THIN SEQUENCES AND THE GRAM MATRIX

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Abstract. We provide a new proof of Volberg’s Theorem characterizing thin interpolating sequences as those for which the Gram matrix associated to the normalized reproducing kernels is a compact perturbation of the identity. In the same paper, Volberg characterized sequences for which the Gram matrix is a compact perturbation of a unitary as well as those for which the Gram matrix is a Schatten-2 class perturbation of a unitary operator. We extend this characterization from 2 to \( p \), where \( 2 \leq p \leq \infty \).

1. Introduction

Let \( \mathbb{D} \) denote the open unit disk and \( \mathbb{T} \) the unit circle. Given \( \{\alpha_j\} \), a Blaschke sequence of points in \( \mathbb{D} \), we let \( B \) denote the corresponding Blaschke product and \( B_n \) denote the Blaschke product with the zero \( \alpha_n \) removed. Further, we let \( \delta_j = |B_j(\alpha_j)| \), \( k_j = \frac{1}{1-\alpha_j z} \) denote the Szegő kernel (the reproducing kernel for \( H^2 \)) at \( \alpha_j \), \( g_j = k_j/\|k_j\| \) the \( H^2 \)-normalized kernel, and \( G \) the Gram matrix with entries \( G_{ij} = \langle g_j, g_i \rangle \). In the second part of \([10, \text{Theorem 2}]\), Volberg’s goal was to develop a condition ensuring that \( \{g_n\} \) is near an orthogonal basis; by this, one means that there exist \( U \) unitary and \( K \) compact such that

\[
g_n = (U + K)e_n,
\]

where \( \{e_n\} \) is the standard orthogonal basis for \( \ell^2 \). By \([7, \text{Section 3}] \) or \([4, \text{Proposition 3.2}] \), this is equivalent to the Gram matrix defining a bounded operator of the form \( I + K \) with \( K \) compact. Following Volberg and anticipating the connection to the Schatten-\( p \) classes, we call such bases \( U + S_\infty \) bases. Volberg showed that \( \{g_n\} \) is a \( U + S_\infty \) basis if and only if \( \lim_n \delta_n = 1 \); in other words, if and only if \( \{\alpha_n\} \) is a thin sequence. Assuming \( \{g_n\} \) is a \( U + S_\infty \) basis, it is not difficult to show that the sequence \( \{\alpha_n\} \) must be thin. But Volberg’s proof of the converse is more difficult and depends on the main lemma of a paper of Axler, Chang and Sarason \([2, \text{Lemma 5}] \), estimating the norm of a certain product of Hankel operators as well as a factorization theorem for Blaschke products. The lemma in \([2]\) uses maximal functions and a certain distribution function inequality. A more direct proof of Volberg’s result is desirable, and we provide a simpler proof of this result in Theorem 3.5 of this paper.

In a second theorem, letting \( S_2 \) denote the class of Hilbert-Schmidt operators, Volberg showed (see \([10, \text{Theorem 3}] \)) that \( \{g_n\} \) is a \( U + S_2 \) basis if and only if \( \prod_{n=1}^\infty \delta_n \) converges. We are interested in estimates for the “in-between” cases. We provide a new proof of Volberg’s theorem for \( p = \infty \) and prove the following theorem.

**Theorem 1.1.** For \( 2 \leq p < \infty \), the operator \( G - I \in S_p \) if and only if \( \sum_n (1 - \delta_n^2)^{p/2} < \infty \).

Volberg’s theorem covered the cases \( p = 2 \) and \( p = \infty \), but our proofs differ in the following way: Instead of using the results of \([2]\) and theorems about Hankel operators, we use the

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relationship between growth estimates of functions that do interpolation on thin sequences (see [5], [6]) and the norm of the Gram matrix. This simplifies previous proofs and provides the best estimates available.

2. Preliminaries and notation

Let \( \{\alpha_j\} \) be a sequence in \( D \) with corresponding Blaschke product \( B \), and \( B_j \) be the Blaschke product with zeroes at every point in the sequence except \( \alpha_j \) and \( \delta_j = |B_j(\alpha_j)| \). The separation constant \( \delta \) is defined to be \( \delta := \inf_j \delta_j \). Carleson’s interpolation theorem says that the sequence \( \{\alpha_j\} \) is interpolating if and only if \( \delta > 0 \), [3]. The sequence \( \{\alpha_j\} \) is said to be thin if \( \lim_{j \to \infty} \delta_j = 1 \). Given a thin sequence we may arrange the \( \delta_j \) in increasing order and rearrange the zeros of the Blaschke product accordingly.

