Wandering Montel Theorems for Hilbert Space Valued Holomorphic Functions

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Abstract: We prove that if $\{u^k\}$ is a sequence of holomorphic functions that takes values in an infinite dimensional Hilbert space $\mathcal{H}$, there are unitaries $\{U^k\}$ on $\mathcal{H}$ so that $U^k u^k$ has a subsequence that converges locally uniformly. We also prove a non-commutative version of this result.

1 Introduction

1.1 Commutative Theory

Let Ω be an open set in $\mathbb{C}^d$ and assume that $\{u^k\}$ is a sequence in $\mathrm{Hol}(\Omega)$, the algebra of of holomorphic functions on Ω equipped with the topology of uniform convergence on compact subsets. The classical Montel Theorem asserts that if $\{u^k\}$ is locally uniformly bounded on Ω, then there exists a subsequence $\{u^{k_l}\}$ that converges in $\mathrm{Hol}(\Omega)$.

It is well known that if $\mathcal{X}$ is an infinite dimensional Banach space, then Montel's Theorem breaks down for $\mathrm{Hol}_{\mathcal{X}}(\Omega)$, the space of $\mathcal{X}$-valued holomorphic functions, see *e.g.* [5, 14]. For example, if $\mathcal{X} = \ell^2$ and $\{f^k\}$ is a locally uniformly bounded sequence of holomorphic functions on Ω, then the sequence

$$\begin{pmatrix} f^1(\lambda) \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ f^2(\lambda) \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ f^3(\lambda) \\ \vdots \end{pmatrix}, \dots$$

is a locally uniformly bounded sequence that will have a convergent subsequence only if there exists a subsequence $\{f^{k_l}\}$ that converges uniformly to 0 on Ω.

Observe that the problem in the example given above is that while for all $\lambda \in \Omega$, u^k converges weakly to 0, it needn't be the case that $u^k(\lambda)$ converges in norm for any $\lambda \in \Omega$. However, just as in the case of the classical proof of Montel's Theorem that uses the Arzela-Ascoli Theorem, if one assumes that $\{u^k\}$ is well behaved pointwise on a large enough set, then one *can* conclude uniform convergence in norm on compact sets. For example, consider the following theorem by Arendt and Nikolski [5, Cor. 2.3]

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Theorem 1.1. Let Ω be an open connected set in $\mathbb{C}$, and let u^k be a sequence in $\mathrm{Hol}_{\mathcal{X}}(\Omega)$ that is locally bounded. Assume that

$$\Omega_0 := \{z \in \Omega : \{u^k(z) : k \in \mathbb{N}\} \text{ is relatively compact in } X\}$$

has an accumulation point in Ω. Then there exists a subsequence which converges to a holomorphic function uniformly on compact subsets of Ω.

Theorem 1.1 deals with the difficulty by making strong additional assumptions about the pointwise behavior of $\{u^k\}$, assumptions that may not hold in desirable applications. The central idea of this paper, for Hilbert space valued functions, is instead to use a sequence of unitaries to push (most of) the range of the functions into a finite-dimensional space. Here is our first main result.

Theorem 1.2. If Ω is an open set in $\mathbb{C}^d$, $\mathcal{H}$ is a Hilbert space, and $\{u^k\}$ is a locally uniformly bounded sequence in $\mathrm{Hol}_{\mathcal{H}}(\Omega)$, then there exists a sequence $\{U^k\}$ of unitary operators on $\mathcal{H}$ such that $\{U^k u^k\}$ has a subsequence that converges in $\mathrm{Hol}_{\mathcal{H}}(\Omega)$.

We prove Theorem 1.2 in Section 2. In Sections 3 and 4 we consider versions for non-commutative functions. These functions have been extensively studied recently—see *e.g.* [17, 8, 4, 10, 11, 13, 15, 6, 12, 9]. Before stating our results, we must spend a little time explaining some definitions.

1.2 Non-commutative theory

In commutative analysis, one studies holomorphic functions defined on domains in $\mathbb{C}^d$. In noncommutative analysis one studies holomorphic functions defined on domains in $\mathbb{M}^d$, the *d-dimensional nc universe*. For each n we let $\mathbb{M}_n^d$ denote the set of d-tuples of $n \times n$ matrices. We then let

$$\mathbb{M}^d = \bigcup_{n=1}^{\infty} \mathbb{M}_n^d.$$

When E is a subset of $\mathbb{M}^d$, then for each n, we adopt the notation

$$E_n = E \cap \mathbb{M}_n^d.$$

In noncommutative analysis one studies *graded* functions, i.e., functions f defined on subsets E of $\mathbb{M}^d$, that satisfy

$$\forall_n \;\; \forall_{\lambda \in E_n} \;\; f(\lambda) \in \mathbb{M}_n. \tag{1.3}$$

$\mathbb{M}^d$ carries a topology, the so-called *finite topology*[1], wherein a set Ω is deemed to be open precisely when

$$\forall_n \;\; \Omega_n \text{ is open in } \mathbb{M}_n^d.$$

With this definition, note that a graded function $f : E \to \mathbb{M}^1$ is finitely continuous if and only if $f|E_n$ is continuous for each n and also that a set $K \subseteq \mathbb{M}^d$ is finitely compact if and only if there exists n such that $E_m = \varnothing$ when $m > n$ and E_m is compact when $m \leq n$.

If Ω is finitely open in $\mathbb{M}^d$, then for each n, Ω_n can be identified with an open set in $\mathbb{C}^{dn^2}$ in an obvious way. If, in addition, f is a graded function on Ω, then we say that f is *holomorphic on* Ω if for each n, $f|\Omega_n$ is a holomorphic mapping of Ω_n into $\mathbb{M}_n$. We let $\mathrm{Hol}(\Omega)$ denote the collection of graded holomorphic functions.

[1]Subsequently, we shall consider other topologies as well.

It is also possible to consider $\mathcal{H}$-valued holomorphic functions in the noncommutative setting. One particularly concrete way to do this is to realize in the scalar case just considered that (1.3) is equivalent to asserting that

$$\forall_n \;\; \forall_{\lambda \in E_n} \;\; f(\lambda) \in \mathcal{B}(\mathbb{C}^n, \mathbb{C}^n).$$

We therefore replace the former definition (that f be graded) with the requirement that f be a *graded $\mathcal{H}$-valued function*, i.e., that

$$\forall_n \;\; \forall_{\lambda \in E_n} \;\; f(\lambda) \in \mathcal{B}(\mathbb{C}^n, \mathbb{C}^n \otimes \mathcal{H}).$$

Just as before, we declare a graded $\mathcal{H}$-valued function defined on a finitely open set Ω in $\mathbb{M}^d$ to be holomorphic if for each n, $f|\Omega_n$ is a holomorphic mapping of Ω_n into $\mathcal{B}(\mathbb{C}^n, \mathbb{C}^n \otimes \mathcal{H})$. We let $\mathrm{Hol}_{\mathcal{H}}(\Omega)$ denote the collection of graded $\mathcal{H}$-valued functions and view $\mathrm{Hol}_{\mathcal{H}}(\Omega)$ as a complete metric space endowed with the topology of uniform convergence on finitely compact subsets of Ω.

A special class of graded functions arise by formalizing certain algebraic properties of free polynomials. If $E \subseteq \mathbb{M}^d$ we say that E is an *nc-set* if E is closed with respect to direct sums. We define the class of *nc-functions* as follows.

Definition 1.4. Let $\mathcal{H}$ be a Hilbert space, E an nc-set, and assume that f is a function defined on E. We say that f is an *$\mathcal{H}$-valued nc-function on E* if the following conditions hold.

