Best Approximation in the Mean by Analytic and Harmonic Functions

Dmitry Khavinson*, John E. McCarthy†, and Harold S. Shapiro

1. Introduction.

For $n \geq 2$, let $\mathbf{B}_n$ denote the unit ball in $\mathbf{R}^n$, and for $p \geq 1$ let L^p denote the Banach space of p-summable functions on $\mathbf{B}_n$. Let $L^p_h(\mathbf{B}_n)$ denote the subspace of harmonic functions on $\mathbf{B}_n$ that are p-summable. When $n = 2$, we often write $\mathbf{D}$ instead of $\mathbf{B}_2$, and we let A^p denote the Bergman space of analytic functions in L^p.

Let ω be a function in L^p. We are interested in finding the best approximation to ω in A^p and $L^p_h(\mathbf{B}_n)$. Existence of a best approximant is straightforward; this paper considers the following two qualitative properties:

(i) Uniqueness of best approximants, when $p = 1$.

(ii) Hereditary regularity of the best approximant $f^\star$ inherited from that of ω, e.g., whether continuity, Hölder continuity, real-analyticity of ω in the closed unit disk enforce those properties in ω's best approximant.

These and many other similar questions have been well-studied for the case when the normalized area measure $dA := \frac{1}{\pi} dxdy$ is replaced by $d\sigma = \frac{d\theta}{2\pi}$ on the unit circle $\mathbf{T}$ and the spaces A^p are replaced, accordingly, by the familiar Hardy spaces H^p (cf., e.g., [Ak], [D], [Ka], [Kh2–6], [RS], [W],and references cited therein). In that situation, the approach based on Hahn–Banach duality and the F. and M. Riesz theorem identifying the annihilator $Ann\,(H^p)$ in $L^q\,(\mathbf{T}, d\theta)$ as $H^q_0 = \{f \in H^q : f(0) = 0\}$, $q = \frac{p}{p-1}$ turns out to be quite successful and answers a number of questions. The difficulty with this approach when using area measure is the tremendous size of the annihilator $Ann\,(A^p)$ of A^p in L^q. The following result, which we shall call Khavin's lemma, characterizes $Ann\,(A^p)$.

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For

$$p : 1 < p < \infty, \quad q : \frac{1}{p} + \frac{1}{q} = 1$$

$$Ann\,(A^p) := \left\{ g \in L^q : \int_{\mathbf{D}} fg dA = 0 \quad for\ all \quad f \in A^p \right\}$$

$$= \left\{ \frac{\partial u}{\partial \overline{z}}\ , u \in W_0^{1,q}(\mathbf{D}) \right\}, \tag{1.1}$$

where $W_0^{1,q}$ is the Sobolev space of functions vanishing on $\mathbf{T}$ (cf. [KS], [Sh 1]). (It can be defined as the closure of compactly-supported test functions in $C_0^\infty(\mathbf{D})$ with respect to the L^q-norm of their gradients, or, equivalently, the L^q-norm of their $\dfrac{\partial}{\partial \overline{z}}$ derivative.)

For $p = 1$, one needs in (1.1) to take the weak-$*$ closure in L^∞ of $\dfrac{\partial u}{\partial \overline{z}}$, $u \in C_0^\infty(\mathbf{D})$. Since the dual of L^p, $p \geq 1$, is L^q, where $\dfrac{1}{p} + \dfrac{1}{q} = 1$, the general Hahn–Banach duality relation for (1.1) then can be written in the following form (cf. e.g., [Kh2–5], [D, Ch.8])

$$\lambda \ := \inf_{f \in A^p} \|\omega - f\|_p \tag{1.2}$$

$$= \max_{\substack{g \in Ann\,(A^p) \\ \|g\|_q \leq 1}} \left| \int_{\mathbf{D}} g\omega \ dA \right|. \tag{1.3}$$

The maximum in the right side of (1.3) indicates that the extremal function $g^\star \in Ann\,(A^p)$ always exists.

