ANALYTIC STRUCTURES FOR SUBNORMAL OPERATORS

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For μ a compactly supported measure on $\mathbb{C}$, we construct a mutually absolutely continuous measure ν so that $P^2(\nu)$ has analytic bounded point evaluations, and the operator of multiplication by z on $P^2(\nu)$ has every invariant subspace hyperinvariant. We also construct an equivalent measure σ so that $R^2(K,\sigma)$ has as analytic bounded point evaluations precisely the interior of the set of weak-star continuous point evaluations of $R^\infty(K,\mu)$. In the course of the proof, we classify weak-star closed super-algebras of $R^\infty(K,\mu)$ when $R(K)$ is hypo-Dirichlet.

INTRODUCTION

Given a measure ν (by which we shall always mean a compactly supported finite Borel measure on $\mathbb{C}$), $P^2(\nu)$ denotes the closure of the polynomials in $L^2(\nu)$, and $P^\infty(\nu)$ the weak-star closure of the polynomials in $L^\infty(\nu)$. If ν is Lebesgue measure $d\theta$ on the circle, then $P^2(\nu)$ is the classical Hardy space H^2, which can also be thought of as a space of analytic functions on the disk. However, if μ is the equivalent (i.e. mutually absolutely continuous) measure $exp(-\theta^{-2})d\theta$, then Szegö's theorem tells us that $P^2(\mu) = L^2(\mu)$, which has no analytic structure. The theme of this paper is going in the reverse direction: given an arbitrary measure μ, find an equivalent measure ν so that $P^2(\nu)$ does have an analytic structure.

This will not always be possible: given any measure μ supported on the unit interval, $P^2(\mu)$ will always be all of $L^2(\mu)$. To have any hope of finding such a ν, it is clearly necessary that $P^\infty(\mu) \neq L^\infty(\mu)$, so that there is some equivalent ν with $P^2(\nu) \neq L^2(\nu)$ (**BOT**). We show (Corollary 3.4) that a necessary and sufficient condition is that $P^\infty(\mu)$

have no L^∞-summand, in which case μ is called completely nonreductive.

To say what we mean by an analytic structure, let us first define a *bounded point evaluation* for $P^2(\mu)$ to be a point ζ such that there exists a constant C_ζ satisfying

$$|p(\zeta)| \leq C_\zeta \|p\|_{2,\mu}$$

for every polynomial p. If ζ is a bounded point evaluation, it makes sense to talk about the value of $f(\zeta)$ for any function f in $P^2(\mu)$, because evaluation at ζ, defined a priori only for polynomials, extends by continuity to all of $P^2(\mu)$. The point ζ is called an *analytic bounded point evaluation* if it is in the interior of the bounded point evaluations, and if, for every $f \in P^2(\mu)$, the function $z \mapsto f(z)$ is analytic in a neighbourhood of ζ.

We shall say that $P^2(\mu)$ has a U-analytic structure, where U is an open subset of $\mathbb{C}$, if

(i) U is contained in the set of analytic bounded point evaluations of $P^2(\mu)$, so every function in this space has an analytic extension to U;

(ii) The map $P^2(\mu) \cap L^\infty(\mu) \to H^\infty(U)$, sending a bounded function to its analytic extension, is a bijection onto the space of all bounded analytic functions on U.

Different measures ν equivalent to μ may have analytic structures corresponding to different sets U; indeed, following a construction of Hastings (**Ha**), if U is any simply connected (but not necessarily connected) subset of the unit disk, of full area, than one can construct a measure ν, equivalent to area measure on the disk, so that $P^2(\nu)$ has a U-analytic structure. For a given measure class $[\mu]$, however, there is an obvious maximal possible open set U, namely the interior of the set of weak-star continuous point evaluations for $P^\infty(\mu)$. If K is the closure of this set, then, by a theorem of Sarason (**Sa**), $R(K)$ (the closure in the sup-norm of $Rat(K)$, the rational functions with poles off K) is a Dirichlet algebra, so before we can prove the existence of $int(K)$-analytic structures, we must first prove a couple of decomposition theorems for $R^2(K,\mu)$ (the closure of $R(K)$ in $L^2(\mu)$, where μ is supported on K). We do this in Section 1. Since it is no extra work, we actually consider the case when $R(K)$ is a hypo-Dirichlet algebra.

In Section 2, we prove that given any measure μ with support contained in any compact set K, there is an equivalent measure ν with the set of analytic bounded point evaluations of $R^2(K,\nu)$ precisely equal to the interior of the set of weak-star continuous point evaluations for $R^\infty(K,\mu)$ (analytic bounded point evaluations for $R^2(K,\nu)$ are defined exactly as in the polynomial case, with polynomials replaced by functions in $Rat(K)$). Unlike in the polynomial case, however, R^∞ does not in general decompose into H^∞ plus L^∞. Indeed, there is an example due to Brennan of a Swiss cheese K, without interior, and a measure μ with $R^\infty(K,\mu) \neq L^\infty(\mu)$ (**Bre**). Thus we need additional hypotheses to find an equivalent measure with an analytic structure; we prove that one exists if K is the support of μ (rather than just a superset) and $R(K)$ is a hypo-Dirichlet algebra (Theorem 2.4).