(i) f is *$\mathcal{H}$-graded*, i.e.,

$$\forall_n \;\; \forall_{\lambda \in E \cap \mathbb{M}_n} \;\; f(\lambda) \in \mathcal{B}(\mathbb{C}^n, \mathbb{C}^n \otimes \mathcal{H}).$$

(ii) f *preserves direct sums*, i.e.,

$$\forall_{\lambda, \mu \in E} \;\; \lambda \oplus \mu \in E \implies f(\lambda \oplus \mu) = f(\lambda) \oplus f(\mu).$$

In this formula, if $\lambda \in E_m$ and $\mu \in E_m$, we identify $\mathbb{C}^m \oplus \mathbb{C}^n$ and $\mathbb{C}^{m+n}$ and identify $(\mathbb{C}^m \otimes \mathcal{H}) \oplus (\mathbb{C}^n \otimes \mathcal{H})$ and $\mathbb{C}^{m+n} \otimes \mathcal{H}$.

(iii) f *preserves similarity*, i.e.,

$$f(S\lambda S^{-1}) = (S \otimes \mathrm{id}_{\mathcal{H}}) f(\lambda) S^{-1}$$

whenever $n \geq 1$, $S \in \mathbb{M}_n$ is invertible, and both λ and $S\lambda S^{-1}$ are in E_n.

When $f : E \to \mathbb{M}^1 \otimes \mathcal{H}$ is an nc-function and E is a finitely open nc-set then Condition (iii) above becomes very strong and yields the following proposition which lies at the heart of nc analysis (see [10] or [13, Thm. 7.2]). We say a function f is bounded on E if $\sup_{\lambda \in E} \|f(\lambda)\| < \infty$.

Proposition 1.5. Let Ω be a finitely open nc-set. If f is a bounded nc-function defined on Ω, then f is holomorphic on Ω.

Proposition 1.5 suggests the following terminology. We say that a set $\Omega \subseteq \mathbb{M}^d$ is an *nc-domain* if Ω is a finitely open nc-set and we say that a topology τ on $\mathbb{M}^d$ is an *nc-topology* if τ has a basis consisting of nc-domains. We then define special classes of functions in noncommuting variables as follows.

Definition 1.6. Let $\Omega \subseteq \mathbb{M}_n^d$, τ be an nc-topology, and assume that $f : \Omega \to \mathbb{M}^1 \otimes \mathcal{H}$ is an $\mathcal{H}$-valued function. We say that f is *τ-holomorphic* if f is a τ-locally bounded nc function on Ω. [2] We let $\mathrm{Hol}^{\tau}_{\mathcal{H}}(\Omega)$ denote the collection of τ-holomorphic $\mathcal{H}$-valued functions defined on Ω.

[2] i.e., f is an nc-function on Ω in the sense of Definition 1.4 and for each $\lambda \in \Omega$, there exists $B \subseteq \Omega$ such that $\lambda \in B \in \tau$ and $f|B$ is bounded.

Evidently, Proposition 1.5 guarantees that if τ is an nc-topology, and f is a τ-holomorphic function in the sense of Definition 1.6, then f is holomorphic, i.e.,

$$\mathrm{Hol}^{\tau}_{\mathcal{H}}(\Omega) \subseteq \mathrm{Hol}^{\mathrm{nc}}_{\mathcal{H}}(\Omega) \subseteq \mathrm{Hol}_{\mathcal{H}}(\Omega),$$

where $\mathrm{Hol}^{\mathrm{nc}}_{\mathcal{H}}(\Omega)$ denotes the set of functions in $\mathrm{Hol}_{\mathcal{H}}(\Omega)$ that are nc.

We can now state our second main result, the non-commutative version of Theorem 1.2.

Theorem 1.7. Assume that τ is an nc-topology, $\Omega \in \tau$, $\mathcal{H}$ is a Hilbert space, and $\{u^k\}$ is a τ-locally uniformly bounded sequence in $\mathrm{Hol}^{\tau}_{\mathcal{H}}(\Omega)$. There exist $u \in \mathrm{Hol}^{\tau}_{\mathcal{H}}(\Omega)$, a sequence $\{U^k\}$ of unitary operators on $\mathcal{H}$, and an increasing sequence of indices $\{k_l\}$ such that $(\mathrm{id}_n \otimes U^{k_l})\, u^{k_l} \to u$ in $\mathrm{Hol}(\Omega)$.

As an application of Theorem 1.7, in Section 5 we prove that the cones

$$\mathcal{P} = \{u(\mu)^* u(\lambda) \,:\, u \in \mathrm{Hol}_{\mathcal{H}}(\Omega) \text{ for some Hilbert space } \mathcal{H}\}$$

and

$$\mathcal{C} \;=\; \left\{ \begin{matrix} \mathrm{id}_{\mathbb{C}^J} \\ \otimes \\ u(\mu)^* \end{matrix} \left(\mathrm{id} - \begin{matrix} \delta(\mu)^*\delta(\lambda) \\ \otimes \\ \mathrm{id}_{\mathcal{H}} \end{matrix} \right) \begin{matrix} \mathrm{id}_{\mathbb{C}^J} \\ \otimes \\ u(\lambda) \end{matrix} \;:\; u \in \mathrm{Hol}_{\mathcal{H}}(B_\delta) \text{ and } u \text{ is nc} \right\}$$

are closed. In this last formula, δ is a J-by-J matrix of free polynomials, and $B_\delta = \{x : \|\delta(x)\| < 1\}$ is a non-commutative polynomial polyhedron. (We adopt the convention of [16] and write the tensors vertically for legibility.)

Proving that the cones are closed is the key step in proving realization formulas for free holomorphic functions—see [2, 1, 7].

In Section 6 we show that the assumptions in Proposition 3.2 below can be weakened to just requiring convergence on a set of uniqueness, which yields a graded version of the Arendt-Nikolski theorem.

2 A Montel Theorem for Hilbert Space Valued Holomorphic Functions

In this section we prove Theorem 1.2 from the introduction.

2.1 Notation and Definitions

If Ω is an open set in $\mathbb{C}^d$, $\mathcal{H}$ is a Hilbert space, we let $\mathrm{Hol}_{\mathcal{H}}(\Omega)$ denote the space of holomorphic $\mathcal{H}$-valued functions on Ω. If $u \in \mathrm{Hol}_{\mathcal{H}}(\Omega)$ and $E \subseteq \Omega$, we let

$$\|u\|_E = \sup_{\lambda \in E} \|u(\lambda)\|_{\mathcal{H}}.$$

If $\|u\|_\Omega < \infty$ then we say that u is *bounded on* Ω. If $\{u^k\}$ is sequence in $\mathrm{Hol}_{\mathcal{H}}(\Omega)$, we say that $\{u^k\}$ is *uniformly bounded on* Ω if

$$\sup_k \|u^k\|_\Omega < \infty,$$

and we say that $\{u^k\}$ is *locally uniformly bounded on* Ω if for each $\lambda \in \Omega$, there exists a neighborhood B of λ such that $\{u^k\}$ is uniformly bounded on B. Recall that if such a neighborhood exists, then a Cauchy Estimate implies that $\{u^k\}$ is *equicontinuous at* λ, i.e., for each $\varepsilon > 0$ there exists a ball B_0 such that $\lambda \in B_0 \subseteq B$ and

$$\forall_{\mu \in B_0}\ \forall_k\ \ \|u^k(\mu) - u^k(\lambda)\| < \varepsilon.$$

We equip $\mathrm{Hol}_{\mathcal{H}}(\Omega)$ with the usual topology of *uniform convergence on compacta*. Thus, a sequence $\{u^k\}$ in $\mathrm{Hol}_{\mathcal{H}}(\Omega)$ is *convergent* precisely when there is a function $u \in \mathrm{Hol}_{\mathcal{H}}(\Omega)$ such that

$$\lim_{k\to\infty} \|u^k - u\|_E = 0$$

for every compact $E \subseteq \Omega$. We say that a sequence $\{u^k\}$ in $\mathrm{Hol}_{\mathcal{H}}(\Omega)$ is a *Cauchy sequence* if for each compact $E \subseteq \Omega$, $\{u^k\}$ is *uniformly Cauchy on* E, i.e., for each $\varepsilon > 0$, there exists N such that

$$k, l \geq N \implies \|u^k - u^l\|_E < \varepsilon.$$

It is well known that $\mathrm{Hol}_{\mathcal{H}}(\Omega)$ is *complete*, i.e., every Cauchy sequence in $\mathrm{Hol}_{\mathcal{H}}(\Omega)$ converges. The following result is proved in [5, Thm. 2.1]; we include a proof that easily generalizes to Proposition 3.2.

