Wandering Montel Theorems for Hilbert Space Valued Holomorphic Functions

Jim Agler ∗
U.C. San Diego
La Jolla, CA 92093

John E. McCarthy †
Washington University
St. Louis, MO 63130

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Abstract: We prove that if \{u_k\} is a sequence of holomorphic functions that takes values in an infinite dimensional Hilbert space \(\mathcal{H}\), there are unitaries \{U_k\} on \(\mathcal{H}\) so that \(U_k u_k\) has a subsequence that converges locally uniformly. We also prove a non-commutative version of this result.

1 Introduction

1.1 Commutative Theory

Let \(\Omega\) be an open set in \(\mathbb{C}^d\) and assume that \{u_k\} is a sequence in Hol(\(\Omega\)), the algebra of holomorphic functions on \(\Omega\) equipped with the topology of uniform convergence on compact subsets. The classical Montel Theorem asserts that if \{u_k\} is locally uniformly bounded on \(\Omega\), then there exists a subsequence \{u_{k_l}\} that converges in Hol(\(\Omega\)).

It is well known that if \(\mathcal{X}\) is an infinite dimensional Banach space, then Montel’s Theorem breaks down for Hol(\(\mathcal{X}\)(\(\Omega\)), the space of \(\mathcal{X}\)-valued holomorphic functions, see e.g. [5, 14]. For example, if \(\mathcal{X} = l^2\) and \(f^k\) is a locally uniformly bounded sequence of holomorphic functions on \(\Omega\), then the sequence

\[
\begin{pmatrix}
    f^1(\lambda) \\
    \vdots
\end{pmatrix},
\begin{pmatrix}
    0 \\
    f^2(\lambda) \\
    \vdots
\end{pmatrix},
\begin{pmatrix}
    0 \\
    0 \\
    f^3(\lambda) \\
    \vdots
\end{pmatrix}, \ldots
\]

is a locally uniformly bounded sequence that will have a convergent subsequence only if there exists a subsequence \(f^{k_l}\) that converges uniformly to 0 on \(\Omega\).

Observe that the problem in the example given above is that while for all \(\lambda \in \Omega\), \(u_k\) converges weakly to 0, it needn’t be the case that \(u_k(\lambda)\) converges in norm for any \(\lambda \in \Omega\). However, just as in the case of the classical proof of Montel’s Theorem that uses the Arzela-Ascoli Theorem, if one assumes that \{u_k\} is well behaved pointwise on a large enough set, then one can conclude uniform convergence in norm on compact sets. For example, consider the following theorem by Arendt and Nikolski [5, Cor. 2.3]

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Theorem 1.1. Let $\Omega$ be an open connected set in $\mathbb{C}$, and let $u^k$ be a sequence in $\text{Hol}_X(\Omega)$ that is locally bounded. Assume that
\[ \Omega_0 := \{ z \in \Omega : \{ u^k(z) : k \in \mathbb{N} \} \text{ is relatively compact in } X \} \]
has an accumulation point in $\Omega$. Then there exists a subsequence which converges to a holomorphic function uniformly on compact subsets of $\Omega$.

Theorem 1.1 deals with the difficulty by making strong additional assumptions about the point-wise behavior of $\{ u^k \}$, assumptions that may not hold in desirable applications. The central idea of this paper, for Hilbert space valued functions, is instead to use a sequence of unitaries to push (most of) the range of the functions into a finite-dimensional space. Here is our first main result.

Theorem 1.2. If $\Omega$ is an open set in $\mathbb{C}^d$, $\mathcal{H}$ is a Hilbert space, and $\{ u^k \}$ is a locally uniformly bounded sequence in $\text{Hol}_\mathcal{H}(\Omega)$, then there exists a sequence $\{ U^k \}$ of unitary operators on $\mathcal{H}$ such that $\{ U^k u^k \}$ has a subsequence that converges in $\text{Hol}_\mathcal{H}(\Omega)$.

We prove Theorem 1.2 in Section 2. In Sections 3 and 4 we consider versions for non-commutative functions. These functions have been extensively studied recently—see e.g. [17, 8, 4, 10, 11, 13, 15, 6, 12, 9]. Before stating our results, we must spend a little time explaining some definitions.

### 1.2 Non-commutative theory

In commutative analysis, one studies holomorphic functions defined on domains in $\mathbb{C}^d$. In noncommutative analysis one studies holomorphic functions defined on domains in $\mathbb{M}^d$, the $d$-dimensional nc universe. For each $n$ we let $\mathbb{M}_n^d$ denote the set of $d$-tuples of $n \times n$ matrices. We then let
\[ \mathbb{M}^d = \bigcup_{n=1}^{\infty} \mathbb{M}_n^d. \]

When $E$ is a subset of $\mathbb{M}^d$, then for each $n$, we adopt the notation
\[ E_n = E \cap \mathbb{M}_n^d. \]

In noncommutative analysis one studies graded functions, i.e., functions $f$ defined on subsets $E$ of $\mathbb{M}^d$, that satisfy
\[ \forall_n \forall_{\lambda \in E_n} f(\lambda) \in \mathbb{M}_n. \] (1.3)

$\mathbb{M}^d$ carries a topology, the so-called finite topology\(^1\), wherein a set $\Omega$ is deemed to be open precisely when
\[ \forall_n \Omega_n \text{ is open in } \mathbb{M}_n. \]

With this definition, note that a graded function $f : E \to \mathbb{M}^1$ is finitely continuous if and only if $f|E_n$ is continuous for each $n$ and also that a set $K \subseteq \mathbb{M}^d$ is finitely compact if and only if there exists $n$ such that $E_m = \emptyset$ when $m > n$ and $E_m$ is compact when $m \leq n$.

If $\Omega$ is finitely open in $\mathbb{M}^d$, then for each $n$, $\Omega_n$ can be identified with an open set in $\mathbb{C}^{dn^2}$, in an obvious way. If, in addition, $f$ is a graded function on $\Omega$, then we say that $f$ is holomorphic on $\Omega$ if for each $n$, $f|\Omega_n$ is a holomorphic mapping of $\Omega_n$ into $\mathbb{M}_n$. We let $\text{Hol}(\Omega)$ denote the collection of graded holomorphic functions.

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\(^1\)Subsequently, we shall consider other topologies as well.
It is also possible to consider $\mathcal{H}$-valued holomorphic functions in the noncommutative setting. One particularly concrete way to do this is to realize in the scalar case just considered that (1.3) is equivalent to asserting that

$$\forall_n \forall_{\lambda \in E_n} f(\lambda) \in \mathcal{B}(\mathbb{C}^n, \mathbb{C}^n).$$

We therefore replace the former definition (that $f$ be graded) with the requirement that $f$ be a graded $\mathcal{H}$-valued function, i.e., that

$$\forall_n \forall_{\lambda \in E_n} f(\lambda) \in \mathcal{B}(\mathbb{C}^n, \mathbb{C}^n \otimes \mathcal{H}).$$

Just as before, we declare a graded $\mathcal{H}$-valued function defined on a finitely open set $\Omega$ in $\mathbb{M}_d^d$ to be holomorphic if for each $n$, $f|_{\Omega_n}$ is a holomorphic mapping of $\Omega_n$ into $\mathcal{B}(\mathbb{C}^n, \mathbb{C}^n \otimes \mathcal{H})$. We let $\text{Hol}_\mathcal{H}(\Omega)$ denote the collection of graded $\mathcal{H}$-valued functions and view $\text{Hol}_\mathcal{H}(\Omega)$ as a complete metric space endowed with the topology of uniform convergence on finitely compact subsets of $\Omega$.