In Section 3 we prove that if μ is completely nonreductive, then an equivalent measure has an analytic structure on the interior of the set of weak-star continuous point evaluations for $P^\infty(\mu)$ (Corollary 3.4). We translate this from the language of function algebras to the language of subnormal operators, and show that this is equivalent to the assertion that any normal operator N of multiplicity one has a full part (restriction to an invariant subspace not properly contained in any reducing subspace) S satisfying:

(i) The spectrum of S is the spectrum of N in the (non-selfadjoint) weakly closed algebra generated by N and the identity.

(ii) The essential spectrum of S is the spectrum of N (as an operator, or, equivalently, in the von Neumann algebra it generates).

(iii) S has Fredholm index -1 whenever it is Fredholm and noninvertible.

An operator is called *reductive* if every invariant subspace is reducing (*i.e.* left invariant by the adjoint). For a normal operator, being reductive clearly implies that the restriction to any invariant subspace is normal, and hence has zero Fredholm index off the essential spectrum. By a theorem of Sarason (**Sa**) the sets in (i) and (ii) are identical if and only if N is reductive, so it follows from the above that a normal operator is reductive if and only if no part has non-vanishing Fredholm index. This result also follows from work of Olin and Thomson (**OT, OT2**), but our approach is more direct.

In Section 3 we also introduce the notion of *ultrareflexivity*, (being reflexive and having all invariant subspaces hyperinvariant), and show that the above subnormal operator S is ultrareflexive. In Section 4 we show that ultrareflexive subnormal operators are star-cyclic, and conjecture that their minimal normal extensions are cyclic.

General references for most of the material used in this paper are Conway's *Subnormal Operators* (**Co**), and Gamelin's book *Uniform Algebras* (**Ga**).

1. STRUCTURE THEOREMS FOR HYPO-DIRICHLET ALGEBRAS

Harmonic measure for a connected open set U is defined in the following way: pick a point a in U. For any continuous function f on the boundary of U, define a function $\hat{f}$ on U by

$$\hat{f}(z) = sup\{g(z) : g \text{ is subharmonic on } U, \text{ and } \limsup_{z\to\zeta} g(z) \leq f(\zeta), \forall\zeta \in \partial U\}.$$

The functional $f \mapsto \hat{f}(a)$ is continuous on $\mathrm{C}(\partial U)$, and therefore comes from a measure, ω_a, which is called harmonic measure for U at a. Whilst a different choice of a will yield a different measure, the two measures will be boundedly mutually absolutely continuous, so we normally don't bother to specify what point a has been chosen.

Throughout this section, K will be a compact subset of $\mathbb{C}$, U will be the interior of K, the components of U will be called U_n, and harmonic measure for each U_n, normalised to have total mass 2^{-n}, will be called ω_n. Harmonic measure for K, the sum of the ω_n, will be called ω.

$R(K)$ is called a *Dirichlet algebra* if $\{Ref|_{\partial K} : f \in R(K)\}$ is dense in $\mathrm{Re}(\mathrm{C}(\partial K))$. $R(K)$ is called *hypo-Dirichlet* if the uniform closure of $\mathrm{Re}(R(K))$ is of finite codimension in $\mathrm{Re}(\mathrm{C}(\partial K))$, and the linear span of $\{log|r| : r, r^{-1} \in R(K)\}$ is dense in $\mathrm{Re}(\mathrm{C}(\partial K))$. (This latter condition, logmodularity, is automatically satisfied in a Dirichlet algebra because $log|exp(r)| = Re(r)$.) For a general reference on hypo-Dirichlet algebras, see (**AS**). As a prototypical example, if the complement of K has finitely many components, then $R(K)$ is a hypo-Dirichlet algebra (**AS2**).

The following theorem basically classifies weak-star closed algebras between $R^\infty(K,\omega)$ and $L^\infty(\mu)$. It may be well-known, at least as a folk theorem; however, we have never seen a proof in the literature, and, moreover, we believe the realisation of such an algebra as $R^2(K,\nu) \cap L^\infty(\nu)$ may be of independent interest. As our techniques do not depend on the Hilbert space structure we prove the following theorems for R^p (the closure of the rational functions in L^p) for p between one and infinity. Nothing is changed, however, if p is always taken to be 2.

THEOREM 1.1: *Suppose $R(K)$ is a hypo-Dirichlet algebra, and μ is a measure on ∂K. Let p be a fixed real number, $1 \leq p < \infty$. Then the only unital weak-star closed subalgebras of $L^\infty(\mu)$, invariant under multiplication by $R(K)$, are of the form $R^p(K,\nu) \cap L^\infty(\nu)$ for some measure ν absolutely continuous with respect to μ. Moreover, any such algebra decomposes into $\oplus_{n \in J} R^\infty(\bar{U}_n, \omega_n) \oplus L^\infty(\nu_s)$, where J is some subset of the positive integers (possibly empty), and ν_s is the singular component of ν with respect to $\sum_{n \in J} \omega_n$.*

PROOF: Let A be such an algebra, i be its unit. Then there is a Borel set E such that $i = \chi_E$ (the characteristic function of E) μ-almost everywhere. If $A = L^\infty(\mu|_E)$, then by a construction of Bram (**Br**) there is a measure ν, equivalent to $\mu|_E$, so that $P^p(\nu) = R^p(K,\nu) = L^p(\nu)$, and then $A = R^p(K,\nu) \cap L^\infty(\nu) = L^\infty(\nu)$, as required.

