The Krzyż Conjecture and an Entropy Conjecture

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Abstract: We show that if the minimum entropy for a polynomial with roots on the unit circle is attained by polynomials with equally spaced roots, then, under a generic hypothesis about the nature of the extremum, the Krzyz conjecture on the maximum modulus of the Taylor coefficients of a holomorphic function that maps the disk to the punctured disk is true.

1 Introduction

Let Ω denote the set of holomorphic functions that map the unit disk $\mathbb{D}$ to $\mathbb{D} \setminus \{0\}$. The Krzyż conjecture, due to J. Krzyż [11], is the following conjecture about the size of Taylor coefficients of functions in Ω.

Conjecture 1.1. *Let n be a positive integer. Then*

$$K_n^\bullet := \sup_{f\in\Omega}\{|\hat{f}(n)| \ : \ f \in \Omega\} \ = \ \frac{2}{e}. \tag{1.2}$$

Moreover, equality is obtained in (1.2) *only for functions of the form*

$$f(z) \ = \ \zeta \exp(\frac{z^n + \omega}{z^n - \omega}) \tag{1.3}$$

where ζ and ω are unimodular constants.

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For any function f defined and holomorphic on a neighborhood of the origin, we use $\hat{f}(k)$ to denote the k^{th} Taylor coefficient at 0, so

$$\hat{f}(k) = \frac{f^{(k)}(0)}{k!}.$$

For a history of the Krzyż conjecture and a summary of known results, see Section 2.

The purpose of this note is to establish a connection between the Krzyż conjecture and the following conjecture about the entropy of polynomials with roots on the unit circle $\mathbb{T}$.

Conjecture 1.4. *Let p be a non-constant polynomial, all of whose roots lie on $\mathbb{T}$, and normalized so that $\frac{1}{2\pi}\int_0^{2\pi}|p(e^{i\theta})|^2 d\theta = 1$. Then*

$$\frac{1}{2\pi}\int_0^{2\pi} |p(e^{i\theta})|^2 \log |p(e^{i\theta})|^2 d\theta \geq 1 - \log(2). \tag{1.5}$$

Moreover, equality occurs in (1.5) *only for polynomials of the form*

$$p(z) = \frac{\zeta}{\sqrt{2}}(\omega + z^n), \tag{1.6}$$

where ζ and ω are unimodular constants, and n is a positive integer.

We shall let $\mathcal{H}$ denote the Herglotz class, the holomorphic functions on the unit disk that have non-negative real part. A function f is in Ω if and only if there is a function $g \in \mathcal{H}$ so that

$$f = e^{-g}.$$

Given an $(n+1)$-tuple $a = (a_0, \ldots, a_n)$ of complex numbers, we shall say that a is *solvable Herglotz data* if there exists $g \in \mathcal{H}$ satisfying

$$\hat{g}(k) = a_k, \text{ for } k = 0, \ldots, n.$$

We shall say that a is *extremal Herglotz data* if it is solvable but for any $r > 1$, the data $(a_0, ra_1, \ldots, r^n a_n)$ is not solvable.

We shall say that f is $K_n^\bullet$-extremal if f is in Ω and $\hat{f}(n) = K_n^\bullet$. It was proved in [10] that if f is $K_n^\bullet$-extremal, and $g = -\log(f)$, then the first $n+1$

Taylor coefficients of g are extremal Herglotz data (we give a proof of this in Lemma 3.3). By a theorem of G. Pick [13], this means g must have the form

$$g(z) \;=\; ai + \sum_{\ell=1}^{m} w_\ell \frac{\tau_\ell + z}{\tau_\ell - z}, \tag{1.7}$$

where $a \in \mathbb{R}$, the number m satisfies $1 \leq m \leq n$, each $w_\ell > 0$, and each τ_ℓ is a distinct point on $\mathbb{T}$. We shall let $\mathcal{R}_m$ denote the set of rational functions that have the form (1.7) (that is, rational functions of degree m that are m-fold covers of the right-half plane by the unit disk), and we shall let $\mathcal{R}_n^\bullet = \cup_{1\leq m\leq n} \mathcal{R}_m$.

Our first main result analyzes the critical points for the Krzyż functional. We prove in Theorem 5.1 that if g is in $\mathcal{R}_n$ (and is normalized in a way described in Section 6), then $f = e^{-g}$ is a critical point for $K_n^\bullet$ with critical value η if and only if

$$e^{-g} \overset{n}{\sim} \eta\gamma^2,$$

where γ is a polynomial of degree n of unit norm in the kernel of $\operatorname{Re} g(S_n)$, where S_n is the compression of the unilateral shift to polynomials of degree less than or equal to n, and the notation $\overset{n}{\sim}$ means that the functions have the same Taylor coefficients up to degree n.

We use this result to prove that if the extremals f for the Krzyż problem have $g = -\log f$ of full degree, then the entropy conjecture implies the Krzyż conjecture.

Theorem 1.8. *Let f be $K_n^\bullet$-extremal, and assume that $g = -\log f$ is in $\mathcal{R}_n$. If Conjecture 1.4 is true, then f has the form* (1.3).

We prove Theorem 1.8 in Section 7. In Section 8 we study critical points of the entropy functional from (1.5). In Section 9, we prove a special case of Conjecture 1.4. Finally, in Section 10, we show how Conjecture 1.4 would follow from Conjecture 2.2, due to A. Baernstein II.

2 History of the Krzyż conjecture

J. Krzyż proved Conjecture 1.1 for $n = 2$, and conjectured it for all n. The $n = 3$ case was proved by J. Hummel, S. Scheinberg and L. Zalcman [10]; they also proved Lemma 3.3 below, and that (1.3) is a strict local maximum for (1.2) (after normalizing so that $f(0)$ and $\hat{f}(n)$ are both positive). The

$n = 4$ case was proved first by D. Tan [15], and later by a different method by J. Brown [5]. The $n = 5$ case was proved by N. Samaris [14].

C. Horowitz [9] proved that there is some constant $H < 1$ such that $K_n^\bullet \le H$ for all n; his proof showed $H \le 0.99987\ldots$. This was improved by R. Ermers to $H \le 0.9991\ldots$ [6].

In [12], M. Martin, E. Sawyer, I. Uriarte-Tuero, and D. Vukotić prove that 16 different conditions are all equivalent to the Krzyż conjecture. The paper also includes a useful historical summary.