Proposition 2.1. Assume that Ω is an open set in $\mathbb{C}^d$, $\{\lambda_i\}$ is a dense sequence in Ω, and $\mathcal{H}$ is a Hilbert space. If $\{u^k\}$ is sequence in $\mathrm{Hol}_{\mathcal{H}}(\Omega)$ that is locally uniformly bounded on Ω, and for each fixed i, $\{u^k(\lambda_i)\}$ is a convergent sequence in $\mathcal{H}$, then $\{u^k\}$ is a convergent sequence in $\mathrm{Hol}_{\mathcal{H}}(\Omega)$.

Proof: Fix a compact set $E \subseteq \Omega$ and $\varepsilon > 0$. Note that as $\{u^k\}$ is assumed to be locally uniformly bounded on Ω, $\{u^k\}$ is equicontinuous at each point of Ω. Hence, as E is compact, we may construct a finite collection $\{B_r : r = 1, \ldots, m\}$ of open balls in $\mathbb{C}^d$ such that

$$E \subseteq \bigcup_{r=1}^{m} B_r \subseteq \Omega \tag{2.2}$$

and

$$\forall_r \ \ \forall_{\mu_1,\mu_2\in B_r} \ \ \forall_k \ \ \|u^k(\mu_1) - u^k(\mu_2)\| < \varepsilon/3 \tag{2.3}$$

As $\{\lambda_i\}$ is assumed dense in Ω,

$$\forall_r \ \ \exists_{i_r} \ \ \lambda_{i_r} \in B_r. \tag{2.4}$$

Consequently, as for each fixed i, we assume that $\{u^k(\lambda_i)\}$ is a convergent (and hence, Cauchy) sequence in $\mathcal{H}$, there exists N such that

$$\forall_r \ \ k, j \geq N \implies \|u^k(\lambda_{i_r}) - u^j(\lambda_{i_r})\| < \varepsilon/3. \tag{2.5}$$

Now, fix $\lambda \in E$. Use (2.2) to choose r such that $\lambda \in B_r$. Use (2.4) to choose i_r such that $\lambda_{i_r} \in B_r$. As λ and λ_{i_r} are both in B_r, we see from (2.3) that

$$\forall_k \ \ \|u^k(\lambda) - u^k(\lambda_{i_r})\| < \varepsilon/3.$$

Hence, using (2.5), we have that if $k, j \geq N$, then

$$\begin{aligned}
&\|u^k(\lambda) - u^j(\lambda)\| \\
\leq\ & \|u^k(\lambda) - u^k(\lambda_{i_r})\| + \|u^k(\lambda_{i_r}) - u^j(\lambda_{i_r})\| + \|u^j(\lambda_{i_r}) - u^j(\lambda)\| \\
<\ & \varepsilon.
\end{aligned}$$

Since the concluding estimate in the previous paragraph holds for an arbitrary point $\lambda \in E$, $\{u^k\}$ is uniformly Cauchy on E. Since E is an arbitrary compact subset of Ω, $\{u^k\}$ is a Cauchy sequence in $\mathrm{Hol}_{\mathcal{H}}(\Omega)$. Therefore, $\{u^k\}$ converges in $\mathrm{Hol}_{\mathcal{H}}(\Omega)$.

2.2 The Proof of Theorem 1.2

Theorem 1.2 follows quickly from Proposition 2.1 and the following lemma.

Lemma 2.6. Wandering Isometry Lemma. Assume that Ω is an open set in $\mathbb{C}^d$, $\{\lambda_i\}$ is a sequence in Ω, and $\mathcal{H}$ is a Hilbert space. If $\{u^k\}$ is sequence in $\mathrm{Hol}_{\mathcal{H}}(\Omega)$ that is locally uniformly bounded on Ω, then there exists a subsequence $\{u^{k_l}\}$ and a sequence $\{V^l\}$ of unitary operators on $\mathcal{H}$ such that for each fixed i, $\{V^l u^{k_l}(\lambda_i)\}$ is a convergent sequence in $\mathcal{H}$.

Proof. If $\mathcal{H}$ is finite dimensional, one can let each unitary be the identity, and the result is the regular Montel theorem. So we shall assume that $\mathcal{H}$ is infinite dimensional. Let $\{e_i\}$ be an orthonormal basis for $\mathcal{H}$. For each fixed k let

$$\mathcal{H}^k = \mathrm{span}\,\{e_1, e_2, \dots, e_k\},$$

$$\mathcal{M}_i^k = \mathrm{span}\,\{u^k(\lambda_1), u^k(\lambda_2), \dots, u^k(\lambda_i)\}, \qquad i = 1, \dots, k.$$

For each k choose a unitary $U^k \in \mathcal{B}(\mathcal{H})$ satisfying

$$U^k \mathcal{M}_i^k \subseteq \mathcal{H}^i, \qquad i = 1, \dots, k$$

Observe that with this construction, for each fixed i,

$$\{U^k u^k(\lambda_i)\}_{k=i}^{\infty}$$

is a bounded sequence in $\mathcal{H}^i$, a finite dimensional Hilbert space. Therefore, there exist $v_i \in \mathcal{H}$ and an increasing sequence of indices $\{k_l\}$ such that

$$U^{k_l} u^{k_l}(\lambda_i) \to v_i \quad \text{in} \quad \mathcal{H} \quad \text{as} \quad l \to \infty.$$

Applying this fact successively with $i = 1$, $i = 2$, and so on, at each stage taking a subsequence of the previously selected subsequence, leads to a sequence $\{v_i\}$ in $\mathcal{H}$ and an increasing sequence of indices $\{k_l\}$ such that

$$U^{k_l} u^{k_l}(\lambda_i) \to v_i \quad \text{in} \quad \mathcal{H} \quad \text{as} \quad l \to \infty.$$

for all i. The lemma then follows if we let $V^l = U^{k_l}$. □

Proof of Theorem 1.2. Assume that Ω is an open set in $\mathbb{C}^d$, $\mathcal{H}$ is a Hilbert space, and $\{u^k\}$ is a locally uniformly bounded sequence in $\mathrm{Hol}_{\mathcal{H}}(\mathbb{D})$. The theorem follows from the classical Montel Theorem (with $U^k = \mathrm{id}_{\mathcal{H}}$ for all k) if $\dim \mathcal{H} < \infty$. Therefore, we may assume that $\dim \mathcal{H} = \infty$.