A special class of graded functions arise by formalizing certain algebraic properties of free polynomials. If $E \subseteq \mathbb{M}_d^d$ we say that $E$ is an nc-set if $E$ is closed with respect to direct sums. We define the class of nc-functions as follows.

**Definition 1.4.** Let $\mathcal{H}$ be a Hilbert space, $E$ an nc-set, and assume that $f$ is a function defined on $E$. We say that $f$ is an $\mathcal{H}$-valued nc-function on $E$ if the following conditions hold.

1. $f$ is $\mathcal{H}$-graded, i.e.,

$$\forall_n \forall_{\lambda \in E \cap \mathbb{M}_n} f(\lambda) \in \mathcal{B}(\mathbb{C}^n, \mathbb{C}^n \otimes \mathcal{H}).$$

2. $f$ preserves direct sums, i.e.,

$$\forall_{\lambda, \mu \in E} \lambda \oplus \mu \in E \implies f(\lambda \oplus \mu) = f(\lambda) \oplus f(\mu).$$

In this formula, if $\lambda \in E_m$ and $\mu \in E_m$, we identify $\mathbb{C}^m \oplus \mathbb{C}^n$ and $\mathbb{C}^{m+n}$ and identify $(\mathbb{C}^m \otimes \mathcal{H}) \oplus (\mathbb{C}^n \otimes \mathcal{H})$ and $\mathbb{C}^{m+n} \otimes \mathcal{H}$.

3. $f$ preserves similarity, i.e.,

$$f(S\lambda S^{-1}) = (S \otimes \text{id}_\mathcal{H})f(\lambda)S^{-1}$$

whenever $n \geq 1$, $S \in \mathbb{M}_n$ is invertible, and both $\lambda$ and $S\lambda S^{-1}$ are in $E_n$.

When $f : E \to \mathbb{M}_1 \otimes \mathcal{H}$ is an nc-function and $E$ is a finitely open nc-set then Condition (iii) above becomes very strong and yields the following proposition which lies at the heart of nc analysis (see [10] or [13, Thm. 7.2]). We say a function $f$ is bounded on $E$ if $\sup_{\lambda \in E} \|f(\lambda)\| < \infty$.

**Proposition 1.5.** Let $\Omega$ be a finitely open nc-set. If $f$ is a bounded nc-function defined on $\Omega$, then $f$ is holomorphic on $\Omega$.

Proposition 1.5 suggests the following terminology. We say that a set $\Omega \subseteq \mathbb{M}_d^d$ is an nc-domain if $\Omega$ is a finitely open nc-set and we say that a topology $\tau$ on $\mathbb{M}_d^d$ is an nc-topology if $\tau$ has a basis consisting of nc-domains. We then define special classes of functions in noncommuting variables as follows.

**Definition 1.6.** Let $\Omega \subseteq \mathbb{M}_d^d$, $\tau$ be an nc-topology, and assume that $f : \Omega \to \mathbb{M}_1 \otimes \mathcal{H}$ is an $\mathcal{H}$-valued function. We say that $f$ is $\tau$-holomorphic if $f$ is a $\tau$-locally bounded nc function on $\Omega$. \footnote{i.e., $f$ is an nc-function on $\Omega$ in the sense of Definition 1.4 and for each $\lambda \in \Omega$, there exists $B \subseteq \Omega$ such that $\lambda \in B \in \tau$ and $f|B$ is bounded.} We let $\text{Hol}^\tau_\mathcal{H}(\Omega)$ denote the collection of $\tau$-holomorphic $\mathcal{H}$-valued functions defined on $\Omega$. 

\[3\]
Evidently, Proposition 1.5 guarantees that if $\tau$ is an nc-topology, and $f$ is a $\tau$-holomorphic function in the sense of Definition 1.6, then $f$ is holomorphic, i.e.,

$$\text{Hol}_H^\tau(\Omega) \subseteq \text{Hol}_H^\mu(\Omega) \subseteq \text{Hol}_H(\Omega),$$

where $\text{Hol}_H^\mu(\Omega)$ denotes the set of functions in $\text{Hol}_H(\Omega)$ that are nc.

We can now state our second main result, the non-commutative version of Theorem 1.2.

**Theorem 1.7.** Assume that $\tau$ is an nc-topology, $\Omega \in \tau$, $\mathcal{H}$ is a Hilbert space, and $\{u^k\}$ is a $\tau$-locally uniformly bounded sequence in $\text{Hol}_H^\tau(\Omega)$. There exist $u \in \text{Hol}_H^\tau(\Omega)$, a sequence $\{U^k\}$ of unitary operators on $\mathcal{H}$, and an increasing sequence of indices $\{k_l\}$ such that $(\text{id}_n \otimes U^{k_l})u^{k_l} \to u$ in $\text{Hol}(\Omega)$.

As an application of Theorem 1.7 in Section 5, we prove that the cones

$$\mathcal{P} = \{u(\mu)^*u(\lambda) : u \in \text{Hol}_H(\Omega) \text{ for some Hilbert space } \mathcal{H}\}$$

and

$$\mathcal{C} = \left\{ \frac{\text{id}_{J,J} \otimes u(\mu)^*}{u(\mu)} \left( \text{id} - \frac{\delta(\mu)^*\delta(\lambda)}{\text{id}_{\mathcal{H}}} \right) \frac{\text{id}_{J,J} \otimes u(\lambda)}{u(\lambda)} : u \in \text{Hol}_H(B_\delta) \text{ and } u \text{ is nc} \right\}$$

are closed. In this last formula, $\delta$ is a $J$-by-$J$ matrix of free polynomials, and $B_\delta = \{x : \|\delta(x)\| < 1\}$ is a non-commutative polynomial polyhedron. (We adopt the convention of [16] and write the tensors vertically for legibility.)

Proving that the cones are closed is the key step in proving realization formulas for free holomorphic functions—see [2, 1, 7].

In Section 6 we show that the assumptions in Proposition 3.2 below can be weakened to just requiring convergence on a set of uniqueness, which yields a graded version of the Arendt-Nikolski theorem.

## 2 A Montel Theorem for Hilbert Space Valued Holomorphic Functions

In this section we prove Theorem 1.2 from the introduction.