The rest of the paper is organized as follows. In Section 2 we prove the existence of best approximations and characterize them. These results are not new (cf. [Kh2–6]), but we include them for the sake of completeness and to set the stage for further discussion.

Section 3 deals with the problem of uniqueness of best approximations by analytic and harmonic functions. The interesting case here is, of course, $p = 1$. We show that if ω is continuous, the best analytic approximant is unique. For harmonic approximation, we can only show that in dimension 2 two different best harmonic approximants to a continuous function on the open disk cannot differ by a bounded function.

In Section 4 we prove two results concerning hereditary smoothness of best approximation by A^p functions, and discuss some open problems.

Section 5 deals with "badly approximable" functions. In the harmonic case, this leads to questions concerning harmonic peak sets, which we investigate.

In Section 6, we give a new proof of the theorem of Armitage, Gardiner, Haussmann and Rogge [AGHR] characterizing best approximation in L^1 to functions continuous on $\overline{\mathbf{B}_n}$ and subharmonic on $\mathbf{B}_n$ by functions continuous on $\overline{\mathbf{B}_n}$ and harmonic on $\mathbf{B}_n$.

Finally in Section 7, we consider best approximation in L^1 to the Newton kernel. We give an explicit example of a smooth function, real-analytic on $\partial\mathbf{B}_n$, whose best harmonic approximant is unbounded. (The first example of this type was given in [GHJ], where the authors showed that the best harmonic approximant to the monomial x^4y^4 in $L^1(\mathbf{D})$ is not continuous on $\overline{\mathbf{D}}$). We also construct a continuous function on the closed disk whose best analytic approximant is unbounded.

Although we carry out the presentation for analytic functions in the unit disk $\mathbf{D}$, a large portion of the results readily extend to arbitrary smoothly bounded domains in $\mathbf{C}$ with merely cosmetic changes to the proofs. Let us also point out that somewhat related topics are discussed in a paper by Vukotić [V].

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2. Existence of best approximations.

Theorem 2.1. *The extremal function $f^\star$ giving the best approximation in (1.2) always exists.*

Proof. Let $p \geq 1$ and let $\{f_n\} \in A^p$ be a minimizing sequence, i.e., $\|\omega - f_n\|_p \to \lambda$. Then $\|f_n\|_p \leq C < +\infty$ for all n and hence, by subharmonicity, f_n's are uniformly bounded on compact subsets of $\mathbf{D}$. Therefore, taking a subsequence, we can assume that $\{f_n\}$ converge uniformly on compact subsets of $\mathbf{D}$ to $f^\star \in A^p$. By Fatou's lemma

$$\lambda^p \leq \|\omega - f^\star\|_p^p \leq \liminf_{n\to\infty} \|\omega - f_n\|_p^p = \lambda^p$$

and, hence, $f^\star$ is the best approximant to ω. Q.E.D. ◁

The following result is based on the Hahn–Banach theorem (cf. [D, Ch.8], [Kh2–5], [RS]) and provides the standard necessary and sufficient conditions for the functions $f^\star, g^\star$ to be the extremals in the respective problems (1.2), (1.3).

Theorem 2.2.

(i) Let $p > 1$, $q : \frac{1}{p} + \frac{1}{q} = 1$. $f^\star \in A^p, g^\star \in Ann\,(A^p)$ are extremals in (1.2) and (1.3) if and only if, for some $\alpha \in \mathbf{R}$,

$$e^{i\alpha} g^\star (\omega - f^\star) \geq 0 \quad a.e.\ in\ \mathbf{D} \tag{2.3a}$$

and

$$\lambda^p |g^\star|^q = |\omega - f^\star|^p \quad a.e.\ in\ \mathbf{D}, \tag{2.3b}$$

where $\lambda := dist_{L^p} (\omega, A^p)$.

