Global holomorphic functions in several non-commuting variables II

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Abstract: We give a new proof that bounded non-commutative functions on polynomial polyhedra can be represented by a realization formula, a generalization of the transfer function realization formula for bounded analytic functions on the unit disk.

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1 Introduction

Let $M_n$ denote the $n$-by-$n$ matrices with complex entries, and let $M^d = \bigcup_{n=1}^{\infty} M_n^d$ be the set of all $d$-tuples of matrices of the same size. An nc-function\(^1\) on a set $E \subseteq M^d$ is a function $\phi : E \to M^1$ that satisfies

(i) $\phi$ is graded, which means that if $x \in E \cap M_n^d$, then $\phi(x) \in M_n$.

(ii) $\phi$ is intertwining preserving, which means if $x, y \in E$ and $S$ is a linear operator satisfying $Sx = yS$, then $S\phi(x) = \phi(y)S$.

The points $x$ and $y$ are $d$-tuples, so we write $x = (x^1, \ldots, x^d)$ and $y = (y^1, \ldots, y^d)$. By $Sx = yS$ we mean that $Sx^r = y^r S$ for each $1 \leq r \leq d$. See [9] for a general reference to nc-functions.

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\(^1\)nc is short for non-commutative
The principal result of [2] was a realization formula for nc-functions that are bounded on polynomial polyhedra; the object of this note is to give a simpler proof of this formula, Theorem 1.5 below.

Let \( \delta \) be an \( I \times J \) matrix whose entries are non-commutative polynomials in \( d \)-variables. If \( x \in \mathbb{M}_d^n \), then \( \delta(x) \) can be naturally thought of as an element of \( \mathcal{B}(\mathbb{C}^I \otimes \mathbb{C}^n, \mathbb{C}^J \otimes \mathbb{C}^n) \), where \( \mathcal{B} \) denotes the bounded linear operators, and all norms we use are operator norms on the appropriate spaces. We define

\[
B_\delta := \{ x \in \mathbb{M}^d : \| \delta(x) \| < 1 \}.
\] (1.1)

Any set of the form (1.1) is called a polynomial polyhedron. Let \( H^\infty(B_\delta) \) denote the nc-functions on \( B_\delta \) that are bounded, and \( H^1_\infty(B_\delta) \) denote the closed unit ball, those nc-functions that are bounded by 1 for every \( x \in B_\delta \).

**Definition 1.2.** A free realization for \( \phi \) consists of an auxiliary Hilbert space \( \mathcal{M} \) and an isometry

\[
\begin{pmatrix}
\mathbb{C} & \mathcal{M} \otimes \mathbb{C}^J \\
\mathcal{M} \otimes \mathbb{C}^J & \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\end{pmatrix}
\] (1.3)

such that, for all \( x \in B_\delta \), we have

\[
\phi(x) = \begin{pmatrix} A & 1 \otimes 1 \\ 1 \otimes \delta(x) & -D \otimes 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \otimes \delta(x) \\ -D \otimes 1 & 1 \otimes 1 \end{pmatrix}^{-1} \begin{pmatrix} C \\ 1 \otimes 1 \end{pmatrix}.
\] (1.4)

The 1’s need to be interpreted appropriately. If \( x \in \mathbb{M}_d^n \), then (1.4) means

\[
\phi(x) = \begin{pmatrix} A \\ id_{\mathbb{C}^n} \end{pmatrix} + \begin{pmatrix} B \\ id_{\mathbb{C}^n} \end{pmatrix} \otimes \delta(x) \begin{pmatrix} id_{\mathcal{M}} & D & id_{\mathcal{M}} \\ id_{\mathcal{M}} & 0 & id_{\mathcal{M}} \\ 0 & id_{\mathcal{M}} & \delta(x) \end{pmatrix}^{-1} \begin{pmatrix} C \\ 0 \\ 0 \end{pmatrix}.
\]

We adopt the convention of [11] and write tensors vertically to enhance legibility. The bottom-most entry corresponds to the space on which \( x \) originally acts; the top corresponds to the intrinsic part of the model on \( \mathcal{M} \).

The following theorem was proved in [2]; another proof appears in [6].