### 2.1 Notation and Definitions

If $\Omega$ is an open set in $\mathbb{C}^d$, $\mathcal{H}$ is a Hilbert space, we let $\text{Hol}_\mathcal{H}(\Omega)$ denote the space of holomorphic $\mathcal{H}$-valued functions on $\Omega$. If $u \in \text{Hol}_\mathcal{H}(\Omega)$ and $E \subseteq \Omega$, we let

$$\|u\|_E = \sup_{\lambda \in E} \|u(\lambda)\|_{\mathcal{H}}.$$

If $\|u\|_\Omega < \infty$ then we say that $u$ is bounded on $\Omega$. If $\{u^k\}$ is sequence in $\text{Hol}_\mathcal{H}(\Omega)$, we say that $\{u^k\}$ is uniformly bounded on $\Omega$ if

$$\sup_k \|u^k\|_\Omega < \infty,$$

and we say that $\{u^k\}$ is locally uniformly bounded on $\Omega$ if for each $\lambda \in \Omega$, there exists a neighborhood $B$ of $\lambda$ such that $\{u^k\}$ is uniformly bounded on $B$. Recall that if such a neighborhood exists, then a Cauchy Estimate implies that $\{u^k\}$ is equicontinuous at $\lambda$, i.e., for each $\varepsilon > 0$ there exists a ball $B_0$ such that $\lambda \in B_0 \subseteq B$ and

$$\forall \mu \in B_0 \forall_k \|u^k(\mu) - u^k(\lambda)\| < \varepsilon.$$
We equip $\text{Hol}_H(\Omega)$ with the usual topology of uniform convergence on compacta. Thus, a sequence $\{u^k\}$ in $\text{Hol}_H(\Omega)$ is convergent precisely when there is a function $u \in \text{Hol}_H(\Omega)$ such that
\[
\lim_{k \to \infty} \|u^k - u\|_E = 0
\]
for every compact $E \subseteq \Omega$. We say that a sequence $\{u^k\}$ in $\text{Hol}_H(\Omega)$ is a Cauchy sequence if for each compact $E \subseteq \Omega$, \{u^k\} is uniformly Cauchy on $E$, i.e., for each $\varepsilon > 0$, there exists $N$ such that
\[
k, l \geq N \implies \|u^k - u^l\|_E < \varepsilon.
\]
It is well known that $\text{Hol}_H(\Omega)$ is complete, i.e., every Cauchy sequence in $\text{Hol}_H(\Omega)$ converges. The following result is proved in [5, Thm. 2.1]; we include a proof that easily generalizes to Proposition 3.2.

**Proposition 2.1.** Assume that $\Omega$ is an open set in $\mathbb{C}^d$, $\{\lambda_i\}$ is a dense sequence in $\Omega$, and $H$ is a Hilbert space. If $\{u^k\}$ is sequence in $\text{Hol}_H(\Omega)$ that is locally uniformly bounded on $\Omega$, and for each fixed $i$, $\{u^k(\lambda_i)\}$ is a convergent (and hence, Cauchy) sequence in $H$, then $\{u^k\}$ is a convergent sequence in $\text{Hol}_H(\Omega)$.

**Proof:** Fix a compact set $E \subseteq \Omega$ and $\varepsilon > 0$. Note that as $\{u^k\}$ is assumed to be locally uniformly bounded on $\Omega$, $\{u^k\}$ is equicontinuous at each point of $\Omega$. Hence, as $E$ is compact, we may construct a finite collection $\{B_r: r = 1, \ldots, m\}$ of open balls in $\mathbb{C}^d$ such that
\[
E \subseteq \bigcup_{r=1}^m B_r \subseteq \Omega
\]
and
\[
\forall r \quad \forall \mu_1, \mu_2 \in B_r \quad \forall k \quad \|u^k(\mu_1) - u^k(\mu_2)\| < \varepsilon/3. \tag{2.3}
\]
As $\{\lambda_i\}$ is assumed dense in $\Omega$,
\[
\forall r \quad \exists i_r \quad \lambda_{i_r} \in B_r. \tag{2.4}
\]
Consequently, as for each fixed $i$, we assume that $\{u^k(\lambda_i)\}$ is a convergent (and hence, Cauchy) sequence in $H$, there exists $N$ such that
\[
\forall r \quad k, j \geq N \implies \|u^k(\lambda_{i_r}) - u^j(\lambda_{i_r})\| < \varepsilon/3. \tag{2.5}
\]
Now, fix $\lambda \in E$. Use (2.2) to choose $r$ such that $\lambda \in B_r$. Use (2.4) to choose $i_r$ such that $\lambda_{i_r} \in B_r$. As $\lambda$ and $\lambda_{i_r}$ are both in $B_r$, we see from (2.3) that
\[
\forall k \quad \|u^k(\lambda) - u^k(\lambda_{i_r})\| < \varepsilon/3.
\]
Hence, using (2.5), we have that if $k, j \geq N$, then
\[
\|u^k(\lambda) - u^j(\lambda)\|
\begin{align*}
& \leq \|u^k(\lambda) - u^k(\lambda_{i_r})\| + \|u^k(\lambda_{i_r}) - u^j(\lambda_{i_r})\| + \|u^j(\lambda_{i_r}) - u^j(\lambda)\| \\
& < \varepsilon.
\end{align*}
\]
Since the concluding estimate in the previous paragraph holds for an arbitrary point $\lambda \in E$, $\{u^k\}$ is uniformly Cauchy on $E$. Since $E$ is an arbitrary compact subset of $\Omega$, $\{u^k\}$ is a Cauchy sequence in $\text{Hol}_H(\Omega)$. Therefore, $\{u^k\}$ converges in $\text{Hol}_H(\Omega)$.
2.2 The Proof of Theorem 1.2

Theorem 1.2 follows quickly from Proposition 2.1 and the following lemma.

Lemma 2.6. Wandering Isometry Lemma. Assume that Ω is an open set in \( C^d \), \( \{\lambda_i\} \) is a sequence in Ω, and \( \mathcal{H} \) is a Hilbert space. If \( \{u^k\} \) is a sequence in \( \text{Hol}_\mathcal{H}(\Omega) \) that is locally uniformly bounded on Ω, then there exists a subsequence \( \{u^{k_i}\} \) and a sequence \( \{V^l\} \) of unitary operators on \( \mathcal{H} \) such that for each fixed \( i \), \( \{V^l u^{k_i}(\lambda_i)\} \) is a convergent sequence in \( \mathcal{H} \).

Proof. If \( \mathcal{H} \) is finite dimensional, one can let each unitary be the identity, and the result is the regular Montel theorem. So we shall assume that \( \mathcal{H} \) is infinite dimensional. Let \( \{e_i\} \) be an orthonormal basis for \( \mathcal{H} \). For each fixed \( k \) let

\[
\mathcal{H}^k = \text{span} \{e_1, e_2, \ldots, e_k\},
\]

\[
\mathcal{M}_i^k = \text{span} \{u^k(\lambda_1), u^k(\lambda_2), \ldots, u^k(\lambda_i)\}, \quad i = 1, \ldots, k.
\]

For each \( k \) choose a unitary \( U^k \in \mathcal{B}(\mathcal{H}) \) satisfying

\[
U^k \mathcal{M}_i^k \subseteq \mathcal{H}^i, \quad i = 1, \ldots, k
\]

Observe that with this construction, for each fixed \( i \),

\[
\{U^k u^k(\lambda_i)\}_{k=1}^\infty
\]

is a bounded sequence in \( \mathcal{H}^i \), a finite dimensional Hilbert space. Therefore, there exist \( v_i \in \mathcal{H} \) and an increasing sequence of indices \( \{k_l\} \) such that

\[
U^{k_l} u^{k_l}(\lambda_i) \to v_i \quad \text{in} \quad \mathcal{H} \quad \text{as} \quad l \to \infty.
\]

Applying this fact successively with \( i = 1, i = 2, \) and so on, at each stage taking a subsequence of the previously selected subsequence, leads to a sequence \( \{v_i\} \) in \( \mathcal{H} \) and an increasing sequence of indices \( \{k_l\} \) such that

\[
U^{k_l} u^{k_l}(\lambda_i) \to v_i \quad \text{in} \quad \mathcal{H} \quad \text{as} \quad l \to \infty.
\]

for all \( i \). The lemma then follows if we let \( V^l = U^{k_l} \). \( \square \)

Proof of Theorem 1.2. Assume that \( \Omega \) is an open set in \( C^d \), \( \mathcal{H} \) is a Hilbert space, and \( \{u^k\} \) is a locally uniformly bounded sequence in \( \text{Hol}_\mathcal{H}(\Omega) \). The theorem follows from the classical Montel Theorem (with \( U^k = \text{id}_\mathcal{H} \) for all \( k \)) if \( \dim \mathcal{H} < \infty \). Therefore, we may assume that \( \dim \mathcal{H} = \infty \).