Recall that if \( T \) is an operator on a Hilbert space \( H \) and \( \lambda_n \) is the \( n \)th singular value of \( T \), then given \( p \) with \( 1 \leq p < \infty \) the Schatten-\( p \) class, \( S_p \), is defined to be the space of all compact operators with corresponding singular sequence in \( \ell^p \), the space of \( p \)-summable sequences. Then \( S_p \) is a Banach space with norm

\[
\|T\|_p = \left( \sum |\lambda_n|^p \right)^{1/p}.
\]

For \( p = \infty \), we let \( S_\infty \) denote the space of compact operators.

Recall that \( k_j \) denotes the Szegő kernel, \( g_j = k_j/\|k_j\| \), and \( G \) the Gram matrix with entries \( G_{ij} = \langle g_j, g_i \rangle \). (The Gram matrix depends of course on the sequence \( \{\alpha_j\} \), but we suppress this in the notation). For \( \{\alpha_j\} \) interpolating, we let \( D \) be the diagonal matrix with entries \( 1/B_j(\alpha_j) \). It is known (see, for example, formula (26) of [8]) that

\[
G^{-1} = D^* G^t D.
\]

For a given sequence \( \{\alpha_j\} \), the interpolation constant is the infimum of those \( M \) such that for any sequence \( \{a_j\} \) in \( \ell^\infty \), one can find a function \( f \) in \( H^\infty \) with \( f(\alpha_j) = a_j \) and \( \|f\|_\infty \leq M \|a\|_\infty \). We shall let \( M(\delta) \) denote the supremum of the interpolation constants over all sequences \( \{\alpha_j\} \) with separation constant \( \delta \).

The following result is due essentially to A. Shields and H. Shapiro [9]. See [1, Proposition 9.5] for a proof of this version.

**Proposition 2.1.** Let \( \{\alpha_j\} \) be an interpolating sequence in \( D \).

(i) If the interpolation constant is \( M \), then both \( \|G\| \) and \( \|G^{-1}\| \) are bounded by \( M^2 \).

(ii) If \( \|G\| = C_1 \) and \( \|G^{-1}\| = C_2 \), then the interpolation constant is bounded by \( \sqrt{C_1 C_2} \).

We shall use the following estimate of J.P. Earl (see [5] or [6]) to obtain our results.

**Theorem 2.2** (Earl’s Theorem). The interpolation constant \( M(\delta) \) satisfies

\[
M(\delta) \leq \left( \frac{1 + \sqrt{1 - \delta^2}}{\delta} \right)^2.
\]

3. Schatten-\( p \) classes

In this section we provide estimates on the Schatten-\( p \) norm of \( G - I \). We will need the theorem and lemma below.
Theorem 3.1. (see e.g. [11, Theorem 1.33]) Let $T$ be an operator on a separable Hilbert space, $\mathcal{H}$.

If $0 < p \leq 2$ then

$$\|T\|_{S_p}^p = \inf \left\{ \sum_n \|Te_n\|^p : \{e_n\} \text{ is any orthonormal basis in } \mathcal{H} \right\}$$

and if $2 \leq p < \infty$

$$\|T\|_{S_p}^p = \sup \left\{ \sum_n \|Te_n\|^p : \{e_n\} \text{ is any orthonormal basis in } \mathcal{H} \right\}.$$

We say that two sequences $\{x_n\}$ and $\{y_n\}$ of positive numbers are equivalent if there exist constants $c$ and $C$, independent of $n$, such that $cy_n \leq x_n \leq Cy_n$ for all $n$. We write $x_n \asymp y_n$.

We will also write $A \asymp B$ to indicate that there exists a constant $C$ such that $A \leq CB$.