(ii) The above assertion is true for $p = 1$ provided (2.3) is replaced by

$$e^{i\alpha} g^\star (\omega - f^\star) = |\omega - f^\star| \quad a.e.\ in\ \mathbf{D}. \tag{2.4}$$

For the reader's convenience we shall sketch a (standard) proof of (2.3)–(2.4) (cf. [D, Ch.8], [Kh2–5]).

(i) By Theorem 2.1 and the Banach–Alaoglu Theorem, there exist extremals $f^\star, g^\star$. We find, applying Hölder's inequality,

$$\begin{aligned}\lambda = \left| \int_{\mathbf{D}} g^\star (\omega - f^\star)\, dA \right| &\leq \int_{\mathbf{D}} |g^\star|\, |\omega - f^\star|\, dA \\ &\leq \|g^\star\|_q \|\omega - f^\star\|_p \leq \|\omega - f^\star\|_p = \lambda. \end{aligned} \tag{2.5}$$

Thus, equalities must occur at each step in (2.5). Combining this with necessary and sufficient conditions for equality in Hölder's inequality we complete the proof of (2.3).

(ii) For $p = 1$, let $f^\star \in A^1$, $g^\star \in L^\infty \cap Ann\left(A^1\right)$ be the extremals. Then, the chain (2.5) becomes

$$\lambda = \left| \int_{\mathbf{D}} g^\star (\omega - f^\star)\, dA \right| \leq \int_{\mathbf{D}} |g^\star|\, |\omega - f^\star|\, dA$$

$$\leq \|\omega - f^\star\|_{L^1} = \lambda, \tag{2.5'}$$

and (2.4) follows.

Conversely, if $f^\star, g^\star$ satisfy (2.3) (or, (2.4)) and $\|g^\star\|_q \leq 1$, we have equality everywhere in (2.5) (or, (2.5′)), and since for any $f \in A^p$, $g \in Ann\,(A^p)$, $\|g\|_p \leq 1$ we have

$$\left|\int_{\mathbf{D}} g\,(\omega - f^\star)\,dA\right| \leq \lambda \leq \|\omega - f\|_p\,,$$

$f^\star, g^\star$ must be extremals. ◁

Remark: For best harmonic approximation, exactly the same result holds with A^p and $Ann(A^p)$ replaced by L^p_h and $Ann(L^p_h)$.

As an application of the theorem, consider the problem of approximating the monomials $\omega = z^n\overline{z}^m$.

Proposition 2.3. *For $m > n$, $f^\star = 0$. When $n \geq m$, $f^\star = cz^{n-m}$, where $c = c(n, m, p)$ is an appropriate constant.*

Proof. First consider the case $m > n$. Note that $g := \dfrac{|z^n\overline{z}^m|^p}{z^n\overline{z}^m} \in Ann\,(A^p)$. (This is checked right away by going to polar coordinates.) Hence, for

$$g^\star = \begin{cases} \frac{g}{\|g\|_q}, & p > 1 \\ g, & p = 1 \end{cases}$$

and $f^\star = 0$, the conditions (2.3) (or, (2.4)) are satisfied and the statement follows.

For $n \geq m$, we first find $c := c(n, m, p)$ so that $g := \dfrac{|z^n\overline{z}^m - cz^{n-m}|^p}{z^{n-m}\left(|z|^{2m} - c\right)} \in Ann\,(A^p)$. Note that (setting $|z| =: r$) $g = \dfrac{r^{p(n-m)}\left|r^{2m} - c\right|^p}{z^{n-m}\left(r^{2m} - c\right)}$. Hence, switching to polar coordinates and integrating with respect to θ first, we observe that g annihilates all monomials z^k, $k \neq n - m$. Choosing $c = c(n, m, p) < 1$ so that

$$\int_0^1 r^{p(n-m)}\left|r^{2m} - c\right|^{p-1} sgn\left(r^{2m} - c\right) r dr = 0,$$

we have $g \in Ann\,(A^p)$. Then, as before, setting

$$g^\star = \begin{cases} \frac{g}{\|g\|_q}, & p > 1 \\ g, & p = 1 \end{cases}$$

and applying (2.3)–(2.4) we complete the proof. ◁

Remark: The same argument shows that the best harmonic approximant to $z^n\overline{z}^m$ is cz^{n-m} if $n \geq m$, and $c\overline{z}^{m-n}$ if $m \geq n$.