Fix a dense sequence \( \{\lambda_i\} \) in \( \Omega \). By Lemma 2.6 there exists a subsequence \( \{u^{k_{i_l}}\} \) and a sequence \( \{V^l\} \) of unitary operators on \( \mathcal{H} \) such that for each fixed \( i \), \( \{V^l u^{k_{i_l}}(\lambda_i)\} \) is a convergent sequence in \( \mathcal{H} \). Furthermore, as \( \{u^k\} \) is locally uniformly bounded, so also, \( \{V^l u^{k_{i_l}}\} \) is locally uniformly bounded. Therefore, Proposition 2.1 implies that \( \{V^l u^{k_{i_l}}\} \) is a convergent sequence in \( \text{Hol}_\mathcal{H}(\Omega) \). The theorem then follows by choosing \( \{U^k\} \) to be any sequence of unitaries in \( \mathcal{B}(\mathcal{H}) \) such that \( U^{k_{i_l}} = V^l \) for all \( l \). \( \square \)

3 Holomorphic Functions in Noncommuting Variables

If \( \Omega \) is finitely open in \( M^d \), we may construct a finitely compact-open exhaustion of \( \Omega \), i.e., an increasing sequence of compact sets \( \{K_i\} \) that satisfy

\[
K_1 \subset \text{int}(K_2) \subset K_2 \subset \text{int}(K_3) \subset \ldots
\]
and with $\Omega = \cup_i K_i$. For a set $E \subseteq \Omega$ and $f \in \text{Hol}(\Omega)$ we let
\[
\|f\|_E = \sup_{\lambda \in E} \|f(\lambda)\|
\]
and then in the usual way define a metric $d$ on $\text{Hol}(\Omega)$ with the formula
\[
d(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|f - g\|_{K_n}}{1 + \|f - g\|_{K_n}}.
\]
It then follows that $f_k \to f$ in the metric space $(\Omega, d)$ if and only if for each finitely compact set $K$ in $\Omega$, $\{f_k\}$ converges uniformly to $f$ on $K$, i.e.,
\[
\lim_{k \to \infty} \|f - f_k\|_K = 0.
\]
Furthermore, $\text{Hol}(\Omega)$ is a complete metric space when endowed with this topology of uniform convergence on finitely compact subsets of $\Omega$.

It is a straightforward exercise to extend Montel’s Theorem to the space $\text{Hol}_H(\Omega)$ when $\dim H$ is finite.

**Proposition 3.1.** If $\Omega$ is a finitely open set in $\mathbb{M}^d$, $H$ is a Hilbert space with $\dim H < \infty$, and $\{u^k\}$ is a finitely locally uniformly bounded sequence in $\text{Hol}_H(\Omega)$, then $\{u^k\}$ has a convergent subsequence.

Also, with the setup we have just described, mere notational changes to the proof of Proposition 2.1 yield a proof of the following proposition.

**Proposition 3.2.** Assume that $\Omega$ is a finitely open set in $\mathbb{M}^d$, $\{\lambda_i\}$ is a dense sequence in $\Omega$ with $\lambda_i \in \mathbb{M}^d_{n_i}$ for each $i$, and $H$ is a Hilbert space. If $\{u^k\}$ is sequence in $\text{Hol}_H(\Omega)$ that is finitely locally uniformly bounded on $\Omega$, and for each fixed $i$, $\{u^k(\lambda_i)\}$ is a convergent sequence in $B(\mathbb{C}^{n_i}, \mathbb{C}^{n_i} \otimes H)$, then $\{u^k\}$ is a convergent sequence in $\text{Hol}_H(\Omega)$.

Just as was the case for Proposition 2.1 in [5], it is possible to relax the assumption in Proposition 3.2 that $\{\lambda_i\}$ be a dense sequence in $\Omega$, to the assumption that $\{\lambda_i\}$ merely be a set of uniqueness for $\text{Hol}_H(\Omega)$ (see Proposition 6.2).

We now turn to an analog of Theorem 1.2 in the noncommutative setting.

**Lemma 3.3. Wandering Isometry Lemma (noncommutative case).** Assume that $\Omega$ is a finitely open set in $\mathbb{M}^d$ and $\{\lambda_i\}$ is a sequence in $\Omega$ (where, for each $i$, $\lambda_i \in \mathbb{M}^d_{n_i}$). If $H$ is an infinite dimensional Hilbert space and $\{u^k\}$ is sequence in $\text{Hol}_H(\Omega)$ with the property that $\{u^k(\lambda_i)\}$ is bounded for each $i$, then there exists a subsequence $\{u^{k_i}\}$ and a sequence $\{V^l\}$ of unitary operators on $H$ such that for each fixed $i$, $\{(\text{id}_{n_i} \otimes V^l) u^{k_i}(\lambda_i)\}$ is a convergent sequence in $B(\mathbb{C}^{n_i}, \mathbb{C}^{n_i} \otimes H)$.

**Proof.** Choose an increasing sequence $\{H_i\}$ of subspaces of $H$ with the property that
\[
\dim H_1 = n_1^2 \quad \text{and} \quad \forall i \geq 1 \quad \dim(\text{span}(H_{i+1} \otimes H_i)) = n_{i+1}^2,
\]
and for each $n$, let $\{e_1, \ldots, e_n\}$ denote the standard basis of $\mathbb{C}^n$.