Conjecture 1.4 may be compared with the following sharp inequality, conjectured by I. Hirschman [8] and proved by W. Beckner [4]:

If $f \in L^2(\mathbb{R})$ has norm 1, and $\mathcal{F}f$ denotes the Fourier transform of f, then

$$\int |f|^2 \log |f|^2 \; + \; \int |\mathcal{F}f|^2 \log |\mathcal{F}f|^2 \; \le \; \log(2) - 1. \tag{2.1}$$

Equality is obtained in (2.1) for Gaussians.

A. Baernstein II made the following conjecture in 2008 [2], where the quasi-norms are with respect to normalized Lebesgue measure on the circle. By $\|f\|_0$ we mean $\exp(\int_{\mathbb{T}} \log |f|)$.

Conjecture 2.2. *Let $Q(z) = 1 + z^n$. Then for all $0 \le s \le t \le \infty$, and for all non-constant polynomials p with all their roots on the unit circle,*

$$\frac{\|p\|_s}{\|Q\|_s} \; \le \; \frac{\|p\|_t}{\|Q\|_t}. \tag{2.3}$$

In Section 10 we show how Baernstein's conjecture implies the entropy conjecture.

3 Preliminaries

Suppose f and g are analytic functions on a neighborhood of 0, and $n \in \mathbb{N}$. Say

$$f \overset{n}{\sim} g$$

if $\hat{f}(k) = \hat{g}(k)$ for $0 \le k \le n$. We leave the proof of the following lemma to the reader.

Lemma 3.1. *Suppose f and g are analytic on a neighborhood of zero. Assume that $f(0) = g(0) = b$, and ϕ is analytic in a neighborhood of b. If $f \overset{n}{\sim} g$, then $\phi \circ f \overset{n}{\sim} \phi \circ g$.*

The following result is due to G. Pick [13], and can be found in any book on Pick interpolation such as [1,3,7].

Lemma 3.2. *If $a = (a_0, \ldots, a_n)$ is extremal Herglotz data, then there exists a unique function $g \in \mathcal{H}$ such that $\hat{g}(k) = a_k$, for $0 \leq k \leq n$. Moreover, $g \in \mathcal{R}_n^\bullet$. Conversely, if $g \in \mathcal{R}_n^\bullet$, then $a = (\hat{g}(0), \ldots, \hat{g}(n))$ is extremal Herglotz data.*

Lemma 3.3. *Fix $n \geq 1$ and assume that f is $K_n^\bullet$-extremal. Define g by $f = e^{-g}$. Then $g \in \mathcal{R}_n^\bullet$.*

Proof: We know that g must be in $\mathcal{H}$, so by Lemma 3.2, if g is not in $\mathcal{R}_n^\bullet$, then for some $r > 1$ we have a function $h \in \mathcal{H}$ such that

$$\hat{h}(k) \;=\; r^k \hat{g}(k), \quad 0 \leq k \leq n.$$

Then $\phi = e^{-h}$ is in Ω, and by Lemma 3.1, $\phi(z) \overset{n}{\sim} f(rz)$, so

$$|\hat{\phi}(n)| \;=\; r^n |\hat{f}(n)| \;>\; |\hat{f}(n)|.$$

This contradicts the claim that f is extremal. □

Corollary 3.4.

$$K_n^\bullet \;=\; \sup_{g \in \mathcal{R}_n^\bullet} |\widehat{(e^{-g})}\,(n)|.$$

Given that $\mathcal{R}_n^\bullet = \cup_{m=0}^n \mathcal{R}_m$, Corollary 3.4 suggests the following optimization problem. For each $n \geq 1$, define

$$K_n \;=\; \sup_{g \in \mathcal{R}_n} |\widehat{(e^{-g})}\,(n)|.$$

Remark 3.5. As $\mathcal{R}_n$ is a dense open set in $\mathcal{R}_n^\bullet$, we have $K_n = K_n^\bullet$. However, whereas a normal families argument guarantees that an extremal function for $K_n^\bullet$ always exists, it is not obvious that an extremal for K_n exists. If Krzyż's conjecture is true, then the supremum is attained.

4 The critical points of K_n

4.1 The definition of critical points

For the rest of the paper, n will be a positive integer. There are a number of equivalent ways to view $\mathcal{R}_n$ as a topological space.

1. Using the a, w, τ parameters of (1.7) (where $m = n$).

2. As the subset of the space of extremal Herglotz data points $a = (a_0, \ldots, a_n)$ with the property that $(a_0, \ldots, a_{n-1})$ is not extremal.

3. With the topology of uniform convergence on compact subsets of $\mathbb{D}$.

We would like to consider the local maxima of the function $F : \mathcal{R}_n \to \mathbb{R}$ defined by

$$F(g) \;=\; |\widehat{(e^{-g})}(n)|^2. \tag{4.1}$$

Let $\mathcal{P}_n^\bullet$ denote the set of complex polynomials of degree less than or equal to n, and $\mathcal{P}_n$ the polynomials of degree exactly n.

Definition 4.2. Let $g \in \mathcal{R}_n$. We say d is an *admissible direction* at g if $d \in \mathcal{P}_n^\bullet$ and there exists $\varepsilon > 0$ such that

$$(\hat{g}(0) + t\hat{d}(0), \ldots, \hat{g}(n) + t\hat{d}(n))$$

is solvable Herglotz data for all t in $(0, \varepsilon)$. We say that g is a critical point for K_n if

$$\frac{d}{dt} |\widehat{e^{-(g+td)}}(n)|^2 \Big|_{t=0^+} \leq 0 \tag{4.3}$$

whenever d is an admissible direction at g. If g is a critical point for K_n, then we refer to $\eta = \widehat{e^{-g}}(n)$ as the *critical value.*

4.2 A Hilbert space setting for the analysis of critical points

Let H^2 denote the classical Hardy space on the unit disk. We shall think of $\mathcal{P}_n^\bullet$ as a subspace of H^2, and let P_n be the orthogonal projection from H^2 onto $\mathcal{P}_n^\bullet$. Define an operator S_n on $\mathcal{P}_n^\bullet$ by the formula

$$(S_n q)(z) \;=\; P_n(zq(z)), \qquad q(z) \in \mathcal{P}_n^\bullet.$$

The operator S_n is the truncated shift, and is nilpotent of order $n+1$. Hence if f is any holomorphic function on a neighborhood of 0, we can define $f(S_n)$ by the Riesz functional calculus, or by either of the two equivalent formulas

$$\begin{aligned} f(S_n) &= \sum_{k=0}^{n} \hat{f}(k) S_n^k \\ f(S_n)q &= P_n(fq). \end{aligned}$$

Observe that if f and g are both holomorphic on a neighborhood of 0, then

$$f \overset{n}{\sim} g \;\Leftrightarrow\; f(S_n) = g(S_n).$$

The following two propositions are basically a reformulation of Lemma 3.2 to the Hilbert space interpretation of interpolation. Recall that for a matrix T, its real part $\mathrm{Re}\,(T) = \frac{1}{2}(T + T^*)$. We say T is positive semi-definite if $\langle Tv, v\rangle \geq 0$ for every vector v; this is equivalent to saying that T is self-adjoint and all its eigenvalues are non-negative. T is positive definite if $\langle Tv, v\rangle > 0$ for every non-zero vector v; equivalently it is self-adjoint and all its eigenvalues are strictly positive. It follows that a matrix that is positive semi-definite but not positive definite must be singular, *i.e.* non-invertible.