So assume $A \neq L^\infty(\mu|_E)$, and suppose for the moment that U is connected. If σ is the singular component of $\mu|_E$ with respect to ω, then there is a sequence r_k in $Rat(K)$, $\|r_k\|_\infty \leq 1$, such that r_k tends to one σ-almost everywhere, and r_k tends to zero ω-almost everywhere (this holds because ω is a dominant representing measure (**AS**, 3.1.1), so Forelli's lemma (**Ga**, p.43) guarantees the existence of the r_k). So if F is an ω null set of full σ measure, r_k converges weak-star to χ_F, so χ_F is in A, and A splits into $A|_F \oplus A|_{F^c}$.

The Kolmogoroff-Krein theorem (**Ga**, p.135) says that for any measure ρ singular with respect to ω, for all points λ in U,

$$\inf_{f \in R(K)} \int |1 - (z-\lambda) f(z)|^p d\rho(z) = 0.$$

Therefore $(z-\lambda)^{-1}$ is in $R^p(K,\rho)$ for all ρ equivalent to σ, so by (**BOT**) it is in $R^\infty(K,\sigma)$, which must then be the same as $R^\infty(\partial K, \sigma)$. But by (**AS**, Thm. 3.1.2), the point masses on ∂K are the only representing measures for $R(K)$, and *a fortiori* for $R(\partial K)$, corresponding to evaluations at points of ∂K. Therefore (**Ch**, Thm. 41) yields that $R^\infty(\partial K, \sigma) = L^\infty(\sigma)$, and so $A|_F$, which perforce contains $R^\infty(K,\sigma)$, must be all of $L^\infty(\sigma)$.

If $(z-\lambda)^{-1}$ were also in $A|_{F^c}$, the above argument would force A to be $L^\infty(\mu|_E)$, so evaluation at λ must be a continuous functional on $A|_{F^c}$. The Szegö theorem for hypo-Dirichlet algebras (**AS**, 10.1) then forces $\mu|_{E\cap F^c}$ to be equivalent to ω. But ω is a dominant representing measure, and unique logmodular measure, and so by (**Ga**, p.115), evaluation at λ cannot extend continuously to any super-algebra of $R^\infty(K,\omega)$. Therefore $A|_{F^c}$ is exactly $R^\infty(K,\omega)$, and

$$A = R^p(\omega+\sigma)\cap L^\infty(\omega+\sigma) = R^\infty(K,\omega)\oplus L^\infty(\sigma),$$

as required.

Now, drop the supposition that U be connected. By (**Ga2**, 8.5, and **Ga**, p.153), because each U_n is a Gleason part of $R(K)$, there is a sequence $r_{n,k}$ $(k = 1, 2, \dots)$ in $R(K)$ with $\|r_{n,k}\| \le 1$, and which converges to 1 ω_n-almost everywhere, and to zero $(\omega - \omega_n)$-almost everywhere. We would like to say that $r_{n,k}$ converges μ-weak-star to a projection, but since μ need not be absolutely continuous with respect to ω_n, we must first pass to a subsequence, so that we can assume that $r_{n,k}$ converges μ-weak-star to a function f_n, with $\|f_n\| = 1$; then by taking powers of f_n, we get a sequence converging μ-weak-star to a projection p_n, which is 1 ω_n-almost everywhere and zero $(\omega-\omega_n)$-almost everywhere. Now apply the above argument to each $p_n A$ separately. □

Notice that ν_s can be absolutely continuous with respect to any ω_n for n not in J, thus allowing summands of the form $L^\infty(\omega_n)$.

We can extend this result, breaking up $R^p\cap L^\infty$ into $R^\infty\oplus L^\infty$, to a measure that has mass on the interior of K, if we assume that the interior points of K are analytic bounded point evaluations. The following theorem is an extension of a result of Raphael (**Ra**), in which he considers certain measures on the disk. Our proof is similar to his, but we include it for completeness. We remark that equivalent measures induce the same weak-star topologies on L^∞, so their R^∞ spaces are identical.

THEOREM 1.2: *Suppose $R(K)$ is a hypo-Dirichlet algebra, and μ is a measure on K. Let p be a fixed real number, $1 \le p < \infty$. If every point of int(K) is an analytic bounded point evaluation for $R^p(K,\mu)$, then there is a Borel set E contained in ∂K such that*

$$R^p(K,\mu)\cap L^\infty(\mu) = R^\infty(K,\mu)\oplus L^\infty(\mu|_E).$$

Note: Since μ and $\mu|_E$ are not disjoint, the right-hand side should properly be written as $R^\infty(K,\mu|_{\mathbb{C}\setminus E})\oplus L^\infty(\mu|_E)$, and the further assertion made that the natural map from $R^\infty(K,\mu)$ to $R^\infty(K,\mu|_{\mathbb{C}\setminus E})$ is an isometric isomorphism and weak-star homeomorphism. This is what we mean by the direct sum notation.

PROOF: Let U be the interior of K. As in the previous theorem, the fact that any function in $H^\infty(U)$ (the space of bounded analytic functions on U) can be approximated pointwise by a sequence of functions in $R(K)$ with the same norm (**Ga2**, 8.5 and **AS**, 4.1) means we can assume without loss of generality that U is connected, that if $\nu=\mu|_{\partial K}$, then ν is absolutely continuous with respect to ω (because any singular part will come off as an L^∞-summand), and that $H^\infty(U)$ is contained in $R^\infty(K,\mu)$.

By (**Co**, p.380), there exists a Borel set E such that

$$R^p(K,\mu)\cap L^\infty(\mu)=L^\infty(\mu|_E)\oplus[R^p(K,\mu)\cap L^\infty(\mu)]\,|_{K\setminus E}, \tag{1}$$

where the second summand contains no L^∞ summand. We claim that the second summand is actually $R^\infty(K,\mu)$.

