_0^j F_{z_3}'(0)).$$

From this system of equations we get that $G_{z_1}'(0) + G_{z_2}'(0) = 2t\alpha$ and $G_{z_3}'(0) = t\beta$. This again contradicts the Schwarz lemma. □

Lemma 6.3. *Let us suppose that* 0 *is a regular point of* $\mathcal{V}$, $\dim_0 \mathcal{V} = 2$*, and that, in a neighborhood of* 0*,* $\mathcal{V}$ *is given by*

$$\{(z_1, z_2, f(z_1, z_2))\},$$

where f *is a germ of an analytic function at* 0*. Let* $\alpha_j = f'_{z_j}(0)$*.*

If $\omega \in \mathbb{T}$ *is such that* $|\alpha_1 + \omega\alpha_2| \leq 1$*, then* $\mathcal{V}$ *is single sheeted over* $\{z_2 = \omega z_1\}$*.*

Proof. Step 1: In the first step we shall show that for any $\omega \in \mathbb{T}$ satisfying the assertion there is a holomorphic function $\varphi : \mathbb{D} \to \mathbb{D}$, $\varphi(0) = 0$, such that an analytic disc $\zeta \mapsto (\zeta, \omega\zeta, \varphi(\zeta))$ is contained in $\mathcal{V}$.

Let us suppose that ω is such that $|\alpha_1 + \omega\alpha_2| = 1$. Then it follows from Lemma 6.2 that $\{(\zeta, \omega\zeta, (\alpha_1 + \omega\alpha_2)\zeta) : \zeta \in \mathbb{D}\} \subset \mathcal{V}$.

So we are left with the case $|\alpha_1 + \omega\alpha_2| < 1$. Note that $\mathcal{V} \cap \{z_2 = \omega z_1\}$ is a one-dimensional analytic variety. Let $\mathcal{W}_0$ be its irreducible component containing 0. Note that any point $z \in \mathcal{W}_0 \setminus \{0\}$ sufficiently close to 0 satisfies $|z_3| < |z_1|$. Therefore two possibilities need to be considered: either $\mathcal{W}_0 \setminus \{0\}$ intersects $\{(z_1, \omega z_1, z_3) : |z_1| = |z_3|\}$, or $\mathcal{W}_0 \setminus \{0\}$ is contained in $\{(z_1, \omega z_1, z_3) : |z_3| < |z_1|\}$. In the case of the first possibility, 0 and a point of the intersection form a 3-balanced pair, whence the disc $\zeta \mapsto (\zeta, \omega\zeta, \eta\zeta)$ is contained in $\mathcal{V}$ for some unimodular η, which contradicts the local description of $\mathcal{V}$ near 0 (precisely, the assumption that $|\alpha_1 + \omega\alpha_2| < 1$).

Assume that the second possibility holds. Then the projection $\pi_1 : (z_1, z_2, z_3) \mapsto z_1$ restricted to the variety $\mathcal{W}_0$ is proper. Consequently, $\pi_1(\mathcal{W}_0)$ is a one-dimensional variety in $\mathbb{D}$, whence $\pi_1(\mathcal{W}_0) = \mathbb{D}$, by Proposition 5.2. Therefore it suffices to show that $\mathcal{W}_0$ is single sheeted over $\{(\zeta, \omega\zeta) : \zeta \in \mathbb{D}\}$. Actually, once we get it we shall be able to express $\mathcal{W}_0$ as $\{(\zeta, \omega\zeta, \varphi(\zeta) : \zeta \in \mathbb{D}\}$, where the function φ is, in particular, bounded, and thus holomorphic, according to Proposition 5.5.

To prove that $\mathcal{W}_0$ is single sheeted take $(z_1, \omega z_1, z_3)$ and $(z_1, \omega z_1, w_3)$ in $\mathcal{W}_0$. Recall that $|z_3|, |w_3| < |z_1|$. Let us compose $\mathcal{W}_0$ and points with an idempotent automorphism ϕ of $\mathbb{D}^3$ interchanging 0 and $(z_1, \omega z_1, z_3)$. Then $0 \in \phi(\mathcal{W}_0)$, $z = (z_1, \omega z_1, z_3) \in \phi(\mathcal{W}_0)$ and $y = (0, 0, m_{z_3}(w_3)) \in \phi(\mathcal{W}_0)$. If $\phi(\mathcal{W}_0) \setminus \{0\}$ intersects $\{(z_1, \omega z_1, z_3) : |z_3| = |z_1|\}$ we can find a 3-balanced pair in $\phi(\mathcal{W}_0)$, which implies that the disc $\zeta \mapsto (\zeta, \omega\zeta, m(\zeta))$ is in $\mathcal{W}_0$ for some Möbius map m. Otherwise we can find in $\phi(\mathcal{W}_0)$ two sequences $(x_n, \omega x_n, a_n x_n)$ and $(y_n, \omega y_n, b_n y_n)$ converging to 0 such that $|a_n| < 1$ and