Proposition 4.4. *Let a be an $(n+1)$-tuple of complex numbers. Then a is solvable Herglotz data if and only if*

$$\mathrm{Re} \sum_{k=0}^{n} a_k S_n^k \geq 0.$$

Moreover a is extremal Herglotz data if and only if $\mathrm{Re} \sum_{k=0}^{n} a_k S_n^k$ is positive semi-definite but not positive definite.

Proposition 4.5. *Assume $g \in \mathcal{H}$.*

1. $\mathrm{Re}\, g(S_n) \geq 0$.
2. *The function g is in $\mathcal{R}_n^\bullet$ if and only if $\mathrm{Re}\, g(S_n)$ is singular.*
3. *If $0 \leq m \leq n$, then $g \in \mathcal{R}_m$ if and only if $\mathrm{rank}(g(S_n)) = m$.*

4.3 Local maxima are critical points

We need to show that there is enough smoothness at local maxima to make sense of (4.3), at least when the local maximum is in $\mathcal{R}_n$.

Proposition 4.6. *Let F be defined by (4.1). If $g \in \mathcal{R}_n$ is a local maximum for F, then g is a critical point for K_n.*

Proof: Let d be an admissible direction for g. Thus by Proposition 4.4 there exists $\varepsilon > 0$ so that

$$\mathrm{Re}\,[g(S_n) + t d(S_n)] \;\geq\; 0 \quad \forall\, t \in [0, \varepsilon).$$

Let $\rho(t)$ denote the smallest eigenvalue of $\mathrm{Re}\,[g(S_n) + td(S_n)]$, so for each t we have $\mathrm{Re}\,[g(S_n) + td(S_n) - \rho(t)]$ is positive semidefinite and singular. By Proposition 4.4, for every t there exists $g_t \in \mathcal{R}_n^\bullet$ such that

$$g_t \overset{n}{\sim} [g + td - \rho(t)].$$

Since g is a local maximum, we have $\rho(t) \to 0$ as $t \to 0^+$, and $g_t \to g$. As $\mathcal{R}_n$ is open in $\mathcal{R}_n^\bullet$, this means for some $\delta > 0$, we have $g_t \in \mathcal{R}_n$ for all t in $[0, \delta)$. As g is a local maximum for F, we have

$$F(g_t) \;\leq\; F(g) \quad \forall\, t \in [0, \delta).$$

As

$$F(g + td) \;=\; |\widehat{e^{-(g+td)}}(n)|^2,$$

it is differentiable with respect to t, and as

$$F(g + td) \;=\; e^{-2\rho(t)} F(g_t) \;\leq\; F(g),$$

the derivative of $F(g + td)$ is nonpositive at 0. □

4.4 Some lemmas about critical points

We shall let $\|\gamma\|$ denote the H^2-norm, so

$$\|\gamma\|^2 \;=\; \frac{1}{2\pi} \int_0^{2\pi} |\gamma(e^{i\theta})|^2 d\theta.$$

We shall let n be fixed, and write S for S_n for legibility. Note that $\mathrm{Re}\, g(S)\gamma$ means $\frac{1}{2}(g(S) + g(S)^*)\gamma$, and not $\mathrm{Re}\ [g(S)\gamma]$.

Lemma 4.7. *If $g \in \mathcal{R}_n$, then there exists a unique vector γ in $\mathcal{P}_n$ such that $\hat{\gamma}(n) > 0$, $\|\gamma\| = 1$ and $\mathrm{Re}\, g(S)\gamma = 0$. Furthermore, if*

$$g(z) \;=\; ai + \sum_{\ell=1}^{n} w_\ell \frac{\tau_\ell + z}{\tau_\ell - z},$$

then

$$\gamma(z) \;=\; \frac{1}{\nu} \prod_{\ell=1}^{n} (z - \tau_l),$$

where

$$\nu \;=\; \|\prod_{\ell=1}^{n} (z - \tau_l)\|.$$

PROOF: By Proposition 4.5, $\operatorname{rank}(g(S)) = n$. Hence there exists some nonzero vector q in $\ker \operatorname{Re} g(S)$. We will show that $q(\tau_\ell) = 0$ for each ℓ, and then define

$$\gamma(z) \;=\; \frac{|\hat{q}(n)|}{\hat{q}(n)\|q\|}q.$$

As

$$\begin{aligned}
\operatorname{Re}\frac{\tau + S}{\tau - S} \quad &= \quad \frac{1}{2}\left(\frac{\tau + S}{\tau - S} + \frac{\bar{\tau} + S^*}{\bar{\tau} - S^*}\right) \\
&= \quad (\bar{\tau} - S^*)^{-1}(1 - S^* S)(\tau - S)^{-1} \\
&= \quad (\bar{\tau} - S^*)^{-1}(z^n \otimes z^n)(\tau - S)^{-1} \\
&= \quad [(\bar{\tau} - S^*)^{-1} z^n] \otimes [(\bar{\tau} - S^*)^{-1} z^n],
\end{aligned}$$

we have

$$\operatorname{Re} g(S) \;=\; \sum_{\ell=1}^{n} w_\ell [(\bar{\tau}_\ell - S^*)^{-1} z^n] \otimes [(\bar{\tau}_\ell - S^*)^{-1} z^n].$$

Since each $w_\ell > 0$, we can only have $\langle \operatorname{Re} g(S) q, q\rangle = 0$ if for each $\ell = 1, \ldots, n$ we have

$$\langle q, (\bar{\tau}_\ell - S^*)^{-1} z^n \rangle = 0.$$

As

$$(\bar{\tau}_\ell - S^*)^{-1} z^n \;=\; \tau_\ell^{n+1}(1 + \bar{\tau}_\ell z + \cdots + \bar{\tau}_\ell{}^n z^n),$$

we get

$$\begin{aligned}
\langle q, (\bar{\tau}_\ell - S^*)^{-1} z^n \rangle \quad &= \quad \bar{\tau}_\ell{}^{n+1} \langle \hat{q}(0) + \cdots + \hat{q}(n) z^n, 1 + \bar{\tau}_\ell z + \ldots \bar{\tau}_\ell{}^n z^n \rangle \\
&= \quad \bar{\tau}_\ell{}^{n+1} q(\tau_\ell).
\end{aligned}$$