If

$$R^p(K,\nu)\cap L^\infty(\nu)=R^\infty(K,\omega), \tag{2}$$

then

$$R^p(K,\mu)\cap L^\infty(\mu)=R^\infty(K,\omega)=R^\infty(K,\mu),$$

because if r_n tends to f in $R^p(K,\mu)$, then r_n tends to f in $R^p(K,\nu)$, so $R^p(K,\nu)\cap L^\infty(\nu)$ is contained in $R^\infty(K,\omega)$. But any function in $R^\infty(K,\omega)$ is in $H^\infty(U)$, which is contained in $R^\infty(K,\mu)$, so both containments must be equalities, and the lemma is proved, with E empty.

So we shall assume that 2 is false; this means, by Theorem 1.1, that

$$R^p(K,\nu)\cap L^\infty(\nu)=L^\infty(\nu). \tag{3}$$

Suppose g is in $R^p(K,\mu)\cap L^\infty(\mu)$. Since evaluation at each point of U is continuous, we can assume that g is defined everywhere on U, and that it is bounded and analytic there. Because $H^\infty(U)$ is contained in $R^\infty(K,\mu)$, there is a function f in $R^\infty(K,\mu)$ which agrees with g on U. Put $h = f - g$, so $h \equiv 0$ on U. We want to show that the characteristic function of the set where $h \neq 0$ is in $R^p(K,\mu)$, and hence that L^∞ of this set is contained in $R^p(K,\mu) \cap L^\infty(\mu)$.

To this end, for each positive integer n, let

$$I_n = \{w \in \partial K : |h(w)| > \frac{1}{n}\},$$

and

$$I_\infty = \{w \in \partial K : |h(w)| > 0\}.$$

Define G_n by

$$G_n(w) = \begin{cases} \frac{1}{h} & : w \in I_n \\ 0 & : w \in \partial K \setminus I_n. \end{cases}$$

By (3), there is a function r_n in $Rat(K)$ such that

$$\int_{\partial K} |G_n - r_n|^p d\nu < \frac{1}{n^{p+1}}.$$

Therefore

$$\int_{\partial K} \chi_{I_n} |\frac{1}{h} - r_n|^p d\nu < \frac{1}{n^{p+1}},$$

and so

$$\int_{I_n} |\chi_{I_n} - hr_n|^p d\nu < \frac{1}{n},$$

which gives

$$\begin{aligned}\int_{\partial K} |\chi_{I_\infty} - hr_n|^p d\nu &< \frac{1}{n} + \int_{I_\infty \setminus I_n} |1 - hr_n|^p d\nu \\ &\leq \frac{1}{n} + \nu(I_\infty \setminus I_n) + \frac{1}{n^p} \int_{I_\infty \setminus I_n} |r_n|^p d\nu \\ &\leq \frac{1}{n} + \nu(I_\infty \setminus I_n) + \frac{1}{n^{2p+1}},\end{aligned}$$

which tends to zero as n tends to ∞. But on $int(K)$, h is identically zero, so

$$\lim_{n\to\infty}\int_K |\chi_{I_\infty} - hr_n|^p d\mu = 0.$$

Because h is in $R^p(K,\mu)\cap L^\infty(\mu)$, hr_n is in $R^p(K,\mu)$, so χ_{I_∞} is in $R^p(K,\mu)\cap L^\infty(\mu)$.

(1) now yields that $\chi_{I_\infty\setminus E}$ is in $R^p(K,\mu)\cap L^\infty(\mu)$. From (3) we see that the closure of $\{r.\chi_{I_\infty\setminus E} : r\in Rat(K)\}$ in $L^p(\mu)$ is $L^p(\mu|_{I_\infty\setminus E})$, because the function $r.\chi_{I_\infty\setminus E}$ is the same in $R^p(K,\mu|_{\partial K})$ as in $R^p(K,\mu)$. Therefore $L^\infty(\mu|_{I_\infty\setminus E})$ is a direct summand of $R^p(K,\mu)\cap L^\infty(\mu)$, and by the maximality of (1), $\mu(I_\infty\setminus E)$ must be zero, i.e. h must vanish μ-a.e. off E.

Thus any function g in the second summand of (1) must agree μ-a.e. with the function f in $R^\infty(K,\mu)$ which agrees with g on U, or else $h=f-g$ contradicts the above conclusion. Therefore

$$\left[R^p(K,\mu)\cap L^\infty(\mu)|_{K\setminus E}\right] = R^\infty(K,\mu) = H^\infty(U), \tag{4}$$

and (1) yields the desired result. □

The following corollary is a direct consequence of (4):

COROLLARY 1.3: *Under the hypotheses of Theorem 1.2, if $R^p(K,\mu)$ has no L^p summand, then it has an $int(K)$-analytic structure.*

2. CONSTRUCTION OF ANALYTIC BOUNDED POINT EVALUATIONS

THEOREM 2.1: *Let μ be a finite, positive, Borel measure with support contained in some compact subset K of $\mathbb{C}$. Let p be a real number, greater than or equal to 1. Then there exists a measure ν equivalent to μ such that the set of analytic bounded point evaluations for $R^p(K,\nu)$ is precisely the interior of the set of weak-star continuous point evaluations for $R^\infty(K,\mu)$.*

PROOF: Let W be the set of weak-star continuous point evaluations for $R^\infty(K,\mu)$. We can assume that W has non-empty interior, for otherwise the theorem is trivially true.