**Theorem 1.5.** The function \( \phi \) is in \( H^\infty_1(B_\delta) \) if and only if it has a free realization.

It is a straightforward calculation that any function of the form (1.4) is in \( H^\infty_1(B_\delta) \). We wish to prove the converse. We shall use two other results, Theorems 1.8 and 1.9 below.

If \( E \subset \mathbb{M}^d \), we let \( E_n \) denote \( E \cap \mathbb{M}_d^n \). If \( \mathcal{K} \) and \( \mathcal{L} \) are Hilbert spaces, a \( \mathcal{B}(\mathcal{K}, \mathcal{L}) \)-valued nc function on a set \( E \subseteq \mathbb{M}^d \) is a function \( \phi \) such that
(i) \( \phi \) is \( B(\mathcal{K}, \mathcal{L}) \) graded, which means if \( x \in E_n \), then \( \phi(x) \in B(\mathcal{K} \otimes \mathbb{C}^n, \mathcal{L} \otimes \mathbb{C}^n) \).

(ii) \( \phi \) is intertwining preserving, which means if \( x, y \in E \) and \( S \) is a linear operator satisfying \( Sx = yS \), then

\[
\frac{\text{id}_\mathcal{L}}{S} \phi(x) = \phi(y) \frac{\text{id}_\mathcal{L}}{S}.
\]

**Definition 1.6.** An nc-model for \( \phi \in H^\infty_1(B_\delta) \) consists of an auxiliary Hilbert space \( \mathcal{M} \) and a \( B(\mathcal{C}, \mathcal{M} \otimes \mathbb{C}^J) \)-valued nc-function \( u \) on \( B_\delta \) such that, for all pairs \( x, y \in B_\delta \) that are on the same level (i.e. both in \( B_\delta \cap M_n^d \) for some \( n \))

\[
1 - \phi(y)^* \phi(x) = u(y)^* \left[ \frac{1}{1 - \delta(y)^* \delta(x)} \right] u(x).
\]  

Again, the 1’s have to be interpreted appropriately. If \( x, y \in B_\delta \cap M_n^d \), then (1.7) means

\[
\text{id}_{\mathbb{C}^n} - \phi(y)^* \phi(x) = u(y)^* \left[ \frac{\text{id}_{\mathcal{M}}}{\text{id}_{\mathbb{C}^n} \otimes \mathbb{C}^n - \delta(y)^* \delta(x)} \right] u(x).
\]

**Theorem 1.8.** A graded function on \( B_\delta \) has an nc-model if and only if it has a free realization.

Theorem 1.8 was proved in [2], but a simpler proof is given by S. Balasubramanian in [5]. Let us note for future reference that the functions \( u \) in (1.7) are locally bounded, and therefore holomorphic [2, Thm. 4.6].

The finite topology on \( M_n^d \) (also called the disjoint union topology) is the topology in which a set \( \Omega \) is open if and only if for every \( n \), \( \Omega_n \) is open in the Euclidean topology on \( M_n^d \). If \( \mathcal{H} \) is a Hilbert space, and \( \Omega \) is finitely open, we shall let \( \text{Hol}^\text{nc}_{\mathcal{H}}(\Omega) \) denote the \( B(\mathcal{C}, \mathcal{H}) \) graded nc-functions on \( \Omega \) that are holomorphic on each \( \Omega_n \).\(^2\) A sequence of functions \( u_k \) on \( \Omega \) is finitely locally uniformly bounded if for each point \( \lambda \in \Omega \), there is a finitely open neighborhood of \( \lambda \) inside \( \Omega \) on which the sequence is uniformly bounded.

The following wandering Montel theorem is proved in [1]. If \( u \) is in \( \text{Hol}^\text{nc}_{\mathcal{H}}(\Omega) \) and \( V \) is a unitary operator on \( \mathcal{H} \), define \( V \ast u \) by

\[
\forall_n (V \ast u)|_{\Omega_n} = \frac{V}{\text{id}_{\mathbb{C}^n}} u|_{\Omega_n}.
\]

\(^2\)A function \( u \) is holomorphic in this context if for each \( n \), for each \( x \in \Omega_n \), for each \( h \in M_n^d \), the limit \( \lim_{t \to 0} \frac{1}{t}(u(x + th) - u(x)) \) exists.
Theorem 1.9. Let $\Omega$ be finitely open, $\mathcal{H}$ a Hilbert space, and $\{u^k\}$ a finitely locally uniformly bounded sequence in $\operatorname{Hol}^\infty_{\mathcal{H}}(\Omega)$. Then there exists a sequence $\{U^k\}$ of unitary operators on $\mathcal{H}$ such that $\{U^k \ast u^k\}$ has a subsequence that converges finitely locally uniformly to a function in $\operatorname{Hol}^\infty_{\mathcal{H}}(B_\delta)$.