Fix a dense sequence $\{\lambda_i\}$ in Ω. By Lemma 2.6, there exists a subsequence $\{u^{k_l}\}$ and a sequence $\{V^l\}$ of unitary operators on $\mathcal{H}$ such that for each fixed i, $\{V^l u^{k_l}(\lambda_i)\}$ is a convergent sequence in $\mathcal{H}$. Furthermore, as $\{u^k\}$ is locally uniformly bounded, so also, $\{V^l u^{k_l}\}$ is locally uniformly bounded. Therefore, Proposition 2.1 implies that $\{V^l u^{k_l}\}$ is a convergent sequence in $\mathrm{Hol}_{\mathcal{H}}(\Omega)$. The theorem then follows by choosing $\{U^k\}$ to be any sequence of unitaries in $\mathcal{B}(\mathcal{H})$ such that $U^{k_l} = V^l$ for all l. □

3 Holomorphic Functions in Noncommuting Variables

If Ω is finitely open in $\mathbb{M}^d$, we may construct a *finitely compact-open exhaustion of* Ω, i.e., an increasing sequence of compact sets $\{K_i\}$ that satisfy

$$K_1 \subset \mathrm{int}(K_2) \subset K_2 \subset \mathrm{int}(K_3) \subset \dots$$

and with $\Omega = \cup_i K_i$. For a set $E \subseteq \Omega$ and $f \in \mathrm{Hol}(\Omega)$ we let

$$\|f\|_E = \sup_{\lambda \in E} \|f(\lambda)\|$$

and then in the usual way define a metric d on $\mathrm{Hol}(\Omega)$ with the formula

$$d(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|f-g\|_{K_n}}{1+\|f-g\|_{K_n}}.$$

It then follows that $f_k \to f$ in the metric space (Ω, d) if and only if for each finitely compact set K in Ω, $\{f_k\}$ converges uniformly to f on K, i.e.,

$$\lim_{k\to\infty} \|f - f_k\|_K = 0.$$

Furthermore, $\mathrm{Hol}(\Omega)$ is a complete metric space when endowed with this topology of uniform convergence on finitely compact subsets of Ω.

It is a straightforward exercise to extend Montel's Theorem to the space $\mathrm{Hol}_{\mathcal{H}}(\Omega)$ when $\dim \mathcal{H}$ is finite.

Proposition 3.1. If Ω is a finitely open set in $\mathbb{M}^d$, $\mathcal{H}$ is a Hilbert space with $\dim \mathcal{H} < \infty$, and $\{u^k\}$ is a finitely locally uniformly bounded sequence in $\mathrm{Hol}_{\mathcal{H}}(\Omega)$, then $\{u^k\}$ has a convergent subsequence.

Also, with the setup we have just described, mere notational changes to the proof of Proposition 2.1 yield a proof of the following proposition.

Proposition 3.2. Assume that Ω is a finitely open set in $\mathbb{M}^d$, $\{\lambda_i\}$ is a dense sequence in Ω with $\lambda_i \in \mathbb{M}^d_{n_i}$ for each i, and $\mathcal{H}$ is a Hilbert space. If $\{u^k\}$ is sequence in $\mathrm{Hol}_{\mathcal{H}}(\Omega)$ that is finitely locally uniformly bounded on Ω, and for each fixed i, $\{u^k(\lambda_i)\}$ is a convergent sequence in $\mathcal{B}(\mathbb{C}^{n_i}, \mathbb{C}^{n_i} \otimes \mathcal{H})$, then $\{u^k\}$ is a convergent sequence in $\mathrm{Hol}_{\mathcal{H}}(\Omega)$.

Just as was the case for Proposition 2.1 in [5], it is possible to relax the assumption in Proposition 3.2 that $\{\lambda_i\}$ be a dense sequence in Ω, to the assumption that $\{\lambda_i\}$ merely be a set of uniqueness for $\mathrm{Hol}_{\mathcal{H}}(\Omega)$ (see Proposition 6.2).

We now turn to an analog of Theorem 1.2 in the noncommutative setting.

Lemma 3.3. Wandering Isometry Lemma (noncommutative case). Assume that Ω is a finitely open set in $\mathbb{M}^d$ and $\{\lambda_i\}$ is a sequence in Ω (where, for each i, $\lambda_i \in \mathbb{M}^d_{n_i}$). If $\mathcal{H}$ is an infinite dimensional Hilbert space and $\{u^k\}$ is sequence in $\mathrm{Hol}_{\mathcal{H}}(\Omega)$ with the property that $\{u^k(\lambda_i)\}$ is bounded for each i, then there exists a subsequence $\{u^{k_l}\}$ and a sequence $\{V^l\}$ of unitary operators on $\mathcal{H}$ such that for each fixed i, $\{(\mathrm{id}_{n_i} \otimes V^l)\, u^{k_l}(\lambda_i)\}$ is a convergent sequence in $\mathcal{B}(\mathbb{C}^{n_i}, \mathbb{C}^{n_i} \otimes \mathcal{H})$.

Proof. Choose an increasing sequence $\{\mathcal{H}_i\}$ of subspaces of $\mathcal{H}$ with the property that

$$\dim \mathcal{H}_1 = n_1^2 \quad \text{and } \forall_{i\geq 1} \quad \dim(\mathcal{H}_{i+1} \ominus \mathcal{H}_i) = n_{i+1}^2,$$

and for each n, let $\{e_1, \dots, e_n\}$ denote the standard basis of $\mathbb{C}^n$.

Fix k. For each $i = 1, \dots, k$, as $u^k(\lambda_i) : \mathbb{C}^{n_i} \to \mathbb{C}^{n_i} \otimes \mathcal{H}$, there exist n_i^2 vectors $x^{k,i}_{r,s} \in \mathcal{H}$, $r, s = 1, \dots, n_i$, such that

$$u^k(\lambda_i)e_r = \sum_{s=1}^{n_i} e_s \otimes x^{k,i}_{r,s}, \qquad r = 1, \dots, n_i. \tag{3.4}$$

For each $i = 1, \ldots k$, define $\mathcal{M}_i^k$ by

$$\mathcal{M}_i^k = \operatorname{span}\{x_{r,s}^{k,i} \,:\, r, s = 1, \ldots, n_i\}.$$

and then define a sequence of spaces $\{\mathcal{N}_i^k\}$ by setting $\mathcal{N}_1^k = \mathcal{M}_1^k$ and

$$\mathcal{N}_i^k = \left(\mathcal{M}_1^k + \mathcal{M}_2^k + \ldots + \mathcal{M}_i^k\right) \ominus \left(\mathcal{M}_1^k + \mathcal{M}_2^k + \ldots + \mathcal{M}_{i-1}^k\right),$$

for $i = 2, \ldots, k$. As for each $i = 1, \ldots, k$, $\dim \mathcal{M}_i^k \le n_i^2$, so also, for $i = 1, \ldots, k$, $\dim \mathcal{N}_i^k \le n_i^2$. Consequently, as the spaces $\{\mathcal{N}_i^k\}$ are also pairwise orthogonal, it follows that there exists a unitary $U^k \in \mathcal{B}(\mathcal{H})$ such that

$$U^k(\mathcal{N}_1^k) \subseteq \mathcal{H}_1 \text{ and } U^k(\mathcal{N}_i^k) \subseteq \mathcal{H}_i \ominus \mathcal{H}_{i-1} \text{ for } i = 2, \ldots, k.$$

With this construction it follows using (3.4) that

$$(\mathrm{id}_{n_i} \otimes U^k) u^k(\lambda_i)(\mathbb{C}^{n_i}) \subseteq \mathbb{C}^{n_i} \otimes \mathcal{H}_i, \qquad i = 1, \ldots, k. \tag{3.5}$$