Fix $k$. For each $i = 1, \ldots, k$, as $u^k(\lambda_i) : \mathbb{C}^{n_i} \to \mathbb{C}^{n_i} \otimes H$, there exist $n_i^2$ vectors $x_{r,s}^{k,i} \in H$, $r, s = 1, \ldots, n_i$, such that
\[
u^{k}(\lambda_i)e_r = \sum_{s=1}^{n_i} e_s \otimes x_{r,s}^{k,i}, \quad r = 1, \ldots, n_i.
\]
For each \( i = 1, \ldots, k \), define \( \mathcal{M}_i^k \) by
\[
\mathcal{M}_i^k = \text{span} \{ x_{r,s}^{k,i} : r, s = 1, \ldots, n_i \}.
\]
and then define a sequence of spaces \( \{ \mathcal{N}_i^k \} \) by setting \( \mathcal{N}_1^k = \mathcal{M}_1^k \) and
\[
\mathcal{N}_i^k = (\mathcal{M}_1^k + \mathcal{M}_2^k + \ldots + \mathcal{M}_i^k) \ominus (\mathcal{M}_1^k + \mathcal{M}_2^k + \ldots + \mathcal{M}_{i-1}^k),
\]
for \( i = 2, \ldots, k \). As for each \( i = 1, \ldots, k \), \( \dim \mathcal{M}_i^k \leq n_i^2 \), so also, for \( i = 1, \ldots, k \), \( \dim \mathcal{N}_i^k \leq n_i^2 \). Consequently, as the spaces \( \{ \mathcal{N}_i^k \} \) are also pairwise orthogonal, it follows that there exists a unitary \( U^k \in \mathcal{B}(\mathcal{H}) \) such that
\[
U^k(\mathcal{N}_i^k) \subseteq \mathcal{H}_1 \text{ and } U^k(\mathcal{N}_i^k) \subseteq \mathcal{H}_i \ominus \mathcal{H}_{i-1} \text{ for } i = 2, \ldots, k.
\]
With this construction it follows using (3.4) that
\[
(id_n \otimes U^k)u^k(\lambda_i)(\mathbb{C}^{n_i}) \subseteq \mathbb{C}^{n_i} \otimes \mathcal{H}_i, \quad i = 1, \ldots, k. \tag{3.5}
\]
Now observe that as (3.5) holds for each \( k \), for each fixed \( i \),
\[
(id_n \otimes U^k)u^k(\lambda_i)(\mathbb{C}^{n_i}) \subseteq \mathbb{C}^{n_i} \otimes \mathcal{H}_i, \quad k = i, i+1, \ldots,
\]
i.e.,
\[
\{ (id_n \otimes U^k)u^k(\lambda_i) \}_{k=i}^\infty
\]
is a bounded sequence in \( \mathcal{B}(\mathbb{C}^{n_i}, \mathbb{C}^{n_i} \otimes \mathcal{H}_i) \), a finite dimensional Hilbert space. Therefore, for each fixed \( i \), there exist \( L \in \mathcal{H} \) and an increasing sequence of indices \( \{ k_l \} \) such that
\[
U^{k_l}u^{k_l}(\lambda_i) \rightarrow L \quad \text{in } \mathcal{B}(\mathbb{C}^{n_i}, \mathbb{C}^{n_i} \otimes \mathcal{H}_i) \quad \text{as } l \rightarrow \infty.
\]
Applying this fact successively with \( i = 1, i = 2 \), and so on, at each stage taking a subsequence of the previously selected subsequence, leads to a sequence \( \{ L_i \} \) with \( L_i \in \mathcal{B}(\mathbb{C}^{n_i}, \mathbb{C}^{n_i} \otimes \mathcal{H}_i) \) for each \( i \) and an increasing sequence of indices \( \{ k_l \} \) such that
\[
\forall i \quad U^{k_l}u^{k_l}(\lambda_i) \rightarrow L_i \quad \text{in } \mathcal{B}(\mathbb{C}^{n_i}, \mathbb{C}^{n_i} \otimes \mathcal{H}_i) \quad \text{as } l \rightarrow \infty.
\]
The lemma then follows if we let \( V^l = U^{k_l} \). \hfill \Box

Lemma 3.3 suggests the following notation. Let \( \Omega \) be a finitely open set in \( \mathbb{M}^d \) and let \( \mathcal{H} \) be a Hilbert space. If \( U \) is a unitary acting on \( \mathcal{H} \), and \( f \in \text{Hol}_\mathcal{H}(\Omega) \) then we may define \( U \ast f \in \text{Hol}_\mathcal{H}(\Omega) \) by the formula
\[
\forall n \quad (U \ast f)|\Omega_n = (id_n \otimes U)f|\Omega_n.
\]
With this notation we may formulate a noncommutative analog of Theorem 1.2 in the noncommutative setting.

**Theorem 3.6.** If \( \Omega \) is a finitely open set in \( \mathbb{M}^d \), \( \mathcal{H} \) is a Hilbert space, and \( \{ u^k \} \) is a finitely locally uniformly bounded sequence in \( \text{Hol}_\mathcal{H}(\Omega) \), then there exists a sequence \( \{ U^k \} \) of unitary operators on \( \mathcal{H} \) such that \( \{ U^k \ast u^k \} \) has a convergent subsequence.

**Proof.** Assume that \( \Omega \) is an open set in \( \mathbb{M}^d \), \( \mathcal{H} \) is a Hilbert space, and \( \{ u^k \} \) is a finitely locally uniformly bounded sequence in \( \text{Hol}_\mathcal{H}(\Omega) \). If \( \dim \mathcal{H} < \infty \), then the theorem follows from Proposition 3.1 if we choose \( U^k = id_\mathcal{H} \) for all \( k \). Therefore, we assume that \( \dim \mathcal{H} = \infty \).

Fix a dense sequence \( \{ \lambda_i \} \) in \( \Omega \). By Lemma 3.3 there exists a subsequence \( \{ u^{k_l} \} \) and a sequence \( \{ V^l \} \) of unitary operators on \( \mathcal{H} \) such that for each fixed \( i \), \( \{ V^l u^{k_l}(\lambda_i) \} \) is a convergent sequence in \( \mathcal{B}(\mathbb{C}^{n_i}, \mathbb{C}^{n_i} \otimes \mathcal{H}) \). Furthermore, as \( \{ u^k \} \) is locally uniformly bounded, so also, \( \{ V^l u^{k_l} \} \) is locally uniformly bounded. Therefore, Proposition 2.1 implies that \( \{ V^l u^{k_l} \} \) is a convergent sequence in \( \text{Hol}_\mathcal{H}(\Omega) \). The theorem then follows by choosing \( \{ U^k \} \) to be any sequence of unitaries in \( \mathcal{B}(\mathcal{H}) \) such that \( U^{k_l} = V^l \) for all \( l \). \hfill \Box
4 Locally bounded nc Functions

Properties of \( \tau \)-holomorphic functions can be very sensitive to the choice of nc-topology \( \tau \). For example, if \( \tau \) is the *fat topology* studied in [3], then \( \tau \)-holomorphic functions satisfy a version of the Implicit Function Theorem. On the other hand if \( \tau \) is the *free topology*, studied in [2], then \( \tau \)-holomorphic functions satisfy the Oka-Weil Approximation Theorem. Remarkably, neither of these theorems holds for the other topology.

Also notice that if \( f \) is \( \tau \)-holomorphic in the sense of Definition 1.6, then necessarily \( \Omega \), the domain of \( f \), is an open set in the \( \tau \) topology: for each \( \lambda \in \Omega \) there exists \( B_\lambda \subseteq \Omega \) such that \( \lambda \in B_\lambda \in \tau \); hence, \( \Omega = \bigcup_\lambda B_\lambda \in \tau \).

There are no such subtleties between the nc-topologies when it comes to understanding the implications of local boundedness.

**Definition 4.1.** Assume that \( \tau \) is an nc-topology and \( \Omega \in \tau \). If \( \{u^k\} \) is a sequence in \( \text{Hol}_H(\Omega) \), we say that \( \{u^k\} \) is \( \tau \)-locally uniformly bounded on \( \Omega \) if for each \( \lambda \in \Omega \), there exists a \( \tau \)-open \( B \subseteq \Omega \) such that, \( \lambda \in B \) and
\[
\sup_k \|u^k\|_B < \infty.
\]

**Lemma 4.2.** Assume that \( \tau \) is an nc-topology and \( \Omega \in \tau \). Let \( u \in \text{Hol}(\Omega) \) and \( \{u^k\} \) be a sequence in \( \text{Hol}_H(\Omega) \). If \( \{u^k\} \) is \( \tau \)-locally uniformly bounded on \( \Omega \) and \( u^k \to u \) in \( \text{Hol}_H(\Omega) \), then \( u \in \text{Hol}_H(\Omega) \).