Let $\phi \in H_1^\infty(B_\delta)$. We shall prove Theorem 1.5 in the following steps.

I For every $z \in B_\delta$, show that $\phi(z)$ is in $\operatorname{Alg}(z)$, the unital algebra generated by the elements of $z$.

II Prove that for every finite set $F \subseteq B_\delta$, there is an nc-model for a function $\psi$ that agrees with $\phi$ on $F$.

III Show that these nc-models have a cluster point that gives an nc-model for $\phi$.

IV Use Theorem 1.8 to get a free realization for $\phi$.

Remarks:

1. Step I is noted in [2] as a corollary of Theorem 1.5; proving it independently allows us to streamline the proof of Theorem 1.5.

2. To prove Step II, we use one direction of [3, Thm 1.3] that gives necessary and sufficient conditions to solve a finite interpolation problem on $B_\delta$. The proof of necessity of this theorem used Theorem 1.5 above, but for Step II we only need the sufficiency of the condition, and the proof of this in [3] did not use Theorem 1.5.

3. All three known proofs of Theorem 1.5 start by proving a realization on finite sets, and then somehow taking a limit. In [2], this is done by considering partial nc-functions; in [6], it is done by using non-commutative kernels to get a compact set in which limit points must exist; in the current paper, we use the wandering Montel theorem.

2 Step I

Let $\{e_j\}_{j=1}^n$ be the standard basis for $\mathbb{C}^n$. For $x$ in $\mathbb{M}_n$ or $\mathbb{M}_d^n$, let $x^{(k)}$ denote the direct sum of $k$ copies of $x$. If $x \in \mathbb{M}_d^n$ and $s$ is invertible in $\mathbb{M}_n$, then $s^{-1}xs$ denotes the $d$-tuple $(s^{-1}x^1s, \ldots, s^{-1}x^d s)$. 

Lemma 2.1. Let $z \in M_n^d$, with $\|z\| < 1$. Assume $w \notin \text{Alg}(z)$. Then there is an invertible $s \in M_{n^2}$ such that $\|s^{-1}z^{(n)}s\| < 1$ and $\|s^{-1}w^{(n)}s\| > 1$.

Proof: Let $A = \text{Alg}(z)$. Since $w \notin A$, and $A$ is finite dimensional and therefore closed, the Hahn-Banach theorem says that there is a matrix $K \in M_n$ such that $\text{tr}(aK) = 0 \quad \forall a \in A$ and $\text{tr}(wK) \neq 0$. Let $u \in \mathbb{C}^n \otimes \mathbb{C}^n$ be the direct sum of the columns of $K$, and $v = e_1 \oplus e_2 \oplus \ldots e_n$. Then for any $b \in M_n$ we have

$$\text{tr}(bK) = \langle b^{(n)}u, v \rangle.$$

Let $A \otimes \text{id}$ denote $\{a^{(n)} : a \in A\}$. We have $\langle a^{(n)}u, v \rangle = 0 \quad \forall a \in A$ and $\langle w^{(n)}u, v \rangle \neq 0$.

Let $N = (A \otimes \text{id})u$. This is an $A \otimes \text{id}$-invariant subspace, but it is not $w^{(n)}$ invariant (since $v \perp N$, but $v$ is not perpendicular to $w^{(n)}u$). So decomposing $\mathbb{C}^n \otimes \mathbb{C}^n$ as $N \oplus N^\perp$, every matrix in $A \otimes \text{id}$ has 0 in the (2,1) entry, and $w^{(n)}$ does not.