3.1. Uniqueness of the best analytic approximation.

The following result is originally due to S.Ya. Khavinson [Kh6], where it is a part of a much more general framework. However, for the reader's convenience we give a straightforward independent proof.

Theorem 3.1. *For $p > 1$, the best approximant $f^\star$ in (1.2) is always unique. For $p = 1$ and ω continuous in $\mathbf{D}$, the best approximant $f^\star$ is unique. For discontinuous ω the best approximation need not be unique.*

Proof. For $p > 1$, uniqueness is an immediate consequence of the strict convexity of L^p (cf., e.g., [Ak]). Let $p = 1$ and ω be continuous in $\mathbf{D}$. Let f_1, f_2 be two best approximants to ω, let $g^\star \in L^\infty$ be the extremal in the dual problem so the relations (2.4):

$$\begin{aligned} g^\star(\omega - f_1) &= |\omega - f_1|; \\ g^\star(\omega - f_2) &= |\omega - f_2| \end{aligned} \tag{3.2}$$

hold almost everywhere in $\mathbf{D}$. Let us separate the following assertions.

Assertion 1. *For $z \in \mathbf{D}$, if $|\omega(z) - f_1(z)| = |\omega(z) - f_2(z)|$, then $f_1(z) = f_2(z)$.*

Proof of Assertion 1. If $\omega(z) - f_1(z) = 0$, the conclusion is obvious since then $\omega(z) - f_2(z) = 0$ also. So, suppose $\omega(z) - f_1(z) \neq 0$. Then (since ω is assumed to be continuous in $\mathbf{D}$) there is a disk $\Delta := \Delta(z, \rho)$ centered at z such that $|\omega - f_1|$ and, consequently, also $|\omega - f_2|$ are positive in Δ, and by (3.2) $|g^\star| \neq 0$ almost everywhere in Δ. Thus, (3.2) yields

$$\frac{\omega - f_1}{\omega - f_2} = \frac{|\omega - f_1|}{|\omega - f_2|} \quad a.e. \text{ in } \Delta \tag{3.3}$$

and, hence (since both sides are continuous), (3.3) holds pointwise in Δ. In particular, (3.3) holds at z and the assertion is proved.

Assertion 2. *If $p(z)$ is a real-valued continuous integrable function in $\mathbf{D}$ such that $\int_{\mathbf{D}} p\,dA = 0$, then there is a nontrivial continuum $K \subset \mathbf{D}$ on which $p = 0$.*

Proof of Assertion 2 (obvious)**.** If $p \equiv 0$ in $\mathbf{D}$, there is nothing to prove. If $U_1 := \{z : p(z) > 0\} \neq \emptyset$, then $U_2 := \mathbf{D} \backslash \overline{U_1} \neq \emptyset$ by the hypothesis. Hence $\partial U_1 \cap \mathbf{D}$ is a continuum (since it separates points in U_1 from those in U_2) on which $p = 0$.

End of the proof of the theorem. Let $p(z) := |\omega(z) - f_1(z)| - |\omega(z) - f_2(z)|$. Since f_1, f_2 are both best approximants to ω, the hypothesis of Assertion 2 is satisfied. Hence, there is a continuum K on which, according to Assertion 1, $f_1 = f_2$ and, accordingly, $f_1 = f_2$ everywhere in $\mathbf{D}$.

The following example shows that for discontinuous ω best approximations need not be unique (for $p = 1$, of course).