**Proof.** Under the assumptions of the lemma, we need to prove the following two assertions.

\[ u \text{ is an nc-function on } \Omega. \] (4.3)

\[ u \text{ is } \tau \text{-locally bounded on } \Omega. \] (4.4)

To prove (4.3), note first that as \( u \in \text{Hol}_H(\Omega) \), Condition (i) in Definition 1.4 holds. To verify Condition (ii), assume that \( \lambda, \mu, \lambda \oplus \mu \in \Omega \). Then, as \( u^k \to u \) in \( \text{Hol}_H(\Omega) \) and \( u^k \in \text{Hol}^\tau_H \) for all \( k \),
\[
u(\lambda \oplus \mu) = \lim_{k \to \infty} u^k(\lambda \oplus \mu)
= \lim_{k \to \infty} \left( u^k(\lambda) \oplus u^k(\mu) \right)
= \lim_{k \to \infty} u^k(\lambda) \oplus \lim_{k \to \infty} u^k(\mu)
= u(\lambda) \oplus u(\mu).
\]

Finally, note that if \( n \geq 1 \), \( S \in \mathbb{M}_n \) is invertible, and both \( \lambda \) and \( S\lambda S^{-1} \) are in \( \Omega_n \), then
\[
u(S\lambda S^{-1}) = \lim_{k \to \infty} u^k(S\lambda S^{-1})
= \lim_{k \to \infty} (S \otimes \text{id}_H) u^k(\lambda) S^{-1}
= (S \otimes \text{id}_H) u(\lambda) S^{-1},
\]
which proves Condition (iii).

To prove (4.4), fix \( \lambda \in \Omega \). As \( \{u^k\} \) is \( \tau \)-locally uniformly bounded on \( \Omega \), Definition 4.1 implies that there exist \( B \subseteq \Omega \) and a constant \( \rho \) such that \( \lambda \in B \in \tau \) and
\[
\sup_k \|u^k\|_B \leq \rho.
\]
Fix $\mu \in B$. As we assume that $u^k \to u$ in $\text{Hol}_H(\Omega)$, it follows that
\[ \|u(\mu)\| = \lim_{k \to \infty} \|u^k(\mu)\| \leq \rho. \]
But then,
\[ \|u\|_B \leq \rho. \]
As $B \in \tau$, this proves that $u$ is $\tau$-locally bounded on $\Omega$.

Definition 4.1 and Lemma 4.2, allow one to easily deduce Theorem 1.7 as a corollary of Theorem 3.6.

**Proof of Theorem 1.7.** As we assume that $\{u^k\}$ is a $\tau$-locally uniformly bounded sequence in $\text{Hol}_H^\tau(\Omega)$, $\{u^k\}$ is finitely locally uniformly bounded in $\text{Hol}_H(\Omega)$. Therefore, Theorem 3.6 implies that there exists a sequence $\{U^k\}$ of unitary operators on $\mathcal{H}$ such that $\{U^k \ast u^k\}$ has a subsequence that converges in $\text{Hol}_H(\Omega)$. Consequently, we may choose $u \in \text{Hol}_H(\Omega)$ and an increasing sequence of indices $\{k_i\}$ such that $U^{k_i} \ast u^{k_i} \to u$ in $\text{Hol}(\Omega)$. The proof is completed by observing that Lemma 4.2 implies that $u \in \text{Hol}_H^\tau(\Omega)$. □

Let us emphasize that Theorem 1.7 asserts that $U^{k_i} \ast u^{k_i}$ converges to $u$, which is in $\text{Hol}_H^\tau(\Omega)$, uniformly on sets that are compact in the finite topology; it does not say that it converges uniformly on compact sets in the $\tau$ topology.

Note that the proofs of Lemma 4.2 and Theorem 1.7 work identically if $u^k$ are just assumed to be in $\text{Hol}_H^{nc}(\Omega)$; so we get

**Theorem 4.5.** Let $\Omega$ be a finitely open set in $M^d$, $\mathcal{H}$ is a Hilbert space, and $\{u^k\}$ is a finitely locally uniformly bounded sequence in $\text{Hol}^{nc}_H(\Omega)$, then there exists a sequence $\{U^k\}$ of unitary operators on $\mathcal{H}$ such that $\{U^k \ast u^k\}$ has a subsequence that converges finitely locally uniformly to an element of $\text{Hol}^{nc}_H(\Omega)$.

### 5 Some Applications

A useful construct in the study of $\tau$-holomorphic functions is the duality construction. If $\Omega$ is a finitely open set it is natural to consider the algebraic tensor product $\text{Hol}(\Omega)^* \otimes \text{Hol}(\Omega)$. This space can concretely be realized as the set of functions $A$ defined on
\[ \Omega \boxtimes \Omega = \bigcup_{n=1}^{\infty} (\Omega \cap M^d_n) \times (\Omega \cap M^d_n), \]
and such that there exist a finite dimensional Hilbert space $\mathcal{H}$ and $u, v \in \text{Hol}_H(\Omega)$ such that
\[ A(\lambda, \mu) = v(\mu)^* u(\lambda), \quad (\lambda, \mu) \in \Omega \boxtimes \Omega. \]

As the functions in $\text{Hol}(\Omega)^* \otimes \text{Hol}(\Omega)$ are holomorphic in $\lambda$ for each fixed $\mu$ and anti-holomorphic in $\mu$ for each fixed $\lambda$, we may complete $\text{Hol}(\Omega)^* \otimes \text{Hol}(\Omega)$ in the topology of uniform convergence on finitely compact subsets of $\Omega \boxtimes \Omega$ to obtain the space of hereditary holomorphic functions on $\Omega$, $\text{Her}(\Omega)$. Inside $\text{Her}(\Omega)$, we may define a cone $\mathcal{P}$ by
\[ \mathcal{P} = \{ u(\mu)^* u(\lambda) : u \in \text{Hol}_H(\Omega) \text{ for some Hilbert space } \mathcal{H} \}. \]

**Theorem 5.1.** $\mathcal{P}$ is closed in $\text{Her}(\Omega)$. 

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Proof. Assume that \( \{v^k\} \) is a sequence with \( v^k \in \text{Hol}_\mathcal{H}_k(\Omega) \) for each \( k \) and with \( v^k(\mu)^*v^k(\lambda) \to A \) in \( \text{Her}(\Omega) \). We may assume that \( \mathcal{H}_k \) is separable for each \( k \). Fix a separable infinite dimensional Hilbert space \( \mathcal{H} \) and for each each \( k \) choose an isometry \( V^k : \mathcal{H}_k \to \mathcal{H} \). If for each \( k \) we let \( u^k = V^k * v^k \), then \( \{u^k\} \) is a sequence in \( \text{Hol}_\mathcal{H}(\Omega) \) and \( u^k(\mu)^*u^k(\lambda) \to A \) in \( \text{Her}(\Omega) \).