Let $s = \alpha I_N + \beta I_{N^\perp}$, with $\alpha \gg \beta > 0$. Then

$$s^{-1} \begin{bmatrix} A & B \\ C & D \end{bmatrix} s = \begin{bmatrix} A & \frac{\beta}{\alpha} B \\ \frac{\beta}{\alpha} C & D \end{bmatrix}.$$

If the ratio $\alpha/\beta$ is large enough, then for each of the $d$ matrices $z^r$, the corresponding $s^{-1}(z^{(n)} \otimes \text{id})s$ will have strict contractions in the (1,1) and (2,2) slots, and each (1,2) entry will be small enough so that the whole thing is a contraction.

For $w$, however, as the (2,1) entry is non-zero, the norm of $s^{-1}w^{(n)}s$ can be made arbitrarily large.

Lemma 2.2. Let $z \in B_S \cap M_n^d$, and $w \in M_n$ not be in $A := \text{Alg}(z)$. Then there is an invertible $s \in M_{n^2}$ such that $s^{-1}z^{(n)}s \in B_S$ and $\|s^{-1}w^{(n)}s\| > 1$.

Proof: As in the proof of Lemma 2.1, we can find an invariant subspace $N$ for $A \otimes \text{id}$ that is not $w$-invariant. Decompose $\delta(z^{(n)})$ as a map from $(N \otimes \mathbb{C}^l) \oplus (N^\perp \otimes \mathbb{C}^l)$ into $(N \otimes \mathbb{C}^l) \oplus (N^\perp \otimes \mathbb{C}^l)$. With $s$ as in Lemma 2.1, and $\alpha \gg \beta > 0$, and $P$ the projection from $\mathbb{C}^n \otimes \mathbb{C}^n$ onto $N$, we get

$$\delta(s^{-1}z^{(n)}s) = \begin{bmatrix} P \otimes \text{id} & P \otimes \text{id} \\ P \otimes \text{id} & P \otimes \text{id} \end{bmatrix} \begin{bmatrix} \delta(z^{(n)}) & \frac{\beta}{\alpha} \delta(z^{(n)}) \\ \frac{\beta}{\alpha} \delta(z^{(n)}) & \delta(z^{(n)}) \end{bmatrix} \begin{bmatrix} P \otimes \text{id} & P \otimes \text{id} \\ P \otimes \text{id} & P \otimes \text{id} \end{bmatrix}. \quad (2.3)$$
The matrix is upper triangular because every entry of $\delta$ is a polynomial, and $N$ is $A$-invariant. For $\alpha/\beta$ large enough, every matrix of the form (2.3) with $z \in B_\delta$ is a contraction, so $s^{-1}z^{(n)}s \in B_\delta$. But $s^{-1}w^{(n)}s$ will contain a non-zero entry multiplied by $\frac{a}{\beta}$, so we achieve the claim. \hfill $\Box$

**Theorem 2.4.** If $\phi$ is in $H^\infty(B_\delta)$, then $\forall z \in B_\delta$, we have $\phi(z) \in \text{Alg}(z)$.

**Proof:** We can assume that $z \in B_\delta$ and that $\|\phi\| \leq 1$ on $B_\delta$. Let $w = \phi(z)$. If $w \notin \text{Alg}(z)$, then by Lemma 2.2, there is an $s$ such that $s^{-1}z^{(n)}s \in B_\delta$ and $\|\phi(s^{-1}z^{(n)}s)\| = \|s^{-1}w^{(n)}s\| > 1$, a contradiction. \hfill $\Box$

Note that Theorem 2.4 does not hold for all nc-functions. In [4] it is shown that there is a class of nc-functions, called fat functions, for which the implicit function theorem holds, but Theorem 2.4 fails.

**3 Step II**

Let $F = \{x_1, \ldots, x_N\}$. Define $\lambda = x_1 \oplus \cdots \oplus x_N$, and define $w = \phi(x_1) \oplus \cdots \oplus \phi(x_N)$. As nc functions preserve direct sums (a consequence of being intertwining preserving) we need to find a function $\psi$ in $H^\infty_1(B_\delta)$ that has an nc-model, and satisfies $\psi(\lambda) = w$.