Now observe that as (3.5) holds for each k, for each fixed i,

$$(\mathrm{id}_{n_i} \otimes U^k) u^k(\lambda_i)(\mathbb{C}^{n_i}) \subseteq \mathbb{C}^{n_i} \otimes \mathcal{H}_i, \qquad k = i, i+1, \ldots,$$

i.e.,

$$\left\{(\mathrm{id}_{n_i} \otimes U^k) u^k(\lambda_i)\right\}_{k=i}^{\infty}$$

is a bounded sequence in $\mathcal{B}(\mathbb{C}^{n_i}, \mathbb{C}^{n_i} \otimes \mathcal{H}_i)$, a finite dimensional Hilbert space. Therefore, for each fixed i, there exist $L \in \mathcal{H}$ and an increasing sequence of indices $\{k_l\}$ such that

$$U^{k_l} u^{k_l}(\lambda_i) \to L \quad \text{in} \quad \mathcal{B}(\mathbb{C}^{n_i}, \mathbb{C}^{n_i} \otimes \mathcal{H}_i) \quad \text{as} \quad l \to \infty.$$

Applying this fact successively with $i = 1$, $i = 2$, and so on, at each stage taking a subsequence of the previously selected subsequence, leads to a sequence $\{L_i\}$ with $L_i \in \mathcal{B}(\mathbb{C}^{n_i}, \mathbb{C}^{n_i} \otimes \mathcal{H}_i)$ for each i and an increasing sequence of indices $\{k_l\}$ such that

$$\forall_i \;\; U^{k_l} u^{k_l}(\lambda_i) \to L_i \quad \text{in} \quad \mathcal{B}(\mathbb{C}^{n_i}, \mathbb{C}^{n_i} \otimes \mathcal{H}_i) \quad \text{as} \quad l \to \infty.$$

The lemma then follows if we let $V^l = U^{k_l}$. □

Lemma 3.3 suggests the following notation. Let Ω be a finitely open set in $\mathbb{M}^d$ and let $\mathcal{H}$ be a Hilbert space. If U is a unitary acting on $\mathcal{H}$, and $f \in \mathrm{Hol}_{\mathcal{H}}(\Omega)$ then we may define $U * f \in \mathrm{Hol}_{\mathcal{H}}(\Omega)$ by the formula

$$\forall_n \;\; (U * f) | \, \Omega_n = (\mathrm{id}_n \otimes U) f | \, \Omega_n.$$

With this notation we may formulate a noncommutative analog of Theorem 1.2 in the noncommutative setting.

Theorem 3.6. If Ω is a finitely open set in $\mathbb{M}^d$, $\mathcal{H}$ is a Hilbert space, and $\{u^k\}$ is a finitely locally uniformly bounded sequence in $\mathrm{Hol}_{\mathcal{H}}(\Omega)$, then there exists a sequence $\{U^k\}$ of unitary operators on $\mathcal{H}$ such that $\{U^k * u^k\}$ has a convergent subsequence.

Proof. Assume that Ω is an open set in $\mathbb{M}^d$, $\mathcal{H}$ is a Hilbert space, and $\{u^k\}$ is a finitely locally uniformly bounded sequence in $\mathrm{Hol}_{\mathcal{H}}(\Omega)$. If $\dim \mathcal{H} < \infty$, then the theorem follows from Proposition 3.1 if we choose $U^k = \mathrm{id}_{\mathcal{H}}$ for all k. Therefore, we assume that $\dim \mathcal{H} = \infty$.

Fix a dense sequence $\{\lambda_i\}$ in Ω. By Lemma 3.3, there exists a subsequence $\{u^{k_l}\}$ and a sequence $\{V^l\}$ of unitary operators on $\mathcal{H}$ such that for each fixed i, $\{V^l u^{k_l}(\lambda_i)\}$ is a convergent sequence in $\mathcal{B}(\mathbb{C}^{n_i}, \mathbb{C}^{n_i} \otimes \mathcal{H})$. Furthermore, as $\{u^k\}$ is locally uniformly bounded, so also, $\{V^l u^{k_l}\}$ is locally uniformly bounded. Therefore, Proposition 2.1 implies that $\{V^l u^{k_l}\}$ is a convergent sequence in $\mathrm{Hol}_{\mathcal{H}}(\Omega)$. The theorem then follows by choosing $\{U^k\}$ to be any sequence of unitaries in $\mathcal{B}(\mathcal{H})$ such that $U^{k_l} = V^l$ for all l. □

4 Locally bounded nc Functions

Properties of τ-holomorphic functions can be very sensitive to the choice of nc-topology τ. For example, if τ is the *fat topology* studied in [3], then τ-holomorphic functions satisfy a version of the Implicit Function Theorem. On the other hand if τ is the *free topology*, studied in [2], then τ-holomorphic functions satisfy the Oka-Weil Approximation Theorem. Remarkably, neither of these theorems holds for the other topology.

Also notice that if f is τ-holomorphic in the sense of Definition 1.6, then necessarily Ω, the domain of f, is an open set in the τ topology: for each $\lambda \in \Omega$ there exists $B_\lambda \subseteq \Omega$ such that $\lambda \in B_\lambda \in \tau$; hence, $\Omega = \bigcup_\lambda B_\lambda \in \tau$.

There are no such subtleties between the nc-topologies when it comes to understanding the implications of local boundedness.

Definition 4.1. Assume that τ is an nc-topology and $\Omega \in \tau$. If $\{u^k\}$ is a sequence in $\mathrm{Hol}^\tau_{\mathcal{H}}(\Omega)$, we say that $\{u^k\}$ is *τ-locally uniformly bounded on* Ω if for each $\lambda \in \Omega$, there exists a τ-open $B \subseteq \Omega$ such that, $\lambda \in B$ and

$$\sup_k \|u^k\|_B < \infty.$$

Lemma 4.2. Assume that τ is an nc-topology and $\Omega \in \tau$. Let $u \in \mathrm{Hol}(\Omega)$ and $\{u^k\}$ be a sequence in $\mathrm{Hol}^\tau_{\mathcal{H}}(\Omega)$. If $\{u^k\}$ is τ-locally uniformly bounded on Ω and $u^k \to u$ in $\mathrm{Hol}_{\mathcal{H}}(\Omega)$, then $u \in \mathrm{Hol}^\tau_{\mathcal{H}}(\Omega)$.

Proof. Under the assumptions of the lemma, we need to prove the following two assertions.

$$u \text{ is an nc-function on } \Omega. \tag{4.3}$$

$$u \text{ is } \tau\text{-locally bounded on } \Omega. \tag{4.4}$$

To prove (4.3), note first that as $u \in \mathrm{Hol}_{\mathcal{H}}(\Omega)$, Condition (i) in Definition 1.4 holds. To verify Condition (ii), assume that $\lambda, \mu, \lambda \oplus \mu \in \Omega$. Then, as $u^k \to u$ in $\mathrm{Hol}_{\mathcal{H}}(\Omega)$ and $u^k \in \mathrm{Hol}^\tau_{\mathcal{H}}$ for all k,

$$\begin{aligned} u(\lambda \oplus \mu) &= \lim_{k\to\infty} u^k(\lambda \oplus \mu) \\ &= \lim_{k\to\infty} \big(u^k(\lambda) \oplus u^k(\mu)\big) \\ &= \lim_{k\to\infty} u^k(\lambda) \oplus \lim_{k\to\infty} u^k(\mu) \\ &= u(\lambda) \oplus u(\mu). \end{aligned}$$

Finally, note that if $n \geq 1$, $S \in \mathbb{M}_n$ is invertible, and both λ and $S\lambda S^{-1}$ are in Ω_n, then

$$\begin{aligned} u(S\lambda S^{-1}) &= \lim_{k\to\infty} u^k(S\lambda S^{-1}) \\ &= \lim_{k\to\infty} (S \otimes \mathrm{id}_{\mathcal{H}})\, u^k(\lambda)\, S^{-1} \\ &= (S \otimes \mathrm{id}_{\mathcal{H}})\, u(\lambda)\, S^{-1}, \end{aligned}$$

which proves Condition (iii).