Therefore $\langle \operatorname{Re} g(S) q, q\rangle = 0$ implies that q vanishes at each τ_ℓ, as claimed. □

Lemma 4.8. *Let $g \in \mathcal{R}_n$ and let γ be the vector described in Lemma 4.7. For $d \in \mathcal{P}_n^\bullet$, the following hold:*

1. *If d is an admissible direction at g, then $\langle \operatorname{Re} d(S)\gamma, \gamma\rangle \geq 0$.*
2. *If $\langle \operatorname{Re} d(S)\gamma, \gamma\rangle = 0$, then d is an admissible direction at g if and only if $\operatorname{Re} d(S)\gamma = 0$.*

3. *If* $\langle \mathrm{Re}\, d(S)\gamma, \gamma\rangle = 0$*, then* $d+\varepsilon$ *is an admissible direction for every* $\varepsilon > 0$*.*

PROOF: 1. Since $g + td \in \mathcal{H}$ for t small and positive, we must have

$$\langle \mathrm{Re}\,[g(S) + td(S)]\gamma, \gamma\rangle \;=\; t\langle \mathrm{Re}\, d(S)\gamma, \gamma\rangle \;\geq\; 0.$$

2. If $\beta \perp \gamma$, then

$$\langle \mathrm{Re}\,[g(S)+td(S)](\gamma+\beta), (\gamma+\beta)\rangle \;=\; 2t\langle \mathrm{Re}\, d(S)\gamma, \beta\rangle + \langle \mathrm{Re}\,[g(S)+td(S)]\beta, \beta\rangle.$$

The right-hand side is non-negative for all β and small positive t if and only if $\mathrm{Re}\, d(S)\gamma = 0$.

3. If $\beta \perp \gamma$, then

$$\begin{aligned} &\langle \mathrm{Re}\,[g(S) + t(d(S)+\varepsilon)](a\gamma+\beta), (a\gamma+\beta)\rangle \\ &\quad = \; 2t\langle \mathrm{Re}\, d(S)a\gamma, \beta\rangle + t\varepsilon(\|a\gamma\|^2 + \|\beta\|^2) + \langle \mathrm{Re}\, g(S)\beta, \beta\rangle. \end{aligned} \tag{4.9}$$

As β is perpendicular to the kernel of $\mathrm{Re}\, g(S)$, the right-hand side is non-negative for t positive and sufficiently small. (The requirement that $\varepsilon > 0$ is only needed if $\beta = 0$). □

Lemma 4.10. *Let* $g \in \mathcal{R}_n$ *and let* γ *be the vector described in Lemma 4.7. If* g *is a critical point for* K_n *and* $d \in \mathcal{P}_n^\bullet$ *satisfies* $\mathrm{Re}\,\langle d(S)\gamma, \gamma\rangle = 0$*, then*

$$\frac{d}{dt}F(g+td)|_{t=0^+} \;\leq\; 0.$$

PROOF: By Lemma 4.8, for all $\varepsilon > 0$ we have $d+\varepsilon$ is admissible, so by Proposition 4.6 we have

$$\frac{d}{dt}F(g+t(d+\varepsilon))|_{t=0^+} \;\leq\; 0.$$

Now let $\varepsilon \to 0^+$. □

Lemma 4.11. *If* g *is analytic on a neighborhood of* 0 *and* $d \in \mathcal{P}_n^\bullet$*, then*

$$\frac{d}{dt}|\widehat{e^{-g+td}}(n)|^2\Big|_{t=0} \;=\; -2\mathrm{Re}\,\langle z^n, e^{-g(S)}1\rangle\langle d(S)e^{-g(S)}1, z^n\rangle.$$

PROOF: Computation. □

5 The Critical Point Equation

We fix $n \geq 1$, and write S for S_n. For $p \in \mathcal{P}_n^\bullet$, define $\tilde{p}$ by

$$\tilde{p}(z) = z^n \overline{p(\frac{1}{\bar{z}})}.$$

So if $p(z) = a_0 + a_1 z + \cdots + a_n z^n$, then $\tilde{p}(z) = \bar{a}_n + \bar{a}_{n-1} z + \cdots + \bar{a}_0 z^n$. We shall say p is *self-inversive* if $p = \tilde{p}$.

Theorem 5.1. *Let $g \in \mathcal{R}_n$ and let γ be the vector described in Lemma 4.7. Then g is a critical point of K_n with non-zero critical value η if and only if*

$$e^{-g} \overset{n}{\sim} \eta\gamma\tilde{\gamma}. \tag{5.2}$$

PROOF: Suppose g is a critical point of K_n with critical value η. If $d \in \mathcal{P}_n^\bullet$ and $\mathrm{Re}\,\langle d(S)\gamma, \gamma\rangle = 0$, then by Lemma 4.10,

$$\frac{d}{dt} |\langle e^{-(g+td)}(S)1, z^n\rangle|^2 \Big|_{t=0^+} \;=\; \frac{d}{dt} F(g+td)\Big|_{t=0^+} \;\leq\; 0.$$

Hence if $\langle d(S)\gamma, \gamma\rangle = 0$, so $\mathrm{Re}\,\langle \zeta d(S)\gamma, \gamma\rangle = 0$ for all $\zeta \in \mathbb{T}$, we get by Lemma 4.11

$$-2\mathrm{Re}\,\zeta\bar{\eta}\langle d(S)e^{-g}(S)1, z^n\rangle \;\leq\; 0.$$

As this holds for all $\zeta \in \mathbb{T}$, we get that

$$d \in \mathcal{P}_n^\bullet \text{ and } \langle d(S)\gamma, \gamma\rangle = 0 \;\Rightarrow\; \langle d(S)e^{-g}(S)1, z^n\rangle = 0. \tag{5.3}$$

Equivalently,

$$\sum_{k=0}^{n} \hat{d}(k)\langle S^k\gamma, \gamma\rangle = 0 \;\Rightarrow\; \sum_{k=0}^{n} \hat{d}(k)\langle S^k e^{-g}(S)1, z^n\rangle = 0.$$

By duality (in the finite dimensional space $\mathcal{P}_n^\bullet$), this means there exists $c \in \mathbb{C}$ so that

$$\langle S^k e^{-g}(S)1, z^n\rangle \;=\; c\,\langle S^k\gamma, \gamma\rangle, \quad 0 \leq k \leq n. \tag{5.4}$$