We first need a lemma asserting the measurability of a choice function we shall wish to integrate.

LEMMA 2.2 *With notation as above, there exists a measurable map*

$$\Lambda : int(W) \to L^1(\mu)$$

$$\lambda \mapsto \phi_\lambda$$

and a constant M such that

(i) $\int r\phi_\lambda d\mu = r(\lambda)$ *, for all r in $Rat(K)$;*

(ii)$\|\phi_\lambda\|_{1,\mu} \leq M$.

PROOF: Fix $M > 1$. Let $R^{\infty\perp}$ be the annihilator of $R^\infty(K,\mu)$ in $L^1(\mu)$. If the space of weak-star continuous linear functionals on $R^\infty(K,\mu)$ is R^∞_*, then it can be identified with the quotient space $L^1(\mu)/R^{\infty\perp}$, giving a short exact sequence

$$0 \longrightarrow R^{\infty\perp} \longrightarrow L^1(\mu) \xrightarrow{\pi} R^\infty_* \longrightarrow 0.$$

If $e_\lambda \in R^\infty_*$ is the functional of evaluation at λ, Schwarz' Lemma yields that the map $\lambda \mapsto e_\lambda$ is norm continuous. Let $\tilde{\pi}$ be π restricted to the ball of radius M in $L^1(\mu)$. Every e_λ is in the range of $\tilde{\pi}$. Since $\tilde{\pi}$ is a continuous function from a complete separable metric space onto its range, that maps open sets to Borel sets, it has, by (**Ar**,p.75), a measurable cross-section, κ. Λ is defined by $\Lambda(\lambda) = \kappa(e_\lambda)$. □

We also need a criterion for a point to be an analytic bounded point evaluation, which we state without proof. The disk with center ζ and radius l will be denoted $D(\zeta, l)$.

LEMMA 2.3 (**Co**, p.171) *The point ζ_0 is an analytic bounded point evaluation for $R^p(K,\mu)$, $1 \leq p < \infty$, if and only if there exists $l > 0$ such that all the points in $D(\zeta_0, l)$ are bounded point evaluations and $sup\{C_\zeta : \zeta \in D(\zeta_0, l)\} < \infty$.*

To prove the theorem, let ϕ_λ be as in Lemma 2.2. Define ν by

$$\nu(F) = \int_{int(W)} dArea(\lambda) \int d\mu(z)(|\phi_\lambda(z)| + 1)\chi_F(z).$$

Then Tonelli's theorem (applicable because of part (ii) of Lemma 2.2) yields that ν is absolutely continuous with respect to μ. Because $\nu(F) \geq Area(int(W)).\mu(F)$, μ and ν are mutually absolutely continuous.

Let $\zeta_0 \in W$. There exist $l, C > 0$ such that $D(\zeta_0, l) \subset W$ and

$$|r(\zeta)| \leq C \int_U |r(\lambda)| dArea(\lambda)$$

for any $\zeta \in D(\zeta_0, l)$ and any function r in $Rat(K)$. But (i) of Lemma 2.2 now gives

$$\begin{aligned}|r(\zeta)| &\leq C \int_W dArea(\lambda) \left| \int d\mu(z) \phi_\lambda(z) r(z) \right| \\ &\leq C \int_W dArea(\lambda) \int d\mu(z) |\phi_\lambda(z)| |r(z)| \\ &\leq C \int d\nu(z) |r(z)|.\end{aligned}$$

So, by Lemma 2.3, every point of $int(W)$ is an analytic bounded point evaluation for $R^1(K,\nu)$, and hence, by Hölder's inequality, for $R^p(K,\nu)$. □

This allows us to prove the following theorem on the existence of an analytic structure.

THEOREM 2.4 *Suppose the support of μ is K, that $R(K)$ is a hypo-Dirichlet algebra, and that $R^\infty(K,\mu)$ has no L^∞ summand. Let p be real, $1 \leq p < \infty$. Then there exists an equivalent measure ν such that $R^p(K,\nu)$ has an $int(K)$-analytic structure.*

PROOF : By (**Ch**, Prop. 47) the interior of the support of μ consists of weak-star continuous point evaluations for $R^\infty(K,\mu)$. Apply Theorem 2.1 to get a measure ν having these points as analytic bounded point evaluations. By Theorem 1.2,

$$R^p(K,\nu) \cap L^\infty(\nu) = R^\infty(K,\nu) \oplus L^\infty(\nu|_E). \tag{5}$$

Repeat this construction with every measure $\tilde{\mu}$ in the measure class of μ, obtaining measures $\tilde{\nu}$ and corresponding Borel sets $E(\tilde{\nu})$. Let

$$m = \inf\{\mu(E(\tilde{\nu})) : \tilde{\nu} \text{ is a measure obtained in the above fashion}\}.$$

Then we claim that

(a) m is attained

(b) m is zero.

Proof of (a): There must be a sequence $\tilde{\nu}_n$ such that $\mu(E(\tilde{\nu}_n))$ decrease to m, and $\{\|\tilde{\nu}_n\|\}$ is bounded. Put $\nu = \sum_{n=1}^\infty 2^{-n}\tilde{\nu}_n$. Each point of $int(K)$ is an analytic bounded point evaluation for $R^p(K,\nu)$, so again we can apply Theorem 1.2, to get a decomposition as in (5). Since $R^p(K,\nu)$ is contained in $R^p(K,\tilde{\nu}_n)$ for each n, $E(\nu)$ is contained in $E(\tilde{\nu}_n)$, so $\mu(E(\nu)) \leq \mu(E(\tilde{\nu}_n))$ for all n, so $\mu(E(\nu)) = m$.