To prove (4.4), fix $\lambda \in \Omega$. As $\{u^k\}$ is τ-locally uniformly bounded on Ω, Definition 4.1 implies that there exist $B \subseteq \Omega$ and a constant ρ such that $\lambda \in B \in \tau$ and

$$\sup_k \|u^k\|_B \leq \rho.$$

Fix $\mu \in B$. As we assume that $u^k \to u$ in $\mathrm{Hol}_{\mathcal{H}}(\Omega)$, it follows that

$$\|u(\mu)\| = \lim_{k\to\infty} \|u^k(\mu)\| \le \rho.$$

But then,

$$\|u\|_B \le \rho.$$

As $B \in \tau$, this proves that u is τ-locally bounded on Ω. □

Definition 4.1 and Lemma 4.2, allow one to easily deduce Theorem 1.7 as a corollary of Theorem 3.6.

Proof of Theorem 1.7: As we assume that $\{u^k\}$ is a τ-locally uniformly bounded sequence in $\mathrm{Hol}^{\tau}_{\mathcal{H}}(\Omega)$, $\{u^k\}$ is finitely locally uniformly bounded in $\mathrm{Hol}_{\mathcal{H}}(\Omega)$. Therefore, Theorem 3.6 implies that there exists a sequence $\{U^k\}$ of unitary operators on $\mathcal{H}$ such that $\{U^k * u^k\}$ has a subsequence that converges in $\mathrm{Hol}_{\mathcal{H}}(\Omega)$. Consequently, we may choose $u \in \mathrm{Hol}_{\mathcal{H}}(\Omega)$ and an increasing sequence of indices $\{k_l\}$ such that $U^{k_l} * u^{k_l} \to u$ in $\mathrm{Hol}(\Omega)$. The proof is completed by observing that Lemma 4.2 implies that $u \in \mathrm{Hol}^{\tau}_{\mathcal{H}}(\Omega)$. □

Let us emphasize that Theorem 1.7 asserts that $U^{k_l} * u^{k_l}$ converges to u, which is in $\mathrm{Hol}^{\tau}_{\mathcal{H}}(\Omega)$, uniformly on sets that are compact in the finite topology; it does not say that it converges uniformly on compact sets in the τ topology.

Note that the proofs of Lemma 4.2 and Theorem 1.7 work identically if u^k are just assumed to be in $\mathrm{Hol}^{\mathrm{nc}}_{\mathcal{H}}(\Omega)$; so we get

Theorem 4.5. Let Ω be a finitely open set in $\mathbb{M}^d$, $\mathcal{H}$ is a Hilbert space, and $\{u^k\}$ is a finitely locally uniformly bounded sequence in $\mathrm{Hol}^{\mathrm{nc}}_{\mathcal{H}}(\Omega)$, then there exists a sequence $\{U^k\}$ of unitary operators on $\mathcal{H}$ such that $\{U^k * u^k\}$ has a subsequence that converges finitely locally uniformly to an element of $\mathrm{Hol}^{\mathrm{nc}}_{\mathcal{H}}(\Omega)$.

5 Some Applications

A useful construct in the study of τ-holomorphic functions is the *duality construction*. If Ω is a finitely open set it is natural to consider the algebraic tensor product $\mathrm{Hol}(\Omega)^* \otimes \mathrm{Hol}(\Omega)$. This space can concretely be realized as the set of functions A defined on

$$\Omega \boxtimes \Omega = \bigcup_{n=1}^{\infty} \left(\Omega \cap \mathbb{M}^d_n\right) \times \left(\Omega \cap \mathbb{M}^d_n\right),$$

and such that there exist a finite dimensional Hilbert space $\mathcal{H}$ and $u, v \in \mathrm{Hol}_{\mathcal{H}}(\Omega)$ such that

$$A(\lambda, \mu) = v(\mu)^* u(\lambda), \qquad (\lambda, \mu) \in \Omega \boxtimes \Omega.$$

As the functions in $\mathrm{Hol}(\Omega)^* \otimes \mathrm{Hol}(\Omega)$ are holomorphic in λ for each fixed μ and anti-holomorphic in μ for each fixed λ, we may complete $\mathrm{Hol}(\Omega)^* \otimes \mathrm{Hol}(\Omega)$ in the topology of uniform convergence on finitely compact subsets of $\Omega \boxtimes \Omega$ to obtain the space of *hereditary holomorphic functions on* Ω, $\mathrm{Her}(\Omega)$. Inside $\mathrm{Her}(\Omega)$, we may define a cone $\mathcal{P}$ by

$$\mathcal{P} = \{u(\mu)^* u(\lambda) \,:\, u \in \mathrm{Hol}_{\mathcal{H}}(\Omega) \text{ for some Hilbert space } \mathcal{H}\}.$$

Theorem 5.1. $\mathcal{P}$ is closed in $\mathrm{Her}(\Omega)$.

Proof. Assume that $\{v^k\}$ is a sequence with $v^k \in \mathrm{Hol}_{\mathcal{H}_k}(\Omega)$ for each k and with $v^k(\mu)^*v^k(\lambda) \to A$ in $\mathrm{Her}(\Omega)$. We may assume that $\mathcal{H}_k$ is separable for each k. Fix a separable infinite dimensional Hilbert space $\mathcal{H}$ and for each each k choose an isometry $V^k : \mathcal{H}_k \to \mathcal{H}$. If for each k we let $u^k = V^k * v^k$, then $\{u^k\}$ is a sequence in $\mathrm{Hol}_{\mathcal{H}}(\Omega)$ and $u^k(\mu)^*u^k(\lambda) \to A$ in $\mathrm{Her}(\Omega)$.

Now, as $u^k(\mu)^*u^k(\lambda) \to A$ in $\mathrm{Her}(\Omega)$ it follows that $\{u^k\}$ is a finitely locally uniformly bounded sequence in $\mathrm{Hol}_{\mathcal{H}}(\Omega)$. Hence, by Theorem 3.6, there exists a sequence U^k of unitary operators on $\mathcal{H}$ such that $\{U^k * u^k\}$ has a convergent subsequence, i.e., there exists $u \in \mathrm{Hol}_{\mathcal{H}}(\Omega)$ and an increasing sequence of indices $\{k_l\}$ such that $U^{k_l} * u^{k_l} \to u$. But then, for each $(\lambda, \mu) \in \Omega \boxtimes \Omega$,

$$\begin{aligned} A(\lambda,\mu) &= \lim_{k\to\infty} u^k(\mu)^*u^k(\lambda) \\ &= \lim_{l\to\infty} u^{k_l}(\mu)^*u^{k_l}(\lambda) \\ &= \lim_{l\to\infty} (U^{k_l} * u^{k_l})(\mu)^*(U^{k_l} * u^{k_l})(\lambda) \\ &= u(\mu)^*u(\lambda), \end{aligned}$$

i.e., $A \in \mathcal{P}$. □

We also may use wandering Montel Theorems to study sums of τ-holomorphic dyads. We let $\mathrm{Her}^\tau(\Omega)$ denote the closure of

$$\{v(\mu)^*u(\lambda) \,:\, u, v \in \mathrm{Hol}^\tau_{\mathcal{H}}(\Omega) \text{ for some finite dimensional Hilbert space } \mathcal{H}\}$$

inside $\mathrm{Her}(\Omega)$ and define $\mathcal{P}^\tau$ in $\mathrm{Her}^\tau(\Omega)$ by

$$\mathcal{P}^\tau = \{u(\mu)^*u(\lambda) \,:\, u \in \mathrm{Hol}^\tau_{\mathcal{H}}(\Omega) \text{ for some Hilbert space } \mathcal{H}\}.$$

Theorem 5.2. Let τ be an nc-topology, and $\Omega \in \tau$. Then $\mathcal{P}^\tau$ is closed in $\mathrm{Her}^\tau(\Omega)$.