Letting $k=0$ in (5.4), we get $c=\eta$. So for $0 \le k \le n$, (5.4) gives

$$\begin{aligned}
\widehat{e^{-g}}(n-k) &= \langle S^k e^{-g}(S)1, z^n\rangle \\
&= \eta\langle S^k\gamma, \gamma\rangle \\
&= \eta\langle z^k\gamma, \gamma\rangle_{H^2} \\
&= \eta\int_0^{2\pi} e^{ik\theta}\gamma(e^{i\theta})\overline{\gamma(e^{i\theta})}\frac{d\theta}{2\pi} \\
&= \eta\int_0^{2\pi} e^{ik\theta}\gamma(e^{i\theta})[e^{-in\theta}\tilde{\gamma}(e^{i\theta})]\frac{d\theta}{2\pi} \\
&= \eta\widehat{\gamma\tilde{\gamma}}(n-k).
\end{aligned}$$

Therefore $e^{-g} \overset{n}{\sim} \eta\gamma\tilde{\gamma}$, as desired.

Conversely, suppose (5.2) holds. Reversing the logic, we get that (5.3) holds. This means that on the n-dimensional subspace of $\mathcal{P}_n^\bullet$ given by

$$\{d : \langle d(S)\gamma, \gamma\rangle = 0\}, \tag{5.5}$$

we have

$$\frac{d}{dt}F(g+td)\Big|_{t=0} = 0.$$

The orthocomplement of the subspace (5.5) is spanned by the function $\beta = P_n(|\gamma|^2)$. An arbitrary polynomial in $\mathcal{P}_n^\bullet$ can be written as $d + a\beta$, where d is in (5.5) and $a \in \mathbb{C}$. By Lemma 4.8, this is an admissible direction if and only if $\operatorname{Re} a \ge 0$. By Lemma 4.11, and using (5.2),

$$\begin{aligned}
\frac{d}{dt}F(g+t(d+a\beta))\Big|_{t=0^+} &= -2\operatorname{Re}\langle z^n, \eta\gamma\tilde{\gamma}\rangle\,[a\langle\beta\eta\gamma\tilde{\gamma}, z^n\rangle + \langle d\eta\gamma\tilde{\gamma}, z^n\rangle] \\
&= -2\operatorname{Re} a|\eta|^2\langle z^n, \gamma z^n\bar{\gamma}\rangle\,\langle P_n(|\gamma|^2)\gamma z^n\bar{\gamma}, z^n\rangle \\
&= -2\operatorname{Re} a|\eta|^2\langle P_n(|\gamma|^2), P_n(|\gamma|^2)\rangle
\end{aligned}$$

and this is less than or equal to 0 whenever $\operatorname{Re} a \ge 0$. This means g is a K_n critical point. Finally,

$$\begin{aligned}
\widehat{e^{-g}}(n) &= \langle \eta\gamma\tilde{\gamma}, z^n\rangle \\
&= \eta\langle\gamma z^n\bar{\gamma}, z^n\rangle \\
&= \eta.
\end{aligned}$$

□

6 Normalization

If g is as in (1.7) and is a local maximum for F, then $bi + g(\zeta z)$ is also a local maximum for any unimodular ζ and real b. We can choose ζ and b so that $a = 0$ and $\prod_\ell (-\tau_\ell) = 1$.

Definition 6.1. *If $g \in \mathcal{R}_n$, we say that g is* normalized *if g has the form*

$$g(z) \;=\; \sum_{\ell=1}^{n} w_\ell \frac{\tau_\ell + z}{\tau_\ell - z}, \tag{6.2}$$

where $\prod_{\ell=1}^{n} (-\tau_\ell) = 1$.

Lemma 6.3. *Let $g \in \mathcal{R}_n$ be a local maximum for F with critical value η and let γ be as in Lemma 4.7. Assume $\operatorname{Im} g(0)$ is chosen in the range $[-\pi, \pi)$. Then g is normalized if and only if $\eta > 0$ and γ is self-inversive.*

Proof: We have

$$\prod_\ell (z - \tau_\ell) \;=\; \prod_{\ell=1}^{n} (-\tau_\ell) \; [\prod_\ell (z - \tau_\ell)]\,\tilde{}\,.$$

So by Lemma 4.7, $\gamma = \tilde{\gamma}$ if and only if $\prod_\ell (-\tau_\ell) = 1$. From (1.7), we have

$$g(0) \;=\; ai + \sum_{\ell=1}^{n} w_\ell,$$

and from Theorem 5.1, we have

$$e^{-g(0)} \;=\; \eta \gamma(0) \tilde{\gamma}(0) \;=\; \frac{\eta}{\nu^2} \prod_\ell (-\tau_\ell).$$

So if $\prod_\ell (-\tau_\ell) = 1$, then η is positive if and only if $\operatorname{Im} g(0)$ is a multiple of 2π. $\Box$

Proposition 6.4. *If g is a normalized local maximum for F with critical value η then*

$$-g \overset{n}{\sim} \log \frac{\eta}{\nu^2} + \log \left(\prod_{\ell=1}^{n} (1 - \bar{\tau}_\ell z)^2 \right). \tag{6.5}$$

PROOF: Since g is normalized, we have

$$\gamma(z)\tilde{\gamma}(z) \;=\; \frac{1}{\nu^2}\prod_{\ell=1}^{n}(z-\tau_\ell)\prod_{\ell=1}^{n}(1-\bar{\tau}_\ell z) \;=\; \frac{1}{\nu^2}\prod_{\ell=1}^{n}(1-\bar{\tau}_\ell z)^2.$$

So from Theorem 5.1, we have

$$e^{-g} \stackrel{n}{\sim} \frac{\eta}{\nu^2}\prod_{\ell=1}^{n}(1-\bar{\tau}_\ell z)^2.$$

Then (6.5) follows from Lemma 3.1. □

Proposition 6.6. *If g is a normalized local maximum for F with critical value η then*

$$\sum_{\ell=1}^{n} w_\ell \;=\; -\log\frac{\eta}{\nu^2}.$$

For $k = 1, \ldots, n$,

$$\sum_{\ell=1}^{n} w_\ell \tau_\ell^k \;=\; \frac{1}{k}\sum_{\ell=1}^{n}\tau_\ell^k.$$

PROOF: Expand both sides of (6.5) into power series and equate coefficients. From (6.2),

$$-g(z) \;=\; -\sum_{\ell=1}^{n} w_\ell - 2\sum_{k=1}^{\infty}\left(\sum_{\ell=1}^{n} w_\ell \bar{\tau_\ell}^k\right) z^k.$$

We have

$$\log\left(\prod_{\ell=1}^{n}(1-\bar{\tau}_\ell z)^2\right) \;=\; -2\sum_{k=1}^{\infty}\left(\sum_{\ell=1}^{n}\frac{\bar{\tau_\ell}^k}{k}\right) z^k.$$

Comparing these we get the result. □

7 The proof of Theorem 1.8

Let us assume that the Entropy Conjecture 1.4 holds, and that $g \in \mathcal{R}_n$ is a normalized local maximum for F with critical value η. Let γ be as in Lemma 4.7.