Example 3.2. Let $\mathbf{D}_0 = \left\{ z : |z| < \frac{1}{\sqrt{2}} \right\}$, so $Area\,(\mathbf{D}_0) = \frac{1}{2} Area\,(\mathbf{D})$. Take $\omega = \chi_{\mathbf{D}_0}$, the characteristic function of $\mathbf{D}_0$. Then, for any $c : 0 \leq c \leq 1$, $f^\star \equiv c$ gives the best approximation in A^1 to ω. Indeed, let $g^\star = \begin{cases} 1, & z \in \mathbf{D}_0 \\ -1, & z \in \mathbf{D} \backslash \mathbf{D}_0 \end{cases}$.

Obviously, $\int_{\mathbf{D}} g^\star z^n dA = 0$, $n = 1, 2, \ldots$, and $\int_{\mathbf{D}} g^\star dA = 0$, as well. Thus, $g^\star \in Ann\left(A^1\right)$ and for any $c : 0 \leq c \leq 1$ we have

$$g^\star\,(\omega - c) = |\omega - c| \quad a.e.\ in\ \mathbf{D}.$$

Thus, (2.4) is fulfilled and $f^\star = c$ is the best approximant to ω. The proof of the theorem is now complete. ◁

Remarks.

(i) The proof given of Thm. 3.1 extends word-for-word to arbitrary domains, in particular, to multiply-connected domains. This is in contrast with the situation in the Hardy space setting where an H^1-best approximation even to a real-analytic function on the boundary of a finitely-connected domain (in $L^1\left(|d\xi|\right)$ norm) need not be unique (cf. [Kh2, Section 3]).

(ii) The argument given for the proof of Thm. 3.1 is sufficiently flexible and, as is easily seen, extends, e.g., to ω, whose set of discontinuity has measure zero in $\mathbf{D}$, is relatively closed (in $\mathbf{D}$) and does not separate $\mathbf{D}$ (cf. [Kh6]). A more general result of S.Ya. Khavinson ([Kh6]) allows to extend Thm. 3.1 to functions ω with special kinds of discontinuities: the limit set of ω at each point of discontinuity is either a segment, or contains three noncollinear points. Yet, the crux of all these proofs lies in the fact that the zero sets of analytic functions DO NOT separate the disk. This raises a very intriguing question of finding a plausible analogue of Thm. 3.1 for best harmonic approximation in the $L^1(\mathbf{D})$-metric, which is addressed in the following subsection.

(iii) It can also be shown that if ω is quasi-continuous, as is the case for example for functions in the Sobolev space $W^{1,1}(\mathbf{D})$, then the best approximant is unique. For indeed, the boundary of the set

$$P := \{z : |\omega(z) - f_1(z)| > |\omega(z) - f_2(z)|\}$$

is contained in

$$Z := \{z : |\omega(z) - f_1(z)| = |\omega(z) - f_2(z)|\} \cap \{z : \omega \text{ continuous at } z\}$$

union a set of arbitrarily small 1-capacity. As ∂P has positive length and therefore positive 1-capacity, so must Z; so by the proof of Assertion 1, f_1 must equal f_2 on a set of uniqueness for analytic functions. (For definitions of quasi-continuous and 1-capacity, see [EG]).

(iv) It follows easily from (2.3) that for $p > 1$ the extremal function $g^\star$ in the dual problem (1.3) is unique (up to a unimodular constant factor, of course), similarly to the uniqueness of $f^\star$. Also, (2.4) implies uniqueness of $g^\star$ (up to a unimodular constant factor again) provided that ω does not coincide with an analytic function on a set of positive measure. The following example shows that if this condition is violated, $g^\star$ may not be unique. Let $\omega = \begin{cases} 1, & z : 0 \le |z| \le \frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}} \le |z| < 1 \\ 0, & \textit{elsewhere.} \end{cases}$. Then, taking $f^\star = 1$ we see that for

$$g_1^\star = \begin{cases} 0, & |z| \le \frac{1}{\sqrt{3}} \\ -1, & \frac{1}{\sqrt{3}} < |z| < \sqrt{\frac{2}{3}} \\ 1, & \sqrt{\frac{2}{3}} \le |z| < 1 \end{cases} \quad \textit{and} \quad g_2^\star = \begin{cases} 1, & |z| \le \frac{1}{\sqrt{3}} \\ -1, & \frac{1}{\sqrt{3}} < |z| < \sqrt{\frac{2}{3}} \\ 0, & \sqrt{\frac{2}{3}} \le |z| < 1 \end{cases},$$