Proof. Assume that $u^k(\mu)^*u^k(\lambda) \to A$ in $\mathrm{Her}(\Omega)$, where, as in the proof of Theorem 5.1, we may assume that $u^k \in \mathrm{Hol}^\tau_{\mathcal{H}}(\Omega)$ for each k. By Theorem 1.7, there exist $u \in \mathrm{Hol}^\tau_{\mathcal{H}}(\Omega)$, a sequence U^k of unitary operators on $\mathcal{H}$, and and an increasing sequence of indices $\{k_l\}$ such that $U^{k_l} * u^{k_l} \to u$. But then as in the proof of Theorem 5.1, $A(\lambda, \mu) = u(\mu)^*u(\lambda)$ for all $(\lambda, \mu) \in \Omega \boxtimes \Omega$, i.e., $A \in \mathcal{P}^\tau$. □

Finally, we shall prove that the model cone is closed; this is the key ingredient in the proof of the realization formula for free holomorphic functions [2, 7, 1]. Let δ be a J-by-L matrix whose entries are free polynomials in d variables. We define B_δ to be the polynomial polyhedron

$$B_\delta \;:=\; \{x \in \mathbb{M}^d : \|\delta(x)\| < 1\}.$$

The free topology is the nc-topology generated by the sets B_δ, as δ ranges over all matrices of polynomials. The model cone $\mathcal{C}$ is the set of hereditary functions on B_δ of the form

$$\mathcal{C} \;:=\; \left\{ \begin{array}{c} \mathrm{id}_{\mathbb{C}^J} \\ \otimes \\ u(\mu)^* \end{array} \left(\mathrm{id} - \begin{array}{c} \delta(\mu)^*\delta(\lambda) \\ \otimes \\ \mathrm{id}_{\mathcal{H}} \end{array} \right) \begin{array}{c} \mathrm{id}_{\mathbb{C}^J} \\ \otimes \\ u(\lambda) \end{array} \;:\; u \in \mathrm{Hol}_{\mathcal{H}}(B_\delta) \text{ and } u \text{ is nc}, \;\; \text{for some Hilbert space } \mathcal{H} \right\}. \tag{5.3}$$

We write the tensors vertically just to enhance readability.

Theorem 5.4. The model cone $\mathcal{C}$, defined in (5.3) is closed in $\mathrm{Her}(B_\delta)$.

Proof. Suppose u^k is a sequence of nc functions in $\mathrm{Hol}_{\mathcal{H}}(B_\delta)$ (we may assume the space $\mathcal{H}$ is the same for each u^k, as in the proof of Thm. 5.1), so that

$$\begin{matrix}\mathrm{id}_{\mathbb{C}^J}\\ \otimes \\ u^k(\mu)^*\end{matrix}\left(\mathrm{id}-\begin{matrix}\delta(\mu)^*\delta(\lambda)\\ \otimes\\ \mathrm{id}_{\mathcal{H}}\end{matrix}\right)\begin{matrix}\mathrm{id}_{\mathbb{C}^J}\\ \otimes \\ u^k(\lambda)\end{matrix} \tag{5.5}$$

converges in $\mathrm{Her}(B_\delta)$ to $A(\lambda,\mu)$. On any finitely compact set, $\|\delta(x)\|$ will be bounded by a constant that is strictly less than one. Since (5.5) converges uniformly on finitely compact subsets of $B_\delta \boxtimes B_\delta$, this means that u^k is a finitely locally uniformly bounded sequence. Therefore by Thm. 3.6, there exist unitaries U^k such that $U^k * u^k$ has a convergent subsequence, which converges to some nc function $u \in \mathrm{Hol}_{\mathcal{H}}(B_\delta)$. Then

$$A(\lambda,\mu) \ = \ \begin{matrix}\mathrm{id}_{\mathbb{C}^J}\\ \otimes \\ u(\mu)^*\end{matrix}\left(\mathrm{id}-\begin{matrix}\delta(\mu)^*\delta(\lambda)\\ \otimes\\ \mathrm{id}_{\mathcal{H}}\end{matrix}\right)\begin{matrix}\mathrm{id}_{\mathbb{C}^J}\\ \otimes \\ u(\lambda)\end{matrix}\ ,$$

as desired.

□

6 Sets of Uniqueness

In this section we shall show that the assumption in Propositions 2.1 and 3.2, that $\{\lambda_i\}$ be a dense sequence in Ω, can be relaxed to the assumption that $\{\lambda_i\}$ be a set of uniqueness for $\mathrm{Hol}(\Omega)$. We remark that it is an elementary fact that if $\mathcal{H}$ is a Hilbert space, then $\{\lambda_i\}$ is a set of uniqueness for $\mathrm{Hol}_{\mathcal{H}}(\Omega)$ if and only if $\{\lambda_i\}$ is a set of uniqueness for $\mathrm{Hol}(\Omega)$.

The following proposition is essentially the same as the Arendt-Nikolski Theorem 1.1, so we shall omit the proof.

Proposition 6.1. Assume that Ω is an open set in $\mathbb{C}^d$, $\{\lambda_i\}$ is a sequence in Ω that is a set of uniqueness for $\mathrm{Hol}_{\mathcal{H}}(\Omega)$ [3], and $\mathcal{H}$ is a Hilbert space. If $\{u^k\}$ is sequence in $\mathrm{Hol}_{\mathcal{H}}(\Omega)$ that is locally uniformly bounded on Ω, and for each fixed i, $\{u^k(\lambda_i)\}$ is a convergent sequence in $\mathcal{H}$, then $\{u^k\}$ converges in $\mathrm{Hol}_{\mathcal{H}}(\Omega)$.

Here is the graded version.

Proposition 6.2. Assume that Ω is a finitely open set in $\mathbb{M}^d$, $\{\lambda_i\}$ is a sequence in Ω (with $\lambda_i \in \mathbb{M}^d_{n_i}$ for each i) that is a set of uniqueness for $\mathrm{Hol}_{\mathcal{H}}(\Omega)$, and $\mathcal{H}$ is a Hilbert space. If $\{u^k\}$ is sequence in $\mathrm{Hol}_{\mathcal{H}}(\Omega)$ that is finitely locally uniformly bounded on Ω, and for each fixed i, $\{u^k(\lambda_i)\}$ is a convergent sequence in $\mathcal{H}$, then $\{u^k\}$ converges in $\mathrm{Hol}_{\mathcal{H}}(\Omega)$.

Proof. The theorem will follow if we can show that $\{u^k|\Omega_n\}$ is a convergent sequence for each n. Accordingly, fix n and adopt the notation H_n for the holomorphic $\mathcal{B}(\mathbb{C}^n, \mathbb{C}^n \otimes \mathcal{H})$-valued functions defined on Ω_n. Thus, $\{u^k|\Omega_n\}$ is a locally uniformly bounded sequence in H_n. Furthermore, if $\{\eta_j\}$ is an enumeration of $\{\lambda_i : i \geq 1\} \cap \Omega_n$, as $\{\lambda_i\}$ is a set of uniqueness for $\mathrm{Hol}(\Omega)$, $\{\eta_i\}$ is a set of uniqueness for both $\mathrm{Hol}(\Omega_n)$ and H_n. Finally, let $u^k(\eta_j) \to u_j$ as $k \to \infty$ for each j.