By Proposition 4.6, g is a critical point for K_n, and by Theorem 5.1 and Proposition 6.4, the $(n+1)$-by-$(n+1)$ Toeplitz matrix $-\mathrm{Re}\, g(S_n)$ is the same as the Toeplitz matrix on $\mathcal{P}_n^\bullet$ whose entries come from the Fourier series of $\log \eta + \log |\gamma|^2$. In particular, for any polynomials $p, q \in \mathcal{P}_n^\bullet$, we have

$$\langle -\mathrm{Re}\, g(S_n)p, q\rangle \;=\; \frac{1}{2\pi}\int_0^{2\pi} [\log \eta + \log |\gamma(e^{i\theta})|^2] p(e^{i\theta})\overline{q(e^{i\theta})} d\theta. \qquad (7.1)$$

Indeed, if we let $p(e^{i\theta}) = e^{i\ell\theta}$ and $q(e^{i\theta}) = e^{ij\theta}$, for ℓ and j between 0 and n, then the left-hand side of (7.1) is

$$-\frac{1}{2}\sum_{k=0}^{n} \langle \hat{g}(k) z^{\ell+k} + \overline{\hat{g}(k)} z^{\ell-k}, z^j\rangle,$$

and the right-hand side is

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}(g(e^{i\theta}) + \overline{g(e^{i\theta})})\, e^{i\ell\theta} e^{-ij\theta} d\theta,$$

and these are equal.

Let $p = q = \gamma$, and observe that the left-hand side of (7.1) vanishes, so

$$\frac{1}{2\pi}\int_0^{2\pi} |\gamma(e^{i\theta})|^2 \log |\gamma(e^{i\theta})|^2] d\theta \;=\; -\log \eta.$$

If (1.5) of Conjecture 1.4 holds, then

$$-\log \eta \geq 1 - \log 2,$$

and so $\eta \leq \frac{2}{e}$. If the uniqueness part of the Conjecture also holds, then γ must have equally spaced zeroes, which would in turn imply uniqueness in Conjecture{refconk. $\square$

8 Entropy Conjecture

Let us establish some notation. We shall fix $n \geq 1$ a positive integer. All integrals are integrals over the unit circle with respect to normalized Lebesgue measure, and norms and inner products are in $L^2(\mathbb{T})$ with respect to this measure.

If p is self-inversive, then all its zeroes either occur on $\mathbb{T}$, the unit circle, or occur in pairs $(\zeta, 1/\bar{\zeta})$, or occur at the origin if $\deg(p) < n$. We shall let $\mathcal{P}_n^{\mathbb{T}}$ denote the set of polynomials in $\mathcal{P}_n$ that are self-inversive and have all their zeroes on $\mathbb{T}$, and $\mathcal{P}_{n;1}^{\mathbb{T}}$ denote the unit sphere of $\mathcal{P}_n^{\mathbb{T}}$, *viz.* the polynomials of norm 1 in $\mathcal{P}_n^{\mathbb{T}}$.

We shall let Π_n be the orthogonal projection from $L^2(\mathbb{T})$ onto $\mathcal{P}_n^{\bullet}$, *i.e.*

$$\Pi_n(\sum_{-\infty}^{\infty} c_k z^k) \;=\; \sum_{0}^{n} c_k z^k.$$

If f, g are in $L^2(\mathbb{T})$, we shall write $f \sim g$ to mean $\Pi_n f = \Pi_n g$, *i.e.*

$$f \sim g \;\Leftrightarrow\; \hat{f}(k) = \hat{g}(k) \;\forall\; 0 \leq k \leq n.$$

Note that if $p, q \in \mathcal{P}_n^{\bullet}$, then

$$\langle \tilde{p}, \tilde{q} \rangle \;=\; \langle q, p \rangle;$$

in particular if p and q are both self-inversive, then their inner product is real.

If $p \in \mathcal{P}_n^{\bullet}$, then $p^\sharp$ denotes the polynomial

$$p^\sharp(z) = i\left(p(z) - \frac{2}{n} z p'(z)\right).$$

In terms of Fourier coefficients,

$$\hat{p^\sharp}(k) \;=\; i\left(\frac{n-2k}{n}\right)\hat{p}(k).$$

Lemma 8.1. *If p is in $\mathcal{P}_n^{\mathbb{T}}$, then so is $p^\sharp$.*

Proof: A calculation shows that for p self-inversive

$$\frac{d}{d\theta}|p(e^{i\theta})|^2 \;=\; -n\bar{z}^n p(z) p^\sharp(z). \tag{8.2}$$

So $p^\sharp$ has zeroes at the local maxima of $|p|^2$ on $\mathbb{T}$; these interleave the zeroes of p. If p has a zero of order $k > 1$ at some point τ on $\mathbb{T}$, then (8.2) vanishes to order $2k-1$ at τ, so $p^\sharp$ has a zero of order $k-1$. Counting them all up, we get that $p^\sharp$ has n zeroes on $\mathbb{T}$, and since it is of degree n, it must be in $\mathcal{P}_n^{\mathbb{T}}$. □

Observe that $\langle p, p^\sharp \rangle = 0$ for all self-inversive p in $\mathcal{P}_n$.