Now, as \( u^k(\mu)^*u^k(\lambda) \to A \) in \( \text{Her}(\Omega) \) it follows that \( \{u^k\} \) is a finitely locally uniformly bounded sequence in \( \text{Hol}_\mathcal{H}(\Omega) \). Hence, by Theorem 3.6 there exists a sequence \( U^k \) of unitary operators on \( \mathcal{H} \) such that \( \{U^k * u^k\} \) has a convergent subsequence, i.e., there exists \( u \in \text{Hol}_\mathcal{H}(\Omega) \) and an increasing sequence of indices \( \{k_l\} \) such that \( U^{k_l} * u^{k_l} \to u \). But then, for each \( (\lambda, \mu) \in \Omega \times \Omega \),

\[
A(\lambda, \mu) = \lim_{k \to \infty} u^k(\mu)^*u^k(\lambda)
= \lim_{l \to \infty} u^{k_l}(\mu)^*u^{k_l}(\lambda)
= \lim_{l \to \infty} (U^{k_l} * u^{k_l})(\mu)^*(U^{k_l} * u^{k_l})(\lambda)
= u(\mu)^*u(\lambda),
\]

i.e., \( A \in \mathcal{P} \). \( \square \)

We also may use wandering Montel Theorems to study sums of \( \tau \)-holomorphic dyads. We let \( \text{Her}^\tau(\Omega) \) denote the closure of

\[\{v(\mu)^*u(\lambda) : u, v \in \text{Hol}^\tau_\mathcal{H}(\Omega) \text{ for some finite dimensional Hilbert space } \mathcal{H}\}\]

inside \( \text{Her}(\Omega) \) and define \( \mathcal{P}^\tau \) in \( \text{Her}^\tau(\Omega) \) by

\[\mathcal{P}^\tau = \{u(\mu)^*u(\lambda) : u \in \text{Hol}^\tau_\mathcal{H}(\Omega) \text{ for some Hilbert space } \mathcal{H}\}.\]

**Theorem 5.2.** Let \( \tau \) be an nc-topology, and \( \Omega \in \tau \). Then \( \mathcal{P}^\tau \) is closed in \( \text{Her}^\tau(\Omega) \).

**Proof.** Assume that \( u^k(\mu)^*u^k(\lambda) \to A \) in \( \text{Her}(\Omega) \), where, as in the proof of Theorem 5.1 we may assume that \( u^k \in \text{Hol}^\tau_\mathcal{H}(\Omega) \) for each \( k \). By Theorem 1.7 there exist \( u \in \text{Hol}^\tau_\mathcal{H}(\Omega) \), a sequence \( U^k \) of unitary operators on \( \mathcal{H} \), and an increasing sequence of indices \( \{k_l\} \) such that \( U^{k_l} * u^{k_l} \to u \). But then as in the proof of Theorem 5.1 \( A(\lambda, \mu) = u(\mu)^*u(\lambda) \) for all \( (\lambda, \mu) \in \Omega \times \Omega \), i.e., \( A \in \mathcal{P}^\tau \). \( \square \)

Finally, we shall prove that the model cone is closed; this is the key ingredient in the proof of the realization formula for free holomorphic functions [2, 7, 11]. Let \( \delta \) be a \( J \)-by-\( L \) matrix whose entries are free polynomials in \( d \) variables. We define \( B_\delta \) to be the polynomial polyhedron

\[B_\delta := \{x \in \mathbb{M}^d : \|\delta(x)\| < 1\} \]

The free topology is the nc-topology generated by the sets \( B_\delta \), as \( \delta \) ranges over all matrices of polynomials. The model cone \( \mathcal{C} \) is the set of hereditary functions on \( B_\delta \) of the form

\[
\mathcal{C} := \{ \text{id}_{\mathbb{M}^d} \otimes \left( \text{id} - \delta(\mu)^*\delta(\lambda) \right) \otimes \text{id}_\mathcal{H} \otimes \text{id}_\mathcal{H} : u \in \text{Hol}_\mathcal{H}(B_\delta) \text{ and } u \text{ is nc, for some Hilbert space } \mathcal{H}\}.
\]

We write the tensors vertically just to enhance readability.

**Theorem 5.4.** The model cone \( \mathcal{C} \), defined in (5.3) is closed in \( \text{Her}(B_\delta) \).
Proof. Suppose $u^k$ is a sequence of nc functions in $\text{Hol}_H(B_δ)$ (we may assume the space $H$ is the same for each $u^k$, as in the proof of Thm. 5.1), so that

$$
\text{id}_{c,J} \otimes u^k(\mu)^* \left( \text{id} - \frac{\delta(\mu) \delta(\lambda)}{\text{id}_H} \right) \frac{\text{id}_{c,J}}{d(u(\mu)^*)} \otimes u^k(\lambda)
$$

converges in $\text{Her}(B_δ)$ to $A(\lambda, \mu)$. On any finitely compact set, $\|\delta(x)\|$ will be bounded by a constant that is strictly less than one. Since (5.5) converges uniformly on finitely compact subsets of $B_δ \otimes B_δ$, this means that $u^k$ is a finitely locally uniformly bounded sequence. Therefore by Thm. 3.6 there exist unitaries $U^k$ such that $U^k * u^k$ has a convergent subsequence, which converges to some nc function $u \in \text{Hol}_H(B_δ)$. Then

$$
A(\lambda, \mu) = \frac{\text{id}_{c,J}}{d(u(\mu)^*)} \left( \text{id} - \frac{\delta(\mu) \delta(\lambda)}{\text{id}_H} \right) \frac{\text{id}_{c,J}}{d(u(\lambda))},
$$

as desired.

\[\square\]

6 Sets of Uniqueness

In this section we shall show that the assumption in Propositions 2.1 and 3.2 that $\{\lambda_i\}$ be a dense sequence in $\Omega$, can be relaxed to the assumption that $\{\lambda_i\}$ be a set of uniqueness for $\text{Hol}(\Omega)$. We remark that it is an elementary fact that if $H$ is a Hilbert space, then $\{\lambda_i\}$ is a set of uniqueness for $\text{Hol}_H(\Omega)$ if and only if $\{\lambda_i\}$ is a set of uniqueness for $\text{Hol}(\Omega)$.

The following proposition is essentially the same as the Arendt-Nikolski Theorem 1.1, so we shall omit the proof.

**Proposition 6.1.** Assume that $\Omega$ is an open set in $\mathbb{C}^d$, $\{\lambda_i\}$ is a sequence in $\Omega$ that is a set of uniqueness for $\text{Hol}_H(\Omega)$, and $H$ is a Hilbert space. If $\{u^k\}$ is sequence in $\text{Hol}_H(\Omega)$ that is locally uniformly bounded on $\Omega$, and for each fixed $i$, $\{u^k(\lambda_i)\}$ is a convergent sequence in $H$, then $\{u^k\}$ converges in $\text{Hol}_H(\Omega)$.

Here is the graded version.

**Proposition 6.2.** Assume that $\Omega$ is a finitely open set in $\mathbb{M}^d$, $\{\lambda_i\}$ is a sequence in $\Omega$ (with $\lambda_i \in \mathbb{M}^d$ for each $i$) that is a set of uniqueness for $\text{Hol}_H(\Omega)$, and $H$ is a Hilbert space. If $\{u^k\}$ is sequence in $\text{Hol}_H(\Omega)$ that is finitely locally uniformly bounded on $\Omega$, and for each fixed $i$, $\{u^k(\lambda_i)\}$ is a convergent sequence in $H$, then $\{u^k\}$ converges in $\text{Hol}_H(\Omega)$.

**Proof.** The theorem will follow if we can show that $\{u^k|\Omega_n\}$ is a convergent sequence for each $n$. Accordingly, fix $n$ and adopt the notation $H_n$ for the holomorphic $\mathcal{B}(\mathbb{C}^n, \mathbb{C}^n \otimes H)$-valued functions defined on $\Omega_n$. Thus, $\{u^k|\Omega_n\}$ is a locally uniformly bounded sequence in $H_n$. Furthermore, if $\{\eta_j\}$ is an enumeration of $\{\lambda_i : i \geq 1\} \cap \Omega_n$, as $\{\lambda_i\}$ is a set of uniqueness for $\text{Hol}(\Omega)$, $\{\eta_k\}$ is a set of uniqueness for both $\text{Hol}(\Omega_n)$ and $H_n$. Finally, let $u^k(\eta_j) \to u_j$ as $k \to \infty$ for each $j$.