Proof of (b): By a result of Ball, Olin and Thomson (**BOT**),

$$R^\infty(K,\mu) = \bigcap_{\tilde{\mu}\equiv\mu} R^p(K,\tilde{\mu}) \cap L^\infty(\mu).$$

So if ν and $E(\nu)$ are as constructed in (a), and $m \neq 0$, then because $\chi_{E(\nu)}$ is not in $R^\infty(K,\mu)$, there exists $\tilde{\nu}$ with the property that $\chi_{E(\nu)}$ is not in $R^p(K,\tilde{\nu})$. But then $\mu(E(\nu+\tilde{\nu})) < m$, contradicting the definition of m.

Therefore

$$R^p(K,\nu) \cap L^\infty(\nu) = R^\infty(K,\nu) = R^\infty(K,\mu) = H^\infty(int(K)),$$

and every point of $int(K)$ is an analytic bounded point evaluation, so $R^p(K,\nu)$ has an $int(K)$-analytic structure, as required. □

3. APPLICATIONS TO SUBNORMAL OPERATORS

Given any operator T on a Hilbert space $\mathcal{K}$, which has an invariant subspace $\mathcal{H}$, the restriction of T to $\mathcal{H}$ is called a *part* of T; the part is called *full* if the smallest reducing subspace of T containing $\mathcal{H}$ is $\mathcal{K}$. If T is normal, and S is a part of T, S is called *subnormal.* In other words, S is subnormal if there exist bounded operators X and Y such that

$$\begin{pmatrix} S & X \\ 0 & Y \end{pmatrix}$$

is normal. S is called *pure* if it does not have a restriction to a reducing subspace that is normal.

For an operator T on $\mathcal{K}$, $\mathcal{A}lgLat(T)$ is defined to be the algebra of bounded operators on $\mathcal{K}$ which leave invariant every invariant subspace of T. $\mathcal{A}(T)$, the weakly closed algebra generated by T, is, by an elementary argument, always contained in $\mathcal{A}lgLat(T)$; if these two algebras are equal, T is called reflexive. It is a deep result of Olin and Thomson that all subnormal operators S are reflexive, and that the weak and σ-weak topologies agree on $\mathcal{A}(S)$ (**OT**).

The spectral theorem says that any star-cyclic normal operator is unitarily equivalent to multiplication by the independent variable on the space $L^2(\mu)$ for some measure μ; the analogous theorem for subnormal operators says that any $Rat(K)$-cyclic subnormal operator is unitarily equivalent to multiplication by the independent variable on some $R^2(K,\mu)$, and any cyclic subnormal to multiplication on some $P^2(\mu)$ (**Co**, p.146). We shall call these operators N_μ, $R_{K,\mu}$ and S_μ respectively.

In terms of the operator $R_{K,\mu}$, the set of analytic bounded point evaluations for $R^2(K,\mu)$ is precisely $\sigma(R_{K,\mu}) \setminus \sigma_{ap}(R_{K,\mu})$ (**Tr**) (σ_{ap} stands for the approximate point spectrum). The commutant of $R_{K,\mu}$ is $R^2(K,\mu) \cap L^\infty(\mu)$ (**Yo**), and the sigma-weakly closed algebra generated by $\{r(R_{K,\mu}) : r \in Rat(K)\}$ is $R^\infty(K,\mu)$ (**CO**). We shall loosely speak of (analytic) bounded point evaluations of $R_{K,\mu}$, when we mean those of $R^2(K,\mu)$.

An invariant subspace M of an operator T is called *hyperinvariant* if M is left invariant by the commutant of T. M is called $Rat(K)$-*invariant* if K contains the spectrum of T, and M is left invariant by $\{r(T) : r \in Rat(K)\}$. If this last condition holds, then M will automatically be left invariant by the sigma-weak closure of $\{r(T)\}$.

Using these identifications Theorem 2.4 becomes:

THEOREM 3.1: *Let N be a normal operator, with spectrum K, such that R(K) is hypo-Dirichlet. Then N has a part R such that $\sigma_{ap}(R) = \partial K$, and with the property that every Rat(K)-invariant subspace of R is hyperinvariant.*

PROOF: Let μ be a scalar spectral measure for N (i.e. a measure with the same null sets as the spectral measure of N). As in Theorem 2.4, find a measure ν so that the points of $int(K)$ are analytic bounded point evaluations for $R^2(K,\nu)$, and so that (5) holds. Let $\sigma = \nu|_{\mathbb{C}\setminus E}$. Then $R = R_{K,\sigma}$ satisfies the conclusion of the theorem. □

In the polynomial case, we need no restrictions on the spectrum of N, because of Sarason's characterisation of $P^\infty(\mu)$.

THEOREM 3.2 (Sa): *Let μ be a measure. Then there exist mutually orthogonal measures μ_1 and μ_2, $\mu = \mu_1 + \mu_2$, such that $P^\infty(\mu) = P^\infty(\mu_1) \oplus L^\infty(\mu_2)$, where $P^\infty(\mu_1)$ has no L^∞ summand. If K is the spectrum of z as an element of the Banach algebra $P^\infty(\mu_1)$, then:*

(i) $\mu_1(\mathbb{C} \setminus K) = 0$ and $R(K) \subset P^\infty(\mu_1)$.