Let

$$\mathcal{F}(p) \;=\; \int |p|^2 \log |p|^2.$$

If γ is norm one and $\mathcal{F}(\gamma) = m$, then minimizing $\mathcal{F}(c\gamma)$ over all $c \geq 0$, one gets that

$$\mathcal{F}(c\gamma) \;\geq\; -e^{-1-m}, \tag{8.3}$$

with equality when $c^2 = e^{-1-m}$. The entropy conjecture 1.4 is equivalent to the conjecture that the minimum of $\mathcal{F}(p)$ over all $p \in \mathcal{P}_n^{\mathbb{T}}$ (not just those of norm one) is $-2e^{-2}$, and that, up to the normalization of requiring that $\hat{p}(0)$ and $\hat{p}(n)$ are positive, this value is attained uniquely by the polynomial

$$p(z) \;=\; e^{-1}(1 + z^n). \tag{8.4}$$

For any function f in L^2, we shall let $\big[f\big]$ denote the $(n+1)$-by-$(n+1)$ Toeplitz matrix with (i,j) entry $\hat{f}(j-i)$. We shall think of this as acting on $\mathcal{P}_n^\bullet$. If $p(z) = c\prod_{\ell=1}^n (1 - \bar{\tau}_\ell z)$, then

$$\big[\log |p|^2\big] \;=\; \begin{bmatrix} \log|c|^2 & -\sum \tau_\ell & -\frac{1}{2}\sum \tau_\ell^2 & \cdots & -\frac{1}{n}\sum \tau_\ell^n \\ -\sum \bar{\tau}_\ell & \log|c|^2 & -\sum \tau_\ell & \cdots & -\frac{1}{n-1}\sum \tau_\ell^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\frac{1}{n}\sum \bar{\tau}_\ell{}^n & -\frac{1}{n-1}\sum \bar{\tau}_\ell{}^{n-1} & -\frac{1}{n-2}\sum \bar{\tau}_\ell{}^{n-2} & \cdots & \log|c|^2 \end{bmatrix}$$

Theorem 8.5. *Suppose γ is a local minimum for $\mathcal{F}$ on $\mathcal{P}_n^{\mathbb{T}}$, and that all the zeroes of γ are distinct. Then*

$$\big[\log |\gamma|^2\big]\gamma \;=\; -\gamma. \tag{8.6}$$

$$\big[\log |\gamma|^2\big] \;\geq -3. \tag{8.7}$$

PROOF: Since all the zeroes of γ are distinct, it q is any self-inversive polynomial in $\mathcal{P}_n^\bullet$, then for t small and real, $\gamma + tq$ is self-inversive, and the zeroes must be close to the zeroes of γ, so they must all lie on the circle. Therefore if we expand $\mathcal{F}(\gamma + tq)$ in powers of t, the first order term must vanish, since γ is a critical point, and the coefficient of t^2 must be non-negative, since γ is a local minimum.

Calculating, using the fact that if p is self-inversive, then on the unit circle $z^n\overline{p(z)} = p(z)$, and writing

$$\log|\gamma + tq|^2 = \log|\gamma|^2 \ + \ 2\mathrm{Re}\, t\frac{q}{\gamma} \ - \ \mathrm{Re}\, t^2\frac{q^2}{\gamma^2} \ + \ O(t^3),$$

we get,

$$\begin{aligned}
\mathcal{F}(\gamma + tq) \ &= \ \int \big(\log|\gamma|^2 + 2\mathrm{Re}\, t\frac{q}{\gamma} - \mathrm{Re}\, t^2\frac{q^2}{\gamma^2}\big)\big(|\gamma|^2 + 2\mathrm{Re}\, t\gamma\bar{q} + t^2|q|^2\big) \\
&\quad + O(t^3) \\
&= \ \mathcal{F}(\gamma) \ + \ t\Big(2\mathrm{Re}\,\langle\big[\log|\gamma|^2\big]\gamma, q\rangle + 2\mathrm{Re}\,\langle\gamma, q\rangle\Big) \\
&\quad + t^2\Big(\langle\big[\log|\gamma|^2\big]q, q\rangle + 4\langle q, q\rangle - \langle q, q\rangle\Big) \ + \ O(t^3).
\end{aligned}$$

Since at a critical point the coefficient of t must vanish for all q, we get $\big[\log|\gamma|^2\big]\gamma + \gamma = 0$, giving (8.6). The non-negativity of the coefficient of t^2 gives (8.7). □

At a critical point, $\big[\log|\gamma|^2\big]$ will have one eigenvalue equal to -3, so the inequality in (8.7) cannot be strict.

Proposition 8.8. *Suppose γ is in $\mathcal{P}_n^{\mathbb{T}}$ and*

$$\big[\log|\gamma|^2\big]\gamma = \kappa\gamma. \tag{8.9}$$

Then

$$\big[\log|\gamma|^2\big]\gamma^\sharp \ = \ (\kappa - 2)\,\gamma^\sharp. \tag{8.10}$$

Proof: Equation (8.9) can be written as

$$\big(\log|\gamma(e^{i\theta})|^2\big)\gamma(e^{i\theta}) \ \sim \ \kappa\gamma(e^{i\theta}). \tag{8.11}$$

Differentiate both sides with respect to θ. Writing γ' for the derivative with respect to z, then (8.11) becomes

$$\frac{iz\gamma'}{\gamma}\gamma \ - \ \frac{i\bar{z}\bar{\gamma}'}{\bar{\gamma}}\gamma + \log|\gamma|^2(iz\gamma') \ \sim \ \kappa(iz\gamma'). \tag{8.12}$$

If we differentiate the equation $z^n\bar{\gamma} = \gamma$ with respect to θ, and use the fact that $\frac{d}{d\theta}\bar{\gamma} = -i\bar{z}\bar{\gamma}'$, then (8.12) becomes

$$\begin{aligned} \log|\gamma|^2(iz\gamma') &\sim \kappa(iz\gamma') - iz\gamma' + iz^{n-1}\bar{\gamma}' \\ &= \kappa(iz\gamma') - iz\gamma' + in\gamma - iz\gamma'. \end{aligned}$$

Therefore

$$\begin{aligned} \log|\gamma|^2(i\gamma - \frac{2i}{n}z\gamma') &\sim i\kappa\gamma - \frac{2i}{n}\left(\kappa(z\gamma') - z\gamma' + n\gamma - z\gamma'\right) \\ &= i(\kappa - 2)\left(\gamma - \frac{2}{n}z\gamma'\right). \end{aligned}$$

This yields (8.10). □

It is plausible that the only polynomial satisfying (8.6) and (8.7) and with positive 0^{th} and n^{th} coefficients is (8.4), but we cannot resolve whether this is true.

9 A special case of the Entropy Conjecture

Self-inversive polynomials p in $\mathcal{P}_n$ can be written as

$$p = q + \tilde{q}, \tag{9.1}$$

where q is a polynomial in $\mathcal{P}_m$, with $m = \lfloor \frac{n}{2} \rfloor$. Specifically, if n is odd, then q is an arbitrary polynomial in $\mathcal{P}_{\frac{n-1}{2}}$, and defined by

$$q = \Pi_{\frac{n-1}{2}}\, p;$$

if n is even, then q is a polynomial in $\mathcal{P}_{\frac{n}{2}}$ whose $\left(\frac{n}{2}\right)^{\text{th}}$ coefficient is real (and half of the coefficient for p).