$|b_n| > 1$. Using Lemma 6.2 ((1) in the case when (a_n) and (b_n) converge to the same (unimodular) constant, or (2) if it is not the case), we get

$$\mathcal{W}_0 \supseteq \{(m_{z_1}(\zeta), m_{\omega z_1}(\omega\zeta), m_{z_3}(\delta\zeta)) : \zeta \in \mathbb{D}\}$$

for some $\delta \in \overline{\mathbb{D}}$ (one can define δ to be an accumulation point of (a_n)), which means that

$$\mathcal{W}_0 \supseteq \{(\zeta, \omega\zeta, m_{z_3}(\delta m_{z_1}(\zeta))) : \zeta \in \mathbb{D}\},$$

as claimed. Since $\mathcal{W}_0 \setminus \{0\}$ was assumed to be a subset of $\{|z_3| < |z_1|\}$ we get a contradiction (as $m : \zeta \mapsto m_{z_3}(\delta m_{z_1}(\zeta))$ is a Möbius map).

Step 2: Now we shall prove the assertion, that is we shall show that $\mathcal{V}$ is single-sheeted over $\{z_2 = \omega z_1\}$. Seeking a contradiction suppose that $(z_1, \omega z_1, z_3) \in \mathcal{V}$ and $\varphi(z_1) \neq z_3$, where φ is an analytic disc contructed in Step 1. Take λ such that $\rho(\lambda, z_1) = \rho(\varphi(\lambda), z_3)$. To justify that such a λ exists note that it is trivial if φ is not proper (consider the values above for $\lambda = z_1$ and properly chosen λ close to the unit circle). On the other hand if φ is a proper self map of the unit disc, then it is a Blaschke product, so $\varphi^{-1}(z_3)$ is non-empty. Then, considering the values for $\lambda = z_1$ and λ' that is picked from $\varphi^{-1}(z_3)$ the desired existence of λ follows. Then, of course, $(\lambda, \omega\lambda, \varphi(\lambda))$ and $(z_1, \omega z_1, z_3)$ form a 3-balanced pair, which means that there is a geodesic $\zeta \mapsto (\zeta, \omega\zeta, m(\zeta))$, where m is a Möbius map, contained in $\mathcal{V}$ and intersecting $\zeta \mapsto (\zeta, \omega\zeta, \varphi(\zeta))$ exactly at one point. Thus Lemma 6.2, applied at the point of intersection, implies that $\{(\zeta, \omega\zeta, \eta) : \zeta \in \mathbb{D}, \eta \in \mathbb{D}\}$ is contained in $\mathcal{V}$, which gives a contradiction with the local description of $\mathcal{V}$ near 0. □

Corollary 6.4. *Keeping the assumptions and notation from Lemma 6.3: if* $|\alpha_1| + |\alpha_2| \leq 1$*, then* $\mathcal{V}$ *is an analytic retract.*

Proof. It follows from Lemma 6.3 that f extends to $\Delta := \{(z_1, z_2) \in \mathbb{D}^2 : |z_1| = |z_2|\}$ and any $x \in \Delta \times \mathbb{D}$ is an isolated point of $\pi_{12}^{-1}(\pi_{12}(x)) \cap \mathcal{V}$. Applying Proposition 5.4 we find that $\pi_{12}|_{\mathcal{V}}$ is proper when restricted to a small neighborhood of x. Thus, by Proposition 5.3, $\pi_{12}|_{\mathcal{V}}$ is a local k-sheeted covering near x. Since any analytic set containing Δ is two dimensional, we get that $k = 1$. Consequently, f extends holomorphically to a neighborhood of Δ, due to Proposition 5.5.