(ii) R(K) is a Dirichlet algebra.

(iii) $\mu_1|_{\partial K}$ is absolutely continuous with respect to harmonic measure for K.

(iv) $P^\infty(\mu)$ is isometrically isomorphic to $H^\infty(int(K))$.

K is called the Sarason hull of μ (or of the associated operator N_μ), and is written $\Sigma(\mu)$.

This enables us to prove our principle operator theoretic result. We shall call an operator T *ultrareflexive* if it is reflexive, and if every invariant subspace is hyperinvariant, *i.e.* $\mathcal{A}(T) = \mathcal{A}lgLat(T) = \{T\}'$. A better term might be *hyperreflexive,* but this term has already been coined by Arveson in a different context (**Ar2**). For the cyclic subnormal operator S_μ, ultrareflexivity is equivalent, by our earlier remarks, to the equality of $P^\infty(\mu)$ and $P^2(\mu) \cap L^\infty(\mu)$.

THEOREM 3.3: *Let N be normal, of multiplicity one. Let K be its Sarason hull. Then N has a full cyclic part S with the interior of K equal to the set of analytic bounded point evaluations of S. The pure part of any such S is always ultrareflexive, and there exists some such S which is ultrareflexive (and pure whenever N is completely non-reductive).*

PROOF: Since $R(K)$ is Dirichlet, and every point of the interior of K is a weak-star continuous evaluation for $P^\infty(\mu)$, where μ is a scalar spectral measure for N, the proof of 2.4 can be followed directly, if μ_2, in the notation of Theorem 3.2, is zero. If it isn't, first construct a measure ν as in Theorem 3.2 corresponding to μ_1, and let σ be $\nu + \mu_2$. Then σ is equivalent to μ, and $P^2(\sigma) = P^2(\nu) + L^2(\mu_2)$, so $P^2(\sigma) \cap L^\infty(\sigma) = P^\infty(\sigma)$, and S_σ is ultrareflexive, as required. $\square$

This answers affirmatively a question of Conway (**Co**, III.11.14), whether there always exists a single measure ν equivalent to μ and satisfying $P^2(\nu) \cap L^\infty(\nu) = P^\infty(\mu)$. However, both Ted Gamelin and Jim Thomson have used other methods to prove the following more general theorem: if $L^1(\mu)$ is separable, and M is any weak-star closed manifold in $L^\infty(\mu)$, then there exists a measure ν equivalent to μ such that $M^1(\nu) \cap L^\infty(\nu) = M$.

We remark that any operator S produced by Theorem 3.3 also has the property that the set of bounded point evaluations is exactly equal to the set of weak-star continuous point evaluations of $P^\infty(\mu)$. This follows because every point in ∂K is a peak point of $R(K)$ (**Ga**, p.54 and **AS**, 3.1.2), and so the only weak-star continuous point evaluations on ∂K are at atoms of μ. The only weak-star continuous point evaluations coming from

μ_2 also come from atoms. Thus, the weak-star continuous point evaluations for $P^\infty(\mu)$ are precisely the interior of K and the atoms of μ_2. Any full part S of N has these atoms as bounded point evaluations.

Because we believe both points of view are useful, and at the risk of belabouring the point, we shall restate Theorem 3.3 in function theoretic terms:

COROLLARY 3.4 *If $P^\infty(\mu)$ has no L^∞ summand, then there is a measure ν mutually absolutely continuous with respect to μ for which $P^2(\nu)$ has an $int(\Sigma(\mu))$-analytic structure.*

Trent has shown that the analytic bounded point evaluations of $P^2(\mu)$ are $\{\lambda : (S_\mu - \lambda)$ is Fredholm and has index -1$\}$ (**Tr**). We thus obtain, as a corollary, a condition for when a normal operator is reductive:

COROLLARY 3.5: *A normal operator is reductive if and only if no part has non-vanishing Fredholm index.*

PROOF: As stated in the introduction, the condition is clearly necessary. Let μ be a scalar spectral measure for the normal operator N. By Theorem 3.3, and in the notation of Theorem 3.2, if μ doesn't equal μ_2, then N_μ (and consequently N) has a part with an analytic bounded point evaluation, and hence a negative Fredholm index. So if no part of N has non-vanishing Fredholm index, $P^\infty(\mu)$ must equal $L^\infty(\mu)$, so the function $\bar{z}$ is in $P^\infty(\mu)$, so by the spectral theorem, $N^\star$ is in the sigma-weakly closed algebra generated by N, and so leaves invariant every invariant subspace of N, and so N is reductive. $\square$

As we remarked in the Introduction, Corollary 3.5 and the first part of Theorem 3.3 also follow from results of Olin and Thomson (**OT, OT2**). They define a *full analytic subspace* of N_μ to be a subspace corresponding to a part S which has $int(\Sigma(\mu))$ as analytic bounded point evaluations, and show that every normal operator has a full analytic subspace. In our terminology, they find an S_ν which satisfies (i) in the definition of analytic structure, but for which the map in (ii) may have a kernel. It therefore follows from their work that a normal operator which is not reductive has a part with non-vanishing Fredholm index.

