AN H^p SCALE FOR COMPLETE PICK SPACES

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1. Introduction

Let $\mathcal{M}$ be a reproducing kernel Hilbert space on a set X, with kernel function k. Let $\mathrm{Mult}(\mathcal{M})$ denote the multiplier algebra of $\mathcal{M}$. We shall make the following assumption throughout our paper:

$$(A) \qquad \mathrm{Mult}(\mathcal{M}) \text{ is densely contained in } \mathcal{M}.$$

We shall let $\mathcal{M} \odot \mathcal{M}$ denote the weak-product of $\mathcal{M}$ with itself, which is

$$\mathcal{M} \odot \mathcal{M} \ := \ \{\sum_{n=1}^{\infty} f_n g_n \ : \ \sum_n \|f_n\|_{\mathcal{M}} \|g_n\|_{\mathcal{M}} < \infty\}. \tag{1.1}$$

This is a Banach space, where the norm of a function h is the infimum of $\sum_n \|f_n\|_{\mathcal{M}} \|g_n\|_{\mathcal{M}}$ over all representations of h as $\sum_n f_n g_n$.

If we use the complex method of interpolation to interpolate between $\mathcal{M} \odot \mathcal{M}$ and its anti-dual (the space of bounded conjugate linear functionals) we get a scale of Banach spaces, whose mid-point is the Hilbert space $\mathcal{M}$. By analogy with the case where $\mathcal{M}$ is the Hardy space H^2 on the unit disk, where the end-points become H^1 and BMOA and the intermediate spaces are H^p for $1 < p < \infty$, we shall define

$$\mathcal{H}^p \ := \ [\mathcal{M} \odot \mathcal{M}, (\mathcal{M} \odot \mathcal{M})^\dagger]_{[\theta]} \tag{1.2}$$

where $p = \frac{1}{1-\theta}$ and $A^\dagger$ denotes the anti-dual of A.

We consider $\mathcal{H}^p$ to be the H^p scale for the space $\mathcal{M}$. In Section 2 we study properties of the $\mathcal{H}^p$ spaces for general $\mathcal{M}$. In Section 3 we specialize to the case that $\mathcal{M}$ is a complete Pick space.

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2. General Spaces

The space $(\mathcal{M} \odot \mathcal{M})^\dagger$ was described in [1]. Let $\mathcal{M} \otimes_\pi \mathcal{M}$ denote the projective tensor product of $\mathcal{M}$ with itself. Its dual is $\mathcal{B}(\mathcal{M}, \overline{\mathcal{M}})$, where $\overline{\mathcal{M}}$ is the complex conjugate of $\mathcal{M}$. Let $\rho : \mathcal{M} \otimes_\pi \mathcal{M} \to \mathcal{M} \odot \mathcal{M}$ be defined by

$$\rho : \sum f_n \otimes g_n \mapsto \sum f_n(z) g_n(z).$$

Then $(\mathcal{M} \odot \mathcal{M})^*$ can be identified with $(\ker\rho)^\perp$. We can identify $(\mathcal{M} \odot \mathcal{M})^\dagger$ with

$$\mathrm{Han} \ := \ \{\overline{T1} : T \in (\ker\rho)^\perp\}.$$

If $b \in \mathrm{Han}$, which is a subset of $\mathcal{M}$, the corresponding conjugate linear functional on $\mathcal{M} \odot \mathcal{M}$ is given by

$$\Lambda_b : f \mapsto \langle b, f \rangle \quad \forall\, f \in \mathcal{M}.$$

We write H_b for the unique operator $H \in \mathcal{B}(\mathcal{M}, \overline{\mathcal{M}}) \cap (\ker\rho)^\perp$ that satisfies $H_b 1 = \bar{b}$. We put a norm on Han by declaring $\|b\|$ equal to the operator norm of H_b. Let

$$\begin{aligned} \mathcal{X}(\mathcal{M}) \ := \ \{b \in \mathcal{M} :\ \exists\, C \geq 0 \quad & \text{s.t. } |\langle b, \phi f \rangle| \leq C \|\phi\|_{\mathcal{M}} \|f\|_{\mathcal{M}} \\ & \forall\, \phi \in \mathrm{Mult}(\mathcal{M}), f \in \mathcal{M}\}. \end{aligned}$$

Then under assumption (A) it is proved in [1, Thm 2.5] that

$$\mathrm{Han} \subseteq \mathcal{X}(\mathcal{M}).$$

Theorem 2.1. *The complex interpolation space $[\mathcal{M} \odot \mathcal{M}, \mathrm{Han}]_{[\frac{1}{2}]}$ is isometrically isomorphic to $\mathcal{M}$.*

Proof: Assumption (A) implies that $\mathcal{M}$ is dense in $\mathcal{M} \odot \mathcal{M}$. Pisier proved in [5] that if a Hilbert space $\mathcal{M}$ is densely contained in a Banach space A, then $[A, A^\dagger]_{[\frac{1}{2}]} = \mathcal{M}$. His proof is in the context of operator spaces; a direct proof of the fact is given in [6]. See also [4] for another proof. □

We shall let $\mathcal{H}^p$ be defined by (1.2) with $\theta = \frac{p-1}{p}$, and write $\mathcal{H}^1$ for $\mathcal{M} \odot \mathcal{M}$. Notice that since $\mathrm{Han} \subseteq \mathcal{M} \odot \mathcal{M}$, we have

$$\mathcal{H}^p \supseteq \mathcal{H}^q \supseteq \mathrm{Han}$$

whenever $1 \leq p \leq q < \infty$.