For fixed $\alpha \in \mathbb{C}^n$ and $\beta \in \mathbb{C}^n$, define $f^k_{\alpha,\beta} \in \mathrm{Hol}(\Omega_n)$ by

$$f^k_{\alpha,\beta}(\lambda) = \langle u^k(\lambda)\alpha, \beta\rangle_{\mathbb{C}^n\otimes\mathcal{H}}, \qquad \lambda \in \Omega. \tag{6.3}$$

Noting that,

$$|f^k_{\alpha,\beta}(\lambda)| = |\langle u^k(\lambda)\alpha, \beta\rangle| \leq \|u^k(\lambda)\|\|\alpha\|\|\beta\|, \tag{6.4}$$

[3] i.e., if $f \in \mathrm{Hol}(\Omega)$ and $f(\lambda_i) = 0$ for all i, then $f(\lambda) = 0$ for all $\lambda \in \Omega$.

it follows that $\{f^k_{\alpha,\beta}\}$ is locally uniformly bounded on Ω_n. Therefore by Montel's Theorem, $\{f^k_{\alpha,\beta}\}$ has compact closure in $\mathrm{Hol}(\Omega_n)$.

We claim that $\{f^k_{\alpha,\beta}\}$ has a unique cluster point. For assume that $\{f^{k_r}_{\alpha,\beta}\}$ and $\{f^{k_s}_{\alpha,\beta}\}$ are subsequences of $\{f^k_{\alpha,\beta}\}$ with $\{f^{k_r}_{\alpha,\beta}\} \to f$ and $\{f^{k_s}_{\alpha,\beta}\} \to g$. Then, as we assume for each j, $u^k(\eta_j) \to u_j$ as $k \to \infty$,

$$\begin{aligned} f(\eta_i) &= \lim_{r\to\infty} f^{k_r}_{\alpha,\beta}(\eta_i) \\ &= \lim_{r\to\infty} \langle u^{k_r}(\eta_i)\alpha, \beta\rangle \\ &= \langle u_i\alpha, \beta\rangle \\ &= \lim_{s\to\infty} \langle u^{k_s}(\eta_i)\alpha, \beta\rangle \\ &= \lim_{s\to\infty} f^{k_s}_{\alpha,\beta}(\eta_i) \\ &= g(\eta_i). \end{aligned}$$

Hence, as $\{\eta_i\}$ is a set of uniqueness, $f = g$. Since $\{f^k_{\alpha,\beta}\}$ has a unique cluster point, we have shown that for each $\alpha \in \mathbb{C}^n$ and $\beta \in \mathbb{C}^n \otimes \mathcal{H}$, there exists $f_{\alpha,\beta} \in \mathrm{Hol}(\Omega_n)$ such that

$$f^k_{\alpha,\beta} \to f_{\alpha,\beta} \text{ in } \mathrm{Hol}(\Omega_n) \text{ as } k \to \infty. \tag{6.5}$$

Now fix $\lambda \in \Omega_n$ and define L_λ by

$$L_\lambda(\alpha, \beta) = f_{\alpha,\beta}(\lambda), \qquad \alpha \in \mathbb{C}^n, \beta \in \mathbb{C}^n \otimes \mathcal{H}. \tag{6.6}$$

Observe that (6.3) and (6.5) imply that L_λ is a sesqui-linear functional on $\mathbb{C}^n \times (\mathbb{C}^n \otimes \mathcal{H})$. Furthermore, (6.4) and (6.5) imply that L_λ is bounded. Therefore, by the Riesz Representation Theorem, there exists $u(\lambda) \in \mathcal{B}(\mathbb{C}^n, \mathbb{C}^n \otimes \mathcal{H})$ such that

$$\forall_{\alpha\in\mathbb{C}^n} \ \ \forall_{\beta\in\mathbb{C}^n\otimes\mathcal{H}} \ \ L_\lambda(\alpha, \beta) = \langle u(\lambda)\alpha, \beta\rangle,$$

or equivalently,

$$\forall_{\alpha\in\mathbb{C}^n} \ \ \forall_{\beta\in\mathbb{C}^n\otimes\mathcal{H}} \ \ \langle u(\lambda)\alpha, \beta\rangle = f_{\alpha,\beta}(\lambda).$$

The function u constructed in the previous paragraph has the following properties: it is holomorphic,

$$\forall_{\lambda\in\Omega_n} \ \ u^k(\lambda) \to u(\lambda) \quad \text{weakly in } \mathcal{B}(\mathbb{C}^n, \mathbb{C}^n \otimes \mathcal{H}) \quad \text{as} \quad k \to \infty, \tag{6.7}$$

and

$$\forall_j \ \ u^k(\eta_j) \to u(\eta_j) \ \text{ in norm in } \mathcal{B}(\mathbb{C}^n, \mathbb{C}^n \otimes \mathcal{H}) \ \text{ as } \quad k \to \infty. \tag{6.8}$$

Claim 6.9.

$$u^k(\mu)^* u^k(\lambda) \to u(\mu)^* u(\lambda) \quad \text{in} \quad \mathrm{Her}(\Omega_n) \quad \text{as} \quad k \to \infty.$$

To prove this claim, first note that as we are assuming $\{u^k\}$ is a locally uniformly bounded sequence in $\mathrm{Hol}_{\mathcal{H}}(\Omega_n)$, $\{u^k(\mu)^* u^k(\lambda)\}$ is a locally uniformly bounded sequence in $\mathrm{Her}(\Omega_n)$. Therefore, the claim follows from Montel's Theorem if we can show that

$$A(\lambda, \mu) = u(\mu)^* u(\lambda) \tag{6.10}$$

whenever $\{k_r\}$ is a sequence of indices such that

$$u^{k_r}(\mu)^* u^{k_r}(\lambda) \to A(\lambda, \mu) \quad \text{in} \quad \mathrm{Her}(\Omega_n) \quad \text{as} \quad r \to \infty. \tag{6.11}$$

But if (6.11) holds, then (6.8) implies that for each independently chosen i and j,

$$A(\eta_j, \eta_i) = \lim_{r\to\infty} u^{k_r}(\eta_i)^* u^{k_r}(\eta_j) = u^{k_r}(\eta_i)^* u^{k_r}(\eta_j).$$

Since both sides of (6.10) are holomorphic in λ and anti-holomorphic in μ, and $\{\eta_i\}$ is a set of uniqueness, it follows that (6.10) holds for all $\lambda, \mu \in \Omega$. This completes the proof of Claim 6.9.

Finally, fix $\lambda \in \Omega$. By (6.7), $\{u^k(\lambda)\}$ converges weakly in $\mathcal{B}(\mathbb{C}^n, \mathbb{C}^n \otimes \mathcal{H})$ to $u(\lambda)$ and by Claim 6.9, $u^k(\lambda)^* u^k(\lambda) \to u(\lambda)^* u(\lambda)$. Therefore, $u^k(\lambda) \to u(\lambda)$ in norm in $\mathcal{B}(\mathbb{C}^n, \mathbb{C}^n \otimes \mathcal{H})$. Since this holds for all $\lambda \in \Omega$, the proof of Proposition 6.2 may be completed by an application of Proposition 3.2. □

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