Lemma 3.2. Let $\{e_n\}$ denote the standard orthonormal basis for $\ell^2$ and $\{\alpha_j\}$ be an interpolating sequence in $\mathbb{D}$ with corresponding $\delta_j$. Then

$$\|(G-I)e_n\| \asymp \sqrt{1 - \delta_n^2}.$$  

Proof. We have

$$\|(G-I)e_n\|^2 = \langle (G^* - I)(G-I)e_n, e_n \rangle$$

$$= \left\langle \begin{pmatrix} \langle g_n, g_1 \rangle & \langle g_n, g_1 \rangle \\ \vdots & \vdots \\ \langle g_n, g_{n-1} \rangle & \langle g_n, g_{n-1} \rangle \\ 0 & 0 \\ \langle g_n, g_{n+1} \rangle & \langle g_n, g_{n+1} \rangle \end{pmatrix}, \begin{pmatrix} \langle g_n, g_1 \rangle \\ \vdots \\ \langle g_n, g_{n-1} \rangle \\ 0 \\ \langle g_n, g_{n+1} \rangle \end{pmatrix} \right\rangle$$

$$= \sum_{j \neq n} \langle g_n, g_j \rangle \langle g_n, g_j \rangle$$

$$= \sum_{j \neq n} \frac{\sqrt{1 - |\alpha_j|^2} \sqrt{1 - |\alpha_n|^2}}{1 - \bar{\alpha}_n \alpha_j}^2$$

$$= \sum_{j \neq n} 1 - \frac{|\alpha_j - \alpha_n|^2}{1 - \bar{\alpha}_n \alpha_j}.$$  

But $-\log x \geq 1 - x$ for $x > 0$ and $-\log x < c(1-x)$ for $x < 1$ bounded away from 0 and some constant $c$ independent of $x$, so $-\log x \asymp 1 - x$ for $x$ bounded away from 0. Consequently,

$$\|(G-I)e_n\|^2 \asymp \sum_{j \neq n} -\log \left| \frac{\alpha_j - \alpha_n}{1 - \bar{\alpha}_n \alpha_j} \right|^2$$

$$= -\log \prod_{j \neq n} \left| \frac{\alpha_j - \alpha_n}{1 - \bar{\alpha}_n \alpha_j} \right|^2$$

$$= -\log \delta_n^2$$

$$\asymp 1 - \delta_n^2.$$
Note that the constants involved do not depend on $n$. \hfill \Box

Combining the lemma with Theorem 3.1, we obtain the following theorem.

**Theorem 3.3.** The following estimates hold:

- If $2 \leq p < \infty$ then
  \[
  \sum_n (1 - \delta_n)^{\frac{p}{2}} \lesssim \|G - I\|_{\mathcal{S}_p};
  \]

- If $0 < p \leq 2$ then
  \[
  \|G - I\|_{\mathcal{S}_p} \lesssim \sum_n (1 - \delta_n)^{\frac{p}{2}};
  \]

- If $p = 2$ then
  \[
  \sum_n (1 - \delta_n) \asymp \|G - I\|_{\mathcal{S}_2}^2.
  \]

**Lemma 3.4.** Let $\{\alpha_j\}$ be an interpolating sequence and $G$ the corresponding Gram matrix. Let $C = \|G^{-1}\|$. Then $\|G - I\| \leq C - 1$.

*Proof.* By (2.1), we have $G \leq CI$, and as $G$ is a positive operator, we have $G \geq (1/C)I$. Therefore
\[
\left(\frac{1}{C} - 1\right)I \leq G - I \leq (C - 1)I,
\]
and as $C > 1$, we get $\|G - I\| \leq C - 1$. \hfill \Box

In what follows, for a positive integer $N$, we let $G_N$ denote the lower right-hand corner of the Gram matrix obtained by deleting the first $N$ rows and columns of $G$. Thus,
\[
G_N = \begin{pmatrix}
1 & \langle g_{N+1}, g_N \rangle & \cdots & \langle g_{N+j}, g_N \rangle & \cdots \\
\langle g_N, g_{N+1} \rangle & 1 & \cdots & \langle g_{N+j}, g_{N+1} \rangle & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\langle g_{N+j}, g_{N+1} \rangle & \cdots & \cdots & 1 & \langle g_{N+j}, g_{N+1} \rangle \\
\end{pmatrix}
\]
and $\lambda_n \geq 0$ denotes the $n$-th singular value of $G - I$, where the singular values are arranged in decreasing order.

We are now ready to provide our simpler proof of Volberg’s result [10, Theorem 2, p. 215].

**Theorem 3.5.** The sequence $\{\alpha_n\}$ is a thin sequence if and only if the Gram matrix $G$ is the identity plus a compact operator.