Take a Reinhardt domain $R_\Delta \subset \mathbb{D}^2$ that contains Δ such that $f \in \mathcal{O}(R_\Delta)$. The envelope of holomorphy of R_Δ, denoted $\widehat{R_\Delta}$, is a Reinhardt domain, as well. Since the intersections of R_Δ with both axis are non-empty, we infer that the envelope is complete, meaning $(\lambda_1 z_1, \lambda_2 z_2) \in \widehat{R_\Delta}$ for any $\lambda_1, \lambda_2 \in \mathbb{D}$, and $(z_1, z_2) \in \widehat{R_\Delta}$. Consequently, $\widehat{R_\Delta} = \mathbb{D}^2$, as $\Delta \subset R_\Delta$. Therefore, f extends holomorphically to the whole bidisc.

Since the distinguished boundary of $r\mathbb{D}^2$ (equal to $r\mathbb{T}^2$), $r < 0 < 1$, is contained in Δ we get that $|f| < 1$ on $r\mathbb{D}^2$ for any $0 < r < 1$. Consequently, f lies in the open unit ball of $H^\infty(\mathbb{D}^2)$. □

Corollary 6.5. *Let us assume that* 0 *is a regular point* $\mathcal{V}$ *and that its germ near* 0 *is of the form*

$$\{(z_1, z_2, f(z_1, z_2))\}. \tag{6.1}$$

Let us denote $\alpha_j := f'_{z_j}(0)$. *Then* $\mathcal{V}$ *is an analytic retract if one of the possibilities holds:*

- $|\alpha_1| + |\alpha_2| \leq 1$,
- $|\alpha_1| \geq 1 + |\alpha_2|$,
- $|\alpha_2| \geq 1 + |\alpha_1|$.

If $\mathcal{V}$ *is not an analytic retract, then the set of* 2*-dimensional regular points of* $\mathcal{V}$ *is single sheeted in each direction.*

Moreover, for any $x \in \mathcal{V}_{reg}$ *there are two pairs of unimodular constants* (ω_i, η_i) *such that the analytic disc* $\{(\zeta, \omega_i\zeta, \eta_i\zeta) : \zeta \in \mathbb{D}\}$ *lie in* $\phi(\mathcal{V})$, *where* $\phi = (m_{x_1}, m_{x_2}, m_{x_3})$ *is an indempotent automorphism switching* 0 *and* x.

Proof. The first case is covered by Corollary 6.4, while the other two are obtained simply by permuting the coordinates.

To prove the second part, when $\mathcal{V}$ is not an analytic retract, choose a point x in $\mathcal{V}_{reg}$ such that $\dim_x \mathcal{V} = 2$. We want to show that $\pi_{ij}^{-1}(\pi_{ij}(x)) \cap \mathcal{V} = x$ for any choice of coordinates (z_i, z_j). We can make two simple reductions: composing with an automorphism of the tridisc we can assume that $x = 0$, and we can focus only on the coordinates (z_1, z_2).

Since 0 is a regular point we can express $\mathcal{V}$ as in (6.1). Since $\mathcal{V}$ is not a holomorphic retract none of the inequalities listed in the statement of the corollary is satisfied. This, in particular, means that there are two unimodular constants ω_i such that $|\alpha_1 + \omega_i \alpha_2| = 1$. It follows from Lemma 6.3 that $\mathcal{V}$ is single sheeted over the $\{z_2 = \omega_i z_1\}$, whence over 0, as well.

The proof of Step 1 in Lemma 6.3 shows that $\eta_i := \alpha_1 + \omega_i \alpha_2$, $i = 1, 2$, satisfy the last assertion of the corollary. □

Lemma 6.6. *Suppose that $\mathcal{V}$ is not an analytic retract. Let $\mathcal{W}$ be its 2-dimensional connected component (which means that $\mathcal{W}$ is a connected component of $\mathcal{V}$ and $\dim_x \mathcal{W} = 2$ for some $x \in \mathcal{W}$). Then*

$$\mathcal{W} = \{(z_1, z_2, f(z_1, z_2)) : (z_1, z_2) \in D\},$$

where D is an open subset of $\mathbb{D}^2$ and $f \in \mathcal{O}(D, \mathbb{D})$.

This can be done over each pair of coordinate functions.

Proof. Let $\mathcal{W}_0$ be a strictly 2-dimensional component of $\mathcal{V}$ (*i.e.* the union of its two dimensional irreducible components in $\mathcal{W}$). We shall show that $(\mathcal{W}_0)_{sing}$ is empty.