Theorem 2.2. *For $1 < p < \infty$, we have $(\mathcal{H}^p)^\dagger$ is isometrically isomorphic to $\mathcal{H}^{p'}$, where p' is the conjugate index to p.*

Proof: By the reiteration theorem [2, Thm. 4.6.1], if we interpolate between $\mathcal{H}^1$ and $\mathcal{H}^2$ we get $\mathcal{H}^p$ for $1 < p < 2$, and if we interpolate between $\mathcal{H}^2$ and Han we get $\mathcal{H}^p$ for $2 < p < \infty$. Since $\mathcal{H}^2$ is reflexive, we have by the duality theorem [2, Cor. 4.5.2]

$$[\mathcal{H}^1, \mathcal{H}^2]^{\dagger}_{[\theta]} \; = \; [\mathrm{Han}, \mathcal{H}^2]_{[\theta]},$$

which proves the theorem for $1 < p \leq 2$. In [3, 12.2], Calderon proved that if one end point space is reflexive, all the intermediate ones are too. So this proves the theorem for $2 < p < \infty$. □

We define Han_0 by

$$\mathrm{Han}_0 \; := \; \{b \in \mathrm{Han} : H_b \text{ is compact}\}.$$

By [1, Thm. 2.5], Han_0 is the predual of $\mathcal{M} \odot \mathcal{M}$. We think of Han_0 as the analogue of VMOA.

Proposition 2.3. *For $0 < \theta < 1$, we have*

$$[\mathcal{M} \odot \mathcal{M}, \mathrm{Han}_0]_{[\theta]} \; = \; [\mathcal{M} \odot \mathcal{M}, \mathrm{Han}]_{[\theta]}.$$

Proof: By [4, Thm. 4.2],

$$[\mathcal{M} \odot \mathcal{M}, \mathrm{Han}_0]_{[\frac{1}{2}]} \; = \; \mathcal{M}.$$

Need to show that, in their notation, if D_0 is the set of finite linear combinations of kernel functions, then this is dense in $A = \mathrm{Han}_0$. This works out to showing that finite sums

$$\sum c_j \overline{k_{x_j}} \otimes k_{x_j} : f \mapsto \langle f, \sum c_j k_{x_j} \rangle_{\mathcal{M}} \, \overline{k_{x_j}}$$

are dense in Han_0.

This should follow from the Hahn–Banach theorem and the fact that ${\mathrm{Han}_0}^* = \mathcal{M} \odot \mathcal{M}$.

Therefore the reiteration theorem proves the result for $0 < \theta \leq \frac{1}{2}$. It remains to prove that

$$[\mathcal{M}, \mathrm{Han}_0]_{[s]} \; = \; [\mathcal{M}, \mathrm{Han}]_{[s]}$$

for $0 < s < 1$. But applying the duality theorem twice we get

$$[\mathcal{M}, \mathrm{Han}_0]^{**}_{[s]} \; = \; [\mathcal{M}, \mathrm{Han}]_{[s]},$$

and by Calderon's reflexivity theorem again, we have $[\mathcal{M}, \mathrm{Han}_0]_{[s]}$ is reflexive for $0 \leq s < 1$, so we are done. □

Let δ_x be the functional of evaluation at $x \in X$.

Lemma 2.4. *For $1 \leq p \leq 2$, we have $||\delta_x||_{\mathcal{H}^{p*}} \leq k(x,x)^{1/p}$.*

Proof. The complex method of interpolation shows that

$$||\delta_x||_{\mathcal{H}^{p*}} \leq ||\delta_x||^{1-\theta}_{\mathcal{M}^*} ||\delta_x||^{\theta}_{(\mathcal{M} \odot \mathcal{M})^*}, \tag{2.3}$$

where

$$\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{1}.$$

Since $||\delta_x||_{\mathcal{H}^{p*}} = k(x,x)^{1/2}$ and $||\delta_x||_{(\mathcal{M}\odot\mathcal{M})^*} \leq k(x,x)$, the right-hand side of (2.3) is dominated by $k(x,x)^{(\theta+1)/2} = k(x,x)^{1/p}$. □

In Section 3 we shall prove that this estimate is sharp (up to a constant) in complete Pick spaces.

Remark 2.5 There are many interesting Hilbert function spaces for which assumption (A) fails, such as ℓ^2, the Hardy space of the upper half-plane, or the Fock space. One can still define an $\mathcal{H}^p$ scale for these spaces for $p \in [1,2]$ by interpolating between $\mathcal{M} \odot \mathcal{M}$ and $\mathcal{M}$. The tricky part is finding a general method for identifying the anti-duals of these spaces with Banach function spaces on X.

3. Complete Pick Spaces

4. CR property

Have $\mathcal{X}(\mathcal{M}) =$ Han

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