(2.4) are satisfied and hence the dual problem has a non-unique extremal ($g_1^\star, g_2^\star \in Ann\left(A^1\right)$ since they are both radial and have the mean value zero over the disk.) Once again, we remark in passing that the extremal function $g^\star$ in the dual problem in the Hardy space context is always unique [Kh2, RS].

3.2. Uniqueness of the best harmonic approximation.

As before, strict convexity of L^p yields the uniqueness of the best harmonic approximant to a function ω in $L^p_h(\mathbf{B}_n)$ for $p > 1$. For $p = 1$, the complete answer is unknown. Example 3.2 shows that for discontinuous ω the best harmonic approximant need not be unique. Similarly, the example at the end of the previous subsection shows that uniqueness in the dual extremal problem fails if the function ω coincides on a set of positive measure with a harmonic function. Whether an analogue of Thm. 3.1 holds for harmonic functions (*i.e.* whether continuous functions have unique best harmonic approximants in L^1) is unknown to the best of our knowledge. Here, we give some rather special results, which extend somewhat some of those in [GHJ], where ω was assumed to be subharmonic and real-analytic.

Proposition 3.3. *Let $\omega(z) = \omega(|z|)$ be a complex-valued, radial function that is continuous and integrable in $\mathbf{D}$. Then the best harmonic approximant to ω in $L^1(\mathbf{D})$ is unique, and is a constant giving the minimal value in the one-dimensional problem of finding the infimum*

$$\inf \left\{ \int_0^1 |\omega(r) - c|\, r dr, \quad c \in \mathbf{C} \right\}.$$

Let us first separate the following.

Lemma 3.4. *For $f \in L^1(\mathbf{D})$, let $f^\sharp(r) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f\left(re^{i\theta}\right) d\theta$, $0 < r < 1$ be the mean value of f over the circle of radius r. Then, for any $u \in L^1_h(\mathbf{D})$ we have*

$$\int_{\mathbf{D}} |f - u|\, dA \geq 2 \int_0^1 \left|f^\sharp(r) - u(0)\right| r dr. \tag{3.4}$$

Proof of the Lemma. Indeed,

$$\begin{aligned}\int_{\mathbf{D}} |f - u|\, dA &= \frac{1}{\pi} \int_0^1 \left\{ \int_0^{2\pi} \left|f\left(re^{i\theta}\right) - u\left(re^{i\theta}\right)\right| d\theta \right\} r dr \\ &\geq \frac{1}{\pi} \int_0^1 \left| \int_0^{2\pi} \left(f\left(re^{i\theta}\right) - u\left(re^{i\theta}\right)\right) d\theta \right| \Bigg\} r dr\end{aligned}$$

$$= 2\int_0^1 \left|f^\sharp(r) - u(0)\right| rdr.$$

Proof of the Proposition. Observe that (3.4) becomes equality when $h = h^\sharp(r)$ is radial and u is a constant. Also, note that in view of Assertions 1 and 2 in the proof of Thm.3.1 that extend mutatis mutandis to the harmonic approximation setting (in fact, to any setting where the approximating functions are continuous), it follows that any two best approximants always coincide on a whole continuum of points. Thus, to finish the proof it remains to show that a strict inequality holds in (3.4) if u is not constant.

Lemma 3.5. *Let u be a (complex-valued) harmonic function in the closed disk. Then, for any $r \le 1$, we have*

$$|u(0)| \le \frac{1}{2\pi}\int_0^{2\pi} \left|u\left(re^{i\theta}\right)\right| d\theta \tag{3.5}$$

and equality holds if and only if $u = const\ v$, where v is a non-negative harmonic function in $\mathbf{D}_r := \{z : |z| \le r\}$.