Proceeding by contradiction, take a point $a \in (\mathcal{W}_0)_{sing}$. Suppose first that $\dim_a(\mathcal{W}_0 \cap (\{(a_1, a_2)\} \times \mathbb{D})) = 0$, which means that $\pi_{12}^{-1}(a_1, a_2) \cap \mathcal{W}_0 \cap U = \{a\}$ for some neighborhood U of a. According to Proposition 5.4 the projection $\pi_{12}|_A$ is proper in a neighborhood of a. Since $\mathcal{V}_{reg}$ is single sheeted by Corollary 6.5, we see that π_{12} is a single sheeted covering near a (it is a covering according to Proposition 5.3). The fact that the covering is single-sheeted immediately implies that $\mathcal{W}_0$ is smooth there – a contradiction.

Permuting coordinates, we trivially get from the above reasoning the following statement: $\dim_b(\mathcal{W}_0 \cap (\mathbb{D} \times \{(b_2, b_3)\})) = 0$, where $b \in \mathcal{W}_0$, implies that b is a regular point of $\mathcal{W}_0$.

So we need to show that $\dim_a(\mathcal{W}_0 \cap (\{(a_1, a_2)\} \times \mathbb{D})) = 0$. If it were not true, *i.e.* $\dim_a(\mathcal{W}_0 \cap (\{(a_1, a_2)\} \times \mathbb{D})) = 1$, we would be able to find a disc Δ centered at a_3 such that $\{(a_1, a_2)\} \times \Delta \subset \mathcal{W}_0$, whence $\{(a_1, a_2)\} \times \mathbb{D} \subset \mathcal{W}_0$. Since $\mathcal{V}$ is single sheeted over its regular points we find that $\{(a_1, a_2)\} \times \mathbb{D} \subset \mathcal{V}_{sing}$, and thus $\{(a_1, a_2)\} \times \mathbb{D} \subset (\mathcal{W}_0)_{sing}$.

Here we can again permute coordinates in the preceding argument — note that we are able to do it because $\dim_{(a_1,a_2,x)}(\mathcal{W}_0 \cap (\mathbb{D} \times \{(a_2, x)\})) = 1$. In this way we find that $\mathbb{D} \times \{(a_2, x)\} \subset (\mathcal{W}_0)_{sing}$ for any $x \in \mathbb{D}$. Consequently, $(\mathcal{W}_0)_{sing}$ is 2-dimensional, which is impossible.

Thus we have shown that $\mathcal{W}_0$ is smooth, so it is locally a graph. According to Corollary 6.5 for any $x \in \mathcal{W}_0$ that is a regular point of $\mathcal{V}$, the variety $\mathcal{W}_0$ is in a neighborhood of x, a graph over each pair of coordinate functions. In particular, none of the inequalities involving derivatives from that corollary (understood after an automorphism) is satisfied at x_0, and by the continuity none is satisfied at points $x \in \mathcal{W}_0 \cap \mathcal{V}_{sing}$ (if there are any), as well. Therefore $\mathcal{W}_0$ is a graph over every choice of the coordinate functions for any $x \in \mathcal{W}_0$.

To prove that $\mathcal{W}_0 = \mathcal{W}$ we proceed by contradiction. Assume that there is $x \in \mathcal{W}_0$ that lies in the analytic set $\mathcal{W}'$ composed of 1 dimensional irreducible components of $\mathcal{W}$. Then x is an isolated point of $\mathcal{W}' \cap \mathcal{W}_0$. Let us take $a \in \mathcal{W}'$ that is sufficiently close to x. Changing coordinates we can suppose that $(a_1, a_2) \neq (x_1, x_2)$. Then $\mathcal{V}$ is smooth at the point of the intersection of $\mathcal{W}$ and $\pi_{12}^{-1}(\pi_{12}(a))$ (note that $\pi_{12}(\mathcal{W}_0)$ is open), and thus it is single-sheeted over $\pi_{12}(a)$; a contradiction. □

Proof of Theorem 6.1. Suppose that $\mathcal{V}$ is not an analytic retract. Let $x \in \mathcal{V}$ be such that $\dim_x \mathcal{V} = 2$. Then, it follows from Lemma 6.6 that the connected component $\mathcal{W}$ containing x is one of the forms listed in (1.8). Therefore, to prove the assertion we need to show that $\mathcal{V}$ is connected.