Theorem 9.2. *Let p be a self-inversive non-constant polynomial of degree n, normalized as in Conjecture 1.4 to have L^2 norm one, and write p as in* (9.1)*. If q has no zeroes in the closed unit disk, then Inequality 1.5 holds, with strict inequality unless p is given by* (1.6).

PROOF: Let us decompose the integral into two pieces, $I + II$:

$$\int |p|^2 \log|p|^2 = \int |p|^2 \log|q|^2 + \int |p|^2 \log|1 + \tilde{q}/q|^2. \tag{9.3}$$

To estimate I, the first term on the right-hand side of (9.3), write

$$\int |p|^2 \log |q|^2 \ = \ \int |q|^2 \log |q|^2 ||1 + \tilde{q}/q|^2.$$

Note that

$$\int |1 + \tilde{q}/q|^2 \ = \ \int 2 + 2\mathrm{Re}\, \frac{\tilde{q}}{q} \ = \ 2,$$

since $\tilde{q}(0) = 0$. So if we apply Jensen's inequality to the convex function $\Phi(x) = x \log x$ and the probability measure $\frac{1}{2}|1 + \tilde{q}/q|^2$, we get

$$\int \Phi(|q|^2) \frac{1}{2} |1 + \tilde{q}/q|^2 \ \geq \ \Phi\left(\int |q|^2 \frac{1}{2} |1 + \tilde{q}/q|^2 \right).$$

This gives

$$\frac{1}{2} \int |q|^2 \log |q|^2 ||1 + \tilde{q}/q|^2 \ \geq \ \Phi(\frac{1}{2}) \ = \ -\frac{1}{2} \log 2.$$

Therefore we have $I \geq -\log 2$.

To estimate II, first assume that n is odd. Note that by the maximum principle, $\frac{\tilde{q}}{q}$ has modulus less than one in the unit disk, so $\log(1 + \tilde{q}/q)$ is analytic on the unit disk and has only logarithmic singularities on the unit circle, and so is in the Hardy space, and therefore its Fourier series agrees with its Maclaurin series. Therefore

$$\log(1 + \tilde{q}/q) \ = \ \tilde{q}/q + O(z^{n+1}),$$

so

$$\begin{aligned} \int |p|^2 \log |1 + \tilde{q}/q|^2 \quad &= \quad 2\mathrm{Re} \int |q + \tilde{q}|^2 \tilde{q}/q \\ &= \quad 2\mathrm{Re} \int (2|q|^2 + z^n \bar{q}^2 + \bar{z}^n q^2)(z^n \bar{q}/q) \\ &= \quad 2\mathrm{Re} \int 2\bar{q}^2 z^n + z^{2n} \bar{q}^3/q + |q|^2 \\ &= \quad 2\mathrm{Re} \int |q|^2 \\ &= \quad 1. \end{aligned}$$

Now assume $n = 2m$ is even. Write

$$q(z) \;=\; a_0 + \cdots + a_m z^m,$$

so

$$\tilde{q}(z) \;=\; a_m z^m + \bar{a}_{m-1} z^{m+1} + \cdots + \bar{a}_0 z^n.$$

When expanding $\log(1 + \tilde{q}/q)$ we get

$$\log(1 + \tilde{q}/q) \;=\; \frac{\tilde{q}}{q} - \frac{1}{2}\frac{a_m^2}{a_0^2} z^n + O(z^{n+1}).$$

Therefore

$$\begin{aligned}\int |p|^2 \log|1 + \tilde{q}/q|^2 \;&=\; 2\mathrm{Re}\int 2\bar{q}^2 z^n + z^{2n}\bar{q}^3/q + |q|^2 - \frac{1}{2}a_m^2 \\ &=\; 3a_m^2 + 1 \\ &\geq\; 1.\end{aligned}$$

Therefore

$$I + II \;\geq\; 1 - \log 2,$$

as required.

Finally, note that the inequality for I using Jensen's inequality is strict unless $|q|$ is constant. □

Note that a simple continuity argument applied to $q(rz)$ shows that (1.5) holds provided q has no zeroes in the open unit disk.

10 Baernstein's conjecture implies the entropy conjecture

Assume Baernstein's conjecture 2.2 holds. Let $Q(z) = (1 + z^n)/\sqrt{2}$ and p be any non-constant polynomial with all its roots on the unit circle, and with $\|p\|_2 = 1$. Let $s \leq 2$, and let $t = \frac{s}{s-1}$ be the conjugate exponent. Then, taking logarithms of (2.3) we have

$$\log\|p\|_s - \log\|p\|_t \;\leq\; \log\|Q\|_s - \log\|Q\|_t.$$

This means

$$\frac{1}{s}\log\int|Q|^s-\frac{s-1}{s}\log\int|Q|^{s/(s-1)}-\frac{1}{s}\log\int|p|^s+\frac{s-1}{s}\log\int|p|^{s/(s-1)}\geq 0. \tag{10.1}$$

Let $\Psi(s)$ denote the left-hand side of (10.1). We have that $\Psi(s)\geq 0$ for $s\leq 2$, and $\Psi(2)=0$. Therefore $\Psi'(2)\leq 0$. Calculating, we get

$$\begin{aligned}\Psi'(s) \quad &= \quad -\frac{1}{s^2}\log\int|Q|^s+\frac{1}{s}\frac{1}{\int|Q|^s}\int|Q|^s\log|Q|-\frac{1}{s^2}\log\int|Q|^{\frac{s}{s-1}}\\ &\quad +\frac{1}{s(s-1)}\frac{1}{\int|Q|^{\frac{s}{s-1}}}\int|Q|^{\frac{s}{s-1}}\log|Q|\\ &\quad +\frac{1}{s^2}\log\int|p|^s-\frac{1}{s}\frac{1}{\int|p|^s}\int|p|^s\log|p|+\frac{1}{s^2}\log\int|p|^{\frac{s}{s-1}}\\ &\quad -\frac{1}{s(s-1)}\frac{1}{\int|p|^{\frac{s}{s-1}}}\int|p|^{\frac{s}{s-1}}\log|p|\end{aligned}$$

Since both p and Q have 2-norm 1, we get that

$$\Psi'(2) \;=\; \int|Q|^2\log|Q|-\int|p|^2\log|p|.$$

Since $\Psi'(2)\leq 0$, we get

$$\int|p|^2\log|p|^2 \;\geq\; \int|Q|^2\log|Q|^2 \;=\; 1-\log 2,$$

which is (1.5). □

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