Indeed, since $|u|$ is subharmonic, for equality to hold in (3.5) u must have a constant argument on $r\mathbf{T} := \partial\mathbf{D}_r$ and, hence, by the Poisson formula, everywhere in $\mathbf{D}_r$.

Now assume ω admits a non-constant best approximant u. First of all, by Lemma 3.4 ω also admits a constant best approximant, namely $u(0)$. Replacing ω by $\omega - u(0)$ we reduce the problem to the following: ω is a radial, continuous function that has zero as a best approximant (i.e., ω is "badly approximable"), and u is another, non-constant best approximant to ω with $u(0) = 0$. By Lemmas 3.4 and 3.5, for each r between 0 and 1 the function $\omega(r) - u(rz) = k(r)v(z)$, where $k(r)$ is a unimodular constant, and v is a non-negative harmonic function in $\mathbf{D}$ which depends on r. Thus, the range of u in $\mathbf{D}_r$ lies on a half-ray passing through $\omega(r)$. Consider two cases:

(i) The range of ω contains three noncollinear points. (Recall that the range of ω is a *continuous* curve). Then let $0 < a < b < c < 1$ be three values of r such that $A := \omega(a)$, $B := \omega(b)$, $C := \omega(c)$ form a non-trivial triangle. Then, the range of u in $\mathbf{D}_a$, $\mathbf{D}_b$, $\mathbf{D}_c$ is contained in half-rays through A, B, and C. Hence, the range of u in the smallest circle $\mathbf{D}_a$ must lie in the intersection of these three rays, i.e., it is at most a point, so u is a constant.

(ii) The range of ω is contained in a line. Translating and rotating we can assume without loss of generality that ω is real-valued and, as before, that one of its possible best approximants is a zero function. If $\omega \equiv 0$, there is nothing to prove. Otherwise, ω must change sign in $\mathbf{D}$ (cf. (2.4)), by which we mean that there exist $r_0 : 0 < r_0 < 1$ such that $\omega(r_0) = 0$ while either to the right or to the left of r_0 and close to r_0 ω has either positive or negative sign. Without loss of generality, assume that for $r : r_0 - \varepsilon < r < r_0$ for some small $\varepsilon > 0$ ω is positive. For all such r the range of u in $\mathbf{D}_r$ is contained in a half-ray on the real axis that ends at $\omega(r)$. Moreover, since $u(0) = 0$ while $\omega(r) > 0$, it must always be the left half-ray. Hence, letting $r \to r_0 - 0$ we obtain that $u \leq 0$ in $\mathbf{D}_{r_0}$. But $u(0) = 0$, so $u \equiv 0$ by the maximum principle and the proof is now complete. $\triangleleft$

Remark: The statement and proof of Proposition 3.3 go through with no difficulty to radial functions on $\mathbf{B}_n$ (with rdr replaced by $r^{n-1}dr$).

Theorem 3.6. *Let ω be a real-valued continuous function in $L^1(\mathbf{D})$. Then ω cannot have two best harmonic approximants in L^1 whose difference is bounded.*

Proof. Suppose h_1 and h_2 are best approximants of ω. Let $f := \omega - \frac{1}{2}(h_1 + h_2)$, and $h := \frac{1}{2}(h_1 - h_2)$. Then f is continuous on $\mathbf{D}$ and has 0, h and $-h$ as best approximants. We wish to prove that if h is bounded then it is identically zero.

As $\int |f|dA = \int |f+h|dA = \int |f-h|dA$, we get that $|f| \geq |h|$ almost everywhere, and so by continuity everywhere, on $\mathbf{D}$. Let $P = \{z \in \mathbf{D} : f(z) > 0\}$ and $N = \{z \in \mathbf{D} : f(z) < 0\}$. Notice that $\mathbf{D} \setminus (P \cup N)$ is contained in the zero-set of h, and is therefore of zero area