Choose $x \in \mathcal{V} \setminus \mathcal{W}$. We can make a few helpful assumptions. First of all, according to Corollary 6.5, it can be assumed that an analytic disc $\{(\lambda, \lambda, \lambda) : \lambda \in \mathbb{D}\}$ lies entirely in $\mathcal{W}$. Changing, if necessary, the coordinates we can also assume that there is a point λ_0 such that

$$\rho(\lambda_0, x_1) = \rho(\lambda_0, x_2) \geq \rho(\lambda_0, x_3).$$

Then we can compose $\mathcal{V}$ with the automorphism Φ of the tridisc that interchanges 0 and $(\lambda_0, \lambda_0, \lambda_0)$ to additionally get that $|x_1| = |x_2| \geq |x_3|$. Let $\omega \in \mathbb{T}$ be such that $x_2 = \omega x_1$. Since, by Corollary 6.5, $\mathcal{V}$ is single sheeted over each point of $\pi_{12}(\mathcal{W})$, we are done, provided that $(x_1, \omega x_1) \in \pi_{12}(\mathcal{W})$. Suppose that it is not true.

Let us consider two values $\rho(\lambda, x_1) = \rho(\omega\lambda, x_2)$ and $\rho(f(\lambda, \omega\lambda), x_3)$. If λ moves from 0 in the direction x_3, then near the first point $\lambda' \in \mathbb{D}$ such that $(\lambda', \omega\lambda') \notin \pi_{12}(\mathcal{W})$ the last value tends to 1. Since the second value is smaller for $\lambda = 0$, we find that there is some a such that $(a, \omega a) \in \pi_{12}(\mathcal{W})$ and the two points $(a, \omega a, f(a, \omega a))$ and x form a 3-balanced pair. In particular, they can be connected with a 3-geodesic that entirely lies in $\mathcal{V}$; a contradiction. □

7 Further properties and examples

Let $\mathcal{V}$ be a relatively polynomially convex set in $\mathbb{D}^3$ that has the extension property and is not a retract. So far two crucial properties have been derived in the preceding section:

a) for each choice of the coordinate functions $\mathcal{V}$ is a graph of a holomorphic function;

b) for any $x \in \mathcal{V}$ there exist two pairs of unimodular constants (ω_i, η_i), $i = 1, 2$, such that $\{(\zeta, \omega_i\zeta, \eta_i\zeta) : \zeta \in \mathbb{D}\}$ lies entirely in $\Phi(\mathcal{V})$, $i = 1, 2$, where Φ is an idempotent automorphism of $\mathbb{D}^3$ interchanging 0 and x.

Example 7.1 Observe that

$$\mathcal{V}_0 := \{z \in \mathbb{D}^3 : \; z_3 = z_1 + z_2\}$$

satisfies a). We shall show that $\mathcal{V}_0$ does not have the extension property.

Let $U := \{(z_1, z_2) \in \mathbb{D}^2 : \; |z_1 + z_2| < 1\}$. Let m_a denote the Möbius map $m_a(\zeta) = \frac{a-\zeta}{1-\bar{a}\zeta}$, where $a \in \mathbb{D}$.

Let us put $h(z_1, z_2) := z_1 + z_2$ and observe that there are points ζ and ξ in the unit disc such that $(\zeta, \zeta m_a(\zeta))$, $(\xi, \xi m_a(\xi))$ lie in U and

$$\rho\left(\frac{h(\zeta, \zeta m_a(\zeta))}{\zeta}, \frac{h(\xi, \xi m_a(\xi))}{\xi}\right) < \rho(\zeta, \xi). \tag{7.1}$$

Indeed, it suffices to take ζ and ξ sufficiently close to $m_a(-1)$ such that $\rho(\zeta, \xi)$ is big enough.

Note that (7.1) implies that there is $\psi_a \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ such that both points $(\zeta, \zeta m_a(\zeta), \zeta\psi_a(\zeta)), (\xi, \xi m_a(\xi), \xi\psi_a(\xi))$ lie in $\mathcal{V}_0$.