4. ULTRAREFLEXIVE SUBNORMAL OPERATORS

In finite dimensions, the condition that every invariant subspace of an operator be hyperinvariant is equivalent to the operator being cyclic. For a discussion, see (**Ong**). However, it is not true that an ultrareflexive subnormal operator is necessarily cyclic: it follows from (**Fr**) that a noninvertible, nonnormal subnormal bilateral weighted shift is ultrareflexive and not cyclic. An example of such an operator is multiplication by z on the closure in $L^2(\mu)$ of functions analytic on the punctured disk $\mathbb{P} = \{z : 0 < |z| < 1\}$, where $d\mu(z) = e^{-\frac{1}{|z|}} dArea(z)$ on $\mathbb{P}$.

We can show, however, that an ultrareflexive subnormal operator S in $B(\mathcal{H})$ is star-cyclic, i.e. there exists a vector ξ in $\mathcal{H}$ such that no proper reducing subspace of S contains ξ. (If S is actually normal, having a star-cyclic vector is equivalent to having a cyclic vector (**Br**)).

PROPOSITION 4.1: *Let S be an ultrareflexive subnormal operator on $\mathcal{H}$. Then S is star-cyclic.*

PROOF: Let M be the von Neumann algebra generated by S, and let $\mathcal{C}$ be its commutant. So

$$\begin{aligned} \mathcal{C} &= \{S\}' \cap \{S^\star\}' \\ &= \mathcal{A}(S)' \cap \mathcal{A}(S)^{\star\prime} \\ &= \mathcal{A}(S) \cap \mathcal{A}(S)^\star. \end{aligned}$$

If N is the minimal normal extension of S, and μ is a scalar spectral measure for N, then, by (**CO**), the map $\Gamma : p \mapsto p(S)$ extends to an isometric isomorphism and weak-star homeomorphism from $P^\infty(\mu)$ onto $\mathcal{A}(S)$, which sends $P^\infty(\mu) \cap \overline{P^\infty(\mu)}$ onto $\mathcal{A}(S) \cap \mathcal{A}(S)^\star$. Moreover, from Theorem 3.2, and using its notation, we see that $P^\infty(\mu) \cap \overline{P^\infty(\mu)}$ consists of functions that are constant on each component U_n of $int(K)$. We can thus decompose $\mathcal{H}$ so that $\mathcal{C}$ has a particularly simple structure.

To do this, for each U_n let E_n be a Borel set in ∂U_n such that $\mu_1(E_n) = \mu_1(\partial U_n)$, $\mu_2(E_n) = 0$, and the $E_n's$ are mutually disjoint (this last condition is possible, because, by a result of Wermer (**We**), the harmonic measures for each component of $int(K)$ are mutually singular when $R(K)$ is a Dirichlet Algebra). Let $F_n = U_n \cup E_n$, $G = K \setminus \cup_n F_n$.

Then

$$\mathcal{H} = \bigoplus_n \Gamma(\chi_{F_n})\mathcal{H} \oplus \Gamma(\chi_G)\mathcal{H}.$$

With respect to this decomposition,

$$\mathcal{C} = \bigoplus_n \mathbb{C}.Id \oplus \Gamma(L^\infty(\mu_2)).$$

So by the Double Commutant Theorem

$$M = \mathcal{C}' = \bigoplus_n B(\Gamma(\chi_{F_n})\mathcal{H}) \oplus [\Gamma(L^\infty(\mu_2))]'.$$

The last summand has a cyclic vector, because it is just the commutant of N_{μ_2}, and each of the other summands has a cyclic vector, so the algebra M must have a cyclic vector, as desired. □

We remark that the existence of a star-cyclic vector is not very surprising, in view of Wogen's still open question, whether the adjoint of every pure subnormal operator be cyclic (**Wo**). We conjecture, however, that more is actually true in this case, namely that the minimal normal extension of an ultrareflexive subnormal operator is always cyclic.

In (**OT2**), Olin and Thomson show that any subnormal operator which is cellular indecomposable (*i.e.* any two invariant subspaces have non-trivial intersection) is ultrareflexive, and its minimal normal extension is always cyclic. Moreover, they construct a cellular indecomposable subnormal operator which is not cyclic. However, their work also suggests (**OT3**) that such operators form a "small" set, excluding, for example, S_σ whenever σ is equivalent to area measure on the disk, although such an operator is ultrareflexive if σ is, say, radial.

5. CONCLUSION

In closing, we would like to remark that, in our opinion, study of the algebra $P^2(\mu) \cap L^\infty(\mu)$ is fundamental to the understanding of the operator S_μ, because it is the largest Banach algebra naturally occurring inside $P^2(\mu)$. In every case known to the author, the algebra is of the form $H^\infty(U) \oplus L^\infty(\mu|_E)$, for some simply connected open set U, and some measurable set E. We would like to know if this is always the case, and, if so, whether the set U consists of analytic bounded point evaluations for $P^2(\mu)$. This latter question is equivalent to asking whether condition (i) in the definition of analytic structure follows automatically from condition (ii) (*i.e.* the existence of a natural isomorphism from $P^2(\mu) \cap L^\infty(\mu)$ onto $H^\infty(U)$). We should point out, however, that it is not even known whether $P^2(\mu)$ has to have any bounded point evaluations, assuming $P^2(\mu) \neq L^2(\mu)$; for some results in this direction see e.g. (**Bre**) and (**Bre2**).

Note added in proof: J. Thomson has recently proved that if $P^2(\mu) \neq L^2(\mu)$ then $P^2(\mu)$ does have analytic bounded point evaluations.

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