For fixed $\alpha \in \mathbb{C}^n$ and $\beta \in \mathbb{C}^n$, define $f^k_{\alpha, \beta} \in \text{Hol}(\Omega_n)$ by

$$
f^k_{\alpha, \beta}(\lambda) = \langle u^k(\lambda) \alpha, \beta \rangle_{\mathbb{C}^n \otimes H}, \quad \lambda \in \Omega.
$$

Noting that,

$$
|f^k_{\alpha, \beta}(\lambda)| = |\langle u^k(\lambda) \alpha, \beta \rangle| \leq \|u^k(\lambda)\| \|\alpha\| \|\beta\|,
$$

\[\text{i.e., if } f \in \text{Hol}(\Omega) \text{ and } f(\lambda_i) = 0 \text{ for all } i, \text{ then } f(\lambda) = 0 \text{ for all } \lambda \in \Omega.\]
it follows that \( \{ f^k_{\alpha,\beta} \} \) is locally uniformly bounded on \( \Omega_n \). Therefore by Montel’s Theorem, \( \{ f^k_{\alpha,\beta} \} \) has compact closure in \( \text{Hol}(\Omega_n) \).

We claim that \( \{ f^k_{\alpha,\beta} \} \) has a unique cluster point. For assume that \( \{ f^{kr}_{\alpha,\beta} \} \) and \( \{ f^{ks}_{\alpha,\beta} \} \) are subsequences of \( \{ f^k_{\alpha,\beta} \} \) with \( f^{kr}_{\alpha,\beta} \to f \) and \( f^{ks}_{\alpha,\beta} \to g \). Then, as we assume for each \( j \), \( u^k(\eta_j) \to u_j \) as \( k \to \infty \),

\[
f(\eta_i) = \lim_{r \to \infty} f^{kr}_{\alpha,\beta}(\eta_i)
= \lim_{r \to \infty} \langle u^k(\eta_i)\alpha, \beta \rangle
= \langle u(\alpha), \beta \rangle
= \lim_{s \to \infty} \langle u^k(\eta_i)\alpha, \beta \rangle
= \lim_{s \to \infty} f^{ks}_{\alpha,\beta}(\eta_i)
= g(\eta_i).
\]

Hence, as \( \{ \eta_i \} \) is a set of uniqueness, \( f = g \). Since \( \{ f^k_{\alpha,\beta} \} \) has a unique cluster point, we have shown that for each \( \alpha \in \mathbb{C}^n \) and \( \beta \in \mathbb{C}^n \otimes \mathcal{H} \), there exists \( f_{\alpha,\beta} \in \text{Hol}(\Omega_n) \) such that

\[
f^k_{\alpha,\beta} \to f_{\alpha,\beta} \text{ in } \text{Hol}(\Omega_n) \text{ as } k \to \infty. \tag{6.5}
\]

Now fix \( \lambda \in \Omega_n \) and define \( L_\lambda \) by

\[
L_\lambda(\alpha, \beta) = f_{\alpha,\beta}(\lambda), \quad \alpha \in \mathbb{C}^n, \beta \in \mathbb{C}^n \otimes \mathcal{H}. \tag{6.6}
\]

Observe that (6.3) and (6.5) imply that \( L_\lambda \) is a sesqui-linear functional on \( \mathbb{C}^n \times (\mathbb{C}^n \otimes \mathcal{H}) \). Furthermore, (6.4) and (6.5) imply that \( L_\lambda \) is bounded. Therefore, by the Riesz Representation Theorem, there exists \( u(\lambda) \in \mathcal{B}(\mathbb{C}^n, \mathbb{C}^n \otimes \mathcal{H}) \) such that

\[
\forall \alpha \in \mathbb{C}^n \forall \beta \in \mathbb{C}^n \otimes \mathcal{H} \quad L_\lambda(\alpha, \beta) = \langle u(\lambda)\alpha, \beta \rangle,
\]

or equivalently,

\[
\forall \alpha \in \mathbb{C}^n \forall \beta \in \mathbb{C}^n \otimes \mathcal{H} \quad \langle u(\lambda)\alpha, \beta \rangle = f_{\alpha,\beta}(\lambda).
\]

The function \( u \) constructed in the previous paragraph has the following properties: it is holomorphic,

\[
\forall \lambda \in \Omega_n \quad u^k(\lambda) \to u(\lambda) \quad \text{weakly in } \mathcal{B}(\mathbb{C}^n, \mathbb{C}^n \otimes \mathcal{H}) \quad \text{as } k \to \infty, \tag{6.7}
\]

and

\[
\forall j \quad u^k(\eta_j) \to u(\eta_j) \quad \text{in norm in } \mathcal{B}(\mathbb{C}^n, \mathbb{C}^n \otimes \mathcal{H}) \quad \text{as } k \to \infty. \tag{6.8}
\]

Claim 6.9.

\[
u^k(\mu)^* u^k(\lambda) \to u(\mu)^* u(\lambda) \quad \text{in } \text{Her}(\Omega_n) \quad \text{as } k \to \infty.
\]

To prove this claim, first note that as we are assuming \( \{ u^k \} \) is a locally uniformly bounded sequence in \( \text{Hol}_H(\Omega_n) \), \( \{ u^k(\mu)^* u^k(\lambda) \} \) is a locally uniformly bounded sequence in \( \text{Her}(\Omega_n) \). Therefore, the claim follows from Montel’s Theorem if we can show that

\[
A(\lambda, \mu) = u(\mu)^* u(\lambda) \tag{6.10}
\]

whenever \( \{ k_r \} \) is a sequence of indices such that

\[
u^{kr}(\mu)^* u^{kr}(\lambda) \to A(\lambda, \mu) \quad \text{in } \text{Her}(\Omega_n) \quad \text{as } r \to \infty. \tag{6.11}
\]
But if (6.11) holds, then (6.8) implies that for each independently chosen $i$ and $j$,
\[ A(\eta_j, \eta_i) = \lim_{r \to \infty} u^{kr}(\eta_i)^* u^{kr}(\eta_j) = u^{kr}(\eta_i)^* u^{kr}(\eta_j). \]

Since both sides of (6.10) are holomorphic in $\lambda$ and anti-holomorphic in $\mu$, and $\{\eta_i\}$ is a set of uniqueness, it follows that (6.10) holds for all $\lambda, \mu \in \Omega$. This completes the proof of Claim 6.9.

Finally, fix $\lambda \in \Omega$. By (6.7), $\{u^k(\lambda)\}$ converges weakly in $\mathcal{B}(\mathbb{C}^n, \mathbb{C}^n \otimes \mathcal{H})$ to $u(\lambda)$ and by Claim 6.9, $u^k(\lambda)^* u^k(\lambda) \to u(\lambda)^* u(\lambda)$. Therefore, $u^k(\lambda) \to u(\lambda)$ in norm in $\mathcal{B}(\mathbb{C}^n, \mathbb{C}^n \otimes \mathcal{H})$. Since this holds for all $\lambda \in \Omega$, the proof of Proposition 6.2 may be completed by an application of Proposition 3.2.

References