Let us consider a 3-point Pick interpolation problem

$$\begin{cases} 0 \mapsto 0, \\ (\zeta, \zeta m_a(\zeta), \zeta\psi_a(\zeta)) \mapsto \zeta m_a(\zeta), \\ (\xi, \xi m_a(\xi), \xi\psi_a(\xi)) \mapsto \xi m_a(\xi). \end{cases}$$

Note that one solution to the above problem is the function

$$F(z_1, z_2, z_3) = (z_1 m_a(z_1) + z_2)/2.$$

Observe that the problem is also extremal (see [15]). Indeed, otherwise we would be able to find a holomorphic function G on the tridisc, with the range relatively compact in $\mathbb{D}$, such that

$$G(x, x m_a(x), x\psi_a(x)) = x m_a(x) \tag{7.2}$$

for $x = 0, \zeta, \xi$. This says that on the disk $\{(x, x m_a(x), x\psi_a(x)) : x \in \mathbb{D}\}$ the degree 2 Blaschke product $x m_a(x)$ solves a three point Pick interpolation problem. By [2, p.77] an n point Pick interpolation problem on the disk has a unique solution if and only if it has a solution that is a Blaschke product of degree strictly less than n. Therefore we find that (7.2) holds for any x, contradicting the fact that that $G(\mathbb{D}^3) \subset\subset \mathbb{D}$.

Consequently, F interpolates extremally, whence $\mathbb{T} \subset F(\overline{\mathcal{V}})$, by Theorem 2.3. Thus there is a point $z \in \overline{\mathcal{V}}$ such that $F(z) = 1$, which means that $z_1 m_a(z_1) = 1$ and $z_2 = 1$. Note that $(1,1) \notin \overline{U}$. Now we easily get a contradiction, as the Möbius map m_a satisfies the following property: any solution of the equation $x m_a(x) = 1$, $x \in \mathbb{D}$, is close to 1 as a approaches 1.

Remark 7.2 The argument from this example can be applied to the algebraic case. To be more precise suppose that $\mathcal{V}$ is an algebraic set with the extension property that is not a retract. Write $h := h_3$ and $U = U_3$, where h_3, U_3 are as in Theorem 1.7 and $h(0,0) = 0$. We shall also write (x, y, z) for the coordinates (z_1, z_2, z_3). Note that $|h|$ extends continuously to $\overline{U}$.

Repeating the idea from the example we can show that: if $(-1, 1) \in \overline{U}$ is such that $|h(-1, 1)| < 1$, then $(1, 1) \in \overline{U}$. Using transitivity of the group of automorphisms of the polydisc we can show slightly more. Choose $\omega, \eta \in \mathbb{T}$ such that $(\zeta, \omega\zeta, \eta\zeta) \in \mathcal{V}$ for $\zeta \in \mathbb{D}$. Put $\Psi_a(x, y) = (\varphi_a(z), \varphi_{\omega a}(y))$, $x, y \in \mathbb{D}^2$, $a \in \mathbb{D}$. Let $\tilde{h} = \varphi_{\eta a} \circ h \circ \Psi_a$ and $\tilde{\mathcal{V}} = \{(x, y) \in \Psi_a(U) : z = \tilde{h}(x, y)\}$. Note that $|\tilde{h}(\varphi_a(-1), \varphi_{\omega a}(1))| < 1$, so $(-\varphi_a(-1), \varphi_{\omega a}(1)) \in \overline{\Psi_a(U)}$. Consequently, $(\varphi_a(-\varphi_a(1)), 1) \in \overline{U}$ for any $a \in \mathbb{D}$. Thus we have shown that $(\omega, 1) \in \overline{U}$ for any $\omega \in \mathbb{T}$.

Remark 7.3 If we apply the previous remark to the case when h is rational, we get that either U is the whole bidisc or $h(\mathbb{T}^2) \subset \mathbb{T}$ (whenever it makes sense). The simplest class of such functions contains among others

$$h : (z_1, z_2) \mapsto \omega \frac{A z_1 + B z_2 + z_1 z_2}{1 + \bar{B} z_1 + \bar{A} z_2},$$

where $\omega \in \mathbb{T}$ and A, B are complex numbers. Observe that if $|A| + |B| \leq 1$, then h is defined on the whole bidisc. Thus we are interested in the question whether for complex numbers A, B such that $|A| + |B| > 1$ and $|A|, |B| \leq 1$ the surface

$$\mathcal{V} := \left\{ (x, y, z) \in \mathbb{D}^3 : \ z = \omega \frac{Ax + By + xy}{1 + \bar{B}x + \bar{A}y} \right\}, \tag{7.3}$$

$\omega \in \mathbb{T}$, has the extension property. Note that $\mathcal{V}$ defined above satisfies properties a) and b).

Remark 7.4 It is interesting that (7.3) appears naturally in another way. Namely, it is the *uniqueness variety* for a three-point Pick interpolation problem in the tridisc.