Let $P_d$ denote the nc polynomials in $d$ variables, and define

$$I_\lambda = \{ q \in P_d : q(\lambda) = 0 \}.$$

Let

$$V_\lambda = \{ x \in M^d : q(x) = 0 \text{ whenever } q \in I_\lambda \}.$$

We need the following theorem from [3]:

**Theorem 3.1.** Let $\lambda \in B_\delta \cap M^d_n$ and $w \in M_n$. There exists a function $\psi$ in the closed unit ball of $H^\infty_1(B_\delta)$ such that $\psi(\lambda) = w$ if (i) $w \in \text{Alg}(\lambda)$, so there exists $p \in P_d$ such that $p(\lambda) = w$.

(ii) $\sup\{\|p(x)\| : x \in V_\lambda \cap B_\delta\} \leq 1$.

Moreover, if the conditions are satisfied, $\psi$ can be chosen to have a free realization.

Since $\phi(\lambda) = w$, by Theorem 2.4, there is a free polynomial $p$ such that $p(\lambda) = w$, so condition (i) is satisfied. To see condition (ii), note that for all $x \in V_\lambda \cap B_\delta$, we have $p(x) = \phi(x)$. Indeed, by Theorem 2.4, there is a
polynomial $q$ so that $q(\lambda \oplus x) = \phi(\lambda \oplus x)$. Therefore $q(\lambda) = p(\lambda)$, so, since $x \in V_\lambda$, we also have $q(x) = p(x)$, and hence $p(x) = \phi(x)$. But $\phi$ is in the unit ball of $H^\infty_1(B_\delta)$, so $\|\phi(x)\| \leq 1$ for every $x$ in $B_\delta$.

So we can apply Theorem 3.1 to conclude that there is a function $\psi$ in $H^\infty(B_\delta)$ that has a free realization, and that agrees with $\phi$ on the finite set $F$.

Remark: The converse of Theorem 3.1 is also true. Given Theorem 2.4, the converse is almost immediate.

4 Steps III and IV

Let $\Lambda = \{x_j\}_{j=1}^\infty$ be a countable dense set in $B_\delta$. For each $k$, let $F_k = \{x_1, \ldots, x_k\}$. By Step II, there is a function $\psi^k \in H^\infty_1(B_\delta)$ that has a free realization and agrees with $\phi$ on $F_k$. By Theorem 1.8, there exists a Hilbert space $\mathcal{M}^k$ and a $\mathcal{B}(\mathbb{C}, \mathcal{M}^k \otimes \mathbb{C}^I)$ valued nc-function $u^k$ on $B_\delta$ so that, for all $n$, for all $x, y \in B_\delta \cap \mathbb{M}_d^n$, we have

$$1 - \psi^k(y)^* \psi^k(x) = u^k(y)^* \left[ \frac{1}{1 - \delta(y)^* \delta(x)} \right] u^k(x).$$

Embed each $\mathcal{M}^k$ in a common Hilbert space $\mathcal{H}$. Since the left-hand side of (4.1) is bounded, it follows that $u^k$ are locally bounded, so we can apply Theorem 1.9 to find a sequence of unitaries $U^k$ so that, after passing to a subsequence, $U^k * u^k$ converges to a function $v$ in $\text{Hol}^\text{nc}_{\mathcal{H}}(\Omega)$. We have therefore that

$$1 - \phi(y)^* \phi(x) = v(y)^* \left[ \frac{1}{1 - \delta(y)^* \delta(x)} \right] v(x)$$

holds for all pairs $(x, y)$ that are both in $\Lambda \cap \mathbb{M}_d^n$ for any $n$, so by continuity, we get that (4.2) is an nc-model for $\phi$ on all $B_\delta$, completing Step III.

Finally, Step IV follows by applying Theorem 1.8.

5 Closing remarks

One can modify the argument to get a realization formula for $\mathcal{B}(\mathcal{K}, \mathcal{L})$-valued bounded nc-functions on $B_\delta$, or to prove Leech theorems (also called Toeplitz-corona theorems—see [10] and [8]). For finite-dimensional $\mathcal{K}$ and $\mathcal{L}$, this was done in [2]; for infinite-dimensional $\mathcal{K}$ and $\mathcal{L}$ the formula was proved in [6], using results from [7].
References