*Proof.* ($\Rightarrow$) Suppose $\{\alpha_n\}$ is thin. By discarding finitely many points in the sequence $(\alpha_n)$, we can assume that the sequence has a positive separation constant, and hence is interpolating.

Let $G_N$ be the Gram matrix of $\{g_j\}$ for $j \geq N$. We shall let $\delta_{N,j}$ denote the $\delta_j$ defined for $G_N$ (that is, corresponding to the Blaschke sequence $\{\alpha_j\}_{j \geq N}$). Note that $\delta_{N,j} \geq \delta_{N+j}$ for $j = 0, 1, 2, \ldots$, and so we have that $\delta(N) := \inf_j \delta_{N,j} \geq \inf_j \delta_{N+j} = \delta_N$. By Theorem 2.2 and Proposition 2.1,
\[
\|G_N^{-1}\| \leq (M(\delta(N)))^2 \leq \frac{(1 + \sqrt{1 - \delta(N)})^4}{\delta(N)^4} \leq \frac{(1 + \sqrt{1 - \delta_N^2})^4}{\delta_N^4}.
\]
Applying Lemma 3.4
\[ \|G_N - I_N\| \leq \left( \frac{\left(1 + \sqrt{1 - \delta_N^2}\right)^4}{\delta_N^4} - 1 \right) \leq C\sqrt{1 - \delta_N}, \]
where \( C \) is a constant independent of \( N \). Since \( \sqrt{1 - \delta_N} \to 0 \) as \( N \to \infty \), we conclude that \( G - I \) is compact.

(\( \Leftarrow \)) From (2.1), we have
\[ (3.1) \quad G^{-1} - I = D^*(G^t - I)D + [D^*D - I]. \]
If \( G - I \) is compact, then so are \( G^t - I \) and \( G^{-1} - I = G^{-1}(I - G) \). Therefore from (3.1), we have \( D^*D - I \) is compact, which means \( \lim_{j \to \infty} \delta_j^2 = 1 \). Consequently, the sequence is thin. \( \square \)

**Theorem 3.6.** For \( 2 \leq p < \infty \), the operator \( G - I \in S_p \) if and only if \( \sum_n (1 - \delta_n^2)^{p/2} < \infty \).

**Proof.** By Theorem 3.3, if \( G - I \in S_p \), then the sum is finite.

Now suppose the sum is finite. Using Lemma 3.4 as in Theorem 3.5, we have
\[ \|G_N - I_N\| \leq C\sqrt{1 - \delta_N}, \]
where \( C \) is independent of \( N \).

By [11, Theorem 1.4.11],
\[ |\lambda_{N+1}| \leq \inf \{\|G - I\| - F\} : F \in F_N, \]
where \( F_N \) is the set of all operators of rank less than or equal to \( N \). Therefore, taking \( F \) to be the matrix with the same first \( N \) rows and columns as \( G - I \), which is of rank at most \( 2N \), we have
\[ |\lambda_{2N+1}| \leq \|G_{N+1} - I_{N+1}\| \leq C\sqrt{1 - \delta_{N+1}}, \]
by our computation above. Therefore
\[ |\lambda_{2N+1}|^p \leq C^p (1 - \delta_{N+1})^{p/2}. \]
Since the singular values are arranged in decreasing order, \( |\lambda_{2n+1}| > |\lambda_{2n}| \) for each \( n \). Thus, if \( \sum_n (1 - \delta_n^2)^{p/2} < \infty \), then \( \sum_n |\lambda_{2n}|^p \leq 2\sum_n |\lambda_{2n+1}|^p < \infty \) and we conclude that \( G - I \in S_p \). \( \square \)

We conclude by remarking that it is possible to trace through the proofs above to determine constants \( c \) and \( C \), which depend only on \( \delta = \inf_n \delta_n \), such that for \( 2 \leq p \leq \infty \),
\[ c\|\sqrt{1 - \delta_n}\|_{\ell^p} \leq \|G - I\|_{S_p} \leq C\|\sqrt{1 - \delta_n}\|_{\ell^p}. \]
In particular, by choosing \( \delta \) close enough to 1, one can choose \( c \) and \( C \) in (3.2) arbitrarily close to \( \sqrt{2} \) and \( 4\sqrt{2}(2^{1/p}) \), respectively.

**Question 3.7.** Is Theorem 3.6 true for \( p < 2 \)?

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References


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