To explain it, take α, β, γ in the unit disc that are not co-linear and let δ be a strict convex combination of these points. For fixed $x, y \in \mathbb{D}$, $x \neq 0$, $y \neq 0$, $x \neq y$, let us consider the following problem:

$$\mathbb{D}^3 \to \mathbb{D}, \quad \begin{cases} 0 \mapsto 0, \\ (x\varphi_\alpha(x), x\varphi_\beta(x), x\varphi_\gamma(x)) \mapsto x\varphi_\delta(x), \\ (y\varphi_\alpha(y), y\varphi_\beta(y), y\varphi_\gamma(y)) \mapsto y\varphi_\delta(y). \end{cases}$$

It is an extremal three point Pick interpolation problem. Moreover, it was shown in [15] that the problem is never uniquely solvable, but there is a set on which all solutions do coincide, namely all interpolating functions are equal on the real surface $\{(\zeta\varphi_{t\alpha}(\zeta), \zeta\varphi_{t\beta}(\zeta), \zeta\varphi_{t\gamma}(\zeta)) : \ \zeta \in \mathbb{D}, t \in (0,1)\}$. This in particular means that the uniqueness variety contains points

$$\left(\frac{\alpha t\zeta - \zeta^2}{1 - \bar{\alpha} t\zeta}, \frac{\beta t\zeta - \zeta^2}{1 - \bar{\beta} t\zeta}, \frac{\gamma t\zeta - \zeta^2}{1 - \bar{\gamma} t\zeta} \right), \tag{7.4}$$

where t and ζ run through an open subset of $\mathbb{C}^2$ (containing $(0,1) \times \mathbb{D}$). Some computations, partially carried out in [15], show that the set composed of points (7.4) coincides with the variety (7.3) with properly chosen ω, A and B.

8 Von Neumann Sets and Spectral Theory

There is a connection between the extension property and the theory of spectral sets for d-tuples of operators. Let $T = (T_1, \ldots, T_d)$ be a d-tuple of commuting operators on some Hilbert space $\mathcal{H}$. We shall call T an *Andô d-tuple* if

$$\|p(T)\| \ \leq \ \sup\{|p(z)| : z \in \mathbb{D}^d\} \quad \forall \ p \in \mathbb{C}[z_1, \ldots, z_d].$$

Let $\mathcal{V}$ be a holomorphic subvariety of $\mathbb{D}^d$. We shall say that a commuting d-tuple T is *subordinate to* $\mathcal{V}$ if $\sigma(T) \subset \mathcal{V}$ and, whenever g is holomorphic on a neighborhood of $\mathcal{V}$ and satisfies $g|\mathcal{V} = 0$, then $g(T) = 0$. If f is any holomorphic function on $\mathcal{V}$, then by Cartan's theorem f can be extended to a function g that is holomorphic not just on a neighborhood of $\mathcal{V}$ but on all of $\mathbb{D}^d$, and if T is subordinate to $\mathcal{V}$ then $f(T)$ can be defined unambiguously as equal to $g(T)$.

Let $\mathcal{A} \subseteq H^\infty(\mathcal{V})$ be an algebra, and assume T is subordinate to $\mathcal{V}$. We shall say that $\mathcal{V}$ is an $\mathcal{A}$-spectral set for T if

$$\|f(T)\| \ \leq \ \sup\{|f(z)| : z \in \mathcal{V}\} \quad \forall\, f \in \mathcal{A}.$$

Definition 8.1. *If $\mathcal{V}$ is a holomorphic subvariety of $\mathbb{D}^d$, and $\mathcal{A} \subseteq H^\infty(\mathcal{V})$, we say $\mathcal{V}$ has the $\mathcal{A}$ von Neumann property if, whenever T is an Andô d-tuple that is subordinate to $\mathcal{V}$, then $\mathcal{V}$ is an $\mathcal{A}$-spectral set for T.*

The von Neumann property is closely related to the extension property. The following theorem was proved for the bidisk in [3].

Theorem 8.2. *Let $\mathcal{V}$ be a holomorphic subvariety of $\mathbb{D}^d$, and $\mathcal{A}$ a subalgebra of $H^\infty(\mathcal{V})$. Then $\mathcal{V}$ has the $\mathcal{A}$ von Neumann property if and only if it has the $\mathcal{A}$ extension property.*

Proof: One direction is easy. Suppose $\mathcal{V}$ has the $\mathcal{A}$ von Neumann property, and T is an Andô d-tuple that is subordinate to $\mathcal{V}$. Let $f \in \mathcal{A}$. By the extension property, there is a function $g \in H^\infty(\mathbb{D}^d)$ that extends f and has the same norm, and $f(T) = g(T)$. Since $\sigma(T) \subseteq \mathbb{D}^d$, we can approximate g uniformly on a neighborhood of $\sigma(T)$ by polynomials p_n with $\|p_n\|_{H^\infty(\mathbb{D}^d)} \leq \|g\|_{H^\infty(\mathbb{D}^d)}$. Therefore

$$\|f(T)\| = \lim \|p_n(T)\| \leq \|g\|_{H^\infty(\mathbb{D}^d)} = \|f\|_{\mathcal{V}}.$$

To prove the other direction, let Λ be a finite set in $\mathbb{D}^d$, with say n elements $\{\lambda_1, \ldots, \lambda_n\}$. Let $\mathcal{K}_\Lambda$ denote the set of n-by-n positive definite matrices K that have 1's down the diagonal, and satisfy

$$[(1 - w_i \bar{w}_j) K_{ij}] \ \geq \ 0$$

whenever there is a function ϕ in the closed unit ball of $H^\infty(\mathbb{D}^d)$ that has $\phi(\lambda_i) = w_i$. We shall need the following result, which was originally proved by E. Amar [4]; see also [19], [7], [22], [2, Thm. 13.36].

Theorem 8.3. *Let $\Lambda = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{D}^d$ and $\{w_1, \ldots, w_n\} \subset \mathbb{C}$. There exists a function ϕ in the closed unit ball of $H^\infty(\mathbb{D}^d)$ that maps each λ_i to the corresponding w_i if and only if*

$$[(1 - w_i \bar{w}_j) K_{ij}] \ \geq \ 0 \quad \forall\, K \in \mathcal{K}_\Lambda.$$

Suppose $\mathcal{V}$ has the $\mathcal{A}$ von Neumann property but not the $\mathcal{A}$ extension property. Then there is some $f \in \mathcal{A}$ with $\|f\|_{\mathcal{V}} = 1$ but no extension of norm 1 to $\mathbb{D}^d$. There must be a finite set Λ and a number $M > 1$ so that every function ϕ in $H^\infty(\mathbb{D}^d)$ that agrees with f on Λ has $\|\phi\| \geq M$. (Otherwise by normal families one would get an extension of f of norm one).

Let $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$, and let $w_i := f(\lambda_i)$ for each i. By Theorem 8.3, there exists some $K \in \mathcal{K}_\Lambda$ so that

$$[(1 - w_i \bar{w}_j) K_{ij}] \quad \text{is not positive semidefinite.} \tag{8.4}$$

Choose unit vectors v_i in $\mathbb{C}^n$ so that

$$\langle v_i, v_j \rangle \ = \ K_{ij}.$$

(This can be done since K is positive definite). For each point λ_i, let its coordinates be given by $\lambda_i = (\lambda_i^1, \ldots, \lambda_i^d)$. Define d commuting matrices T on $\mathbb{C}^n$ by

$$T_j v_i \ = \ \lambda_i^j v_j$$

Then T is an Andô d-tuple, because if p is a polynomial of norm 1 on $\mathbb{D}^d$, then

$$p(T) : v_i \mapsto p(\lambda) v_i,$$

so

$$\begin{aligned} \langle (1 - p(T)^* p(T)) \sum c_i v_i, \sum c_j v_j \rangle \quad &= \quad \sum c_i \bar{c}_j (1 - p(\lambda_i) \overline{p(\lambda_j)} \langle v_i, v_j \rangle \\ &= \quad \sum c_i \bar{c}_j (1 - p(\lambda_i) \overline{p(\lambda_j)} K_{ij}. \end{aligned}$$

This last quantity is positive since $K \in \mathcal{K}_\Lambda$, so $p(T)$ is a contraction, as claimed. But since $\mathcal{V}$ is assumed to have the $\mathcal{A}$ von Neumann property, this means that $f(T)$ is also a contraction, so $I - f(T)^* f(T) \geq 0$. But

$$\langle (1 - f(T)^* f(T)) \sum c_i v_i, \sum c_j v_j \rangle \ = \ \sum c_i \bar{c}_j (1 - w_i \overline{w_j}) K_{ij},$$

and if this is non-negative for every choice of c_i we contradict (8.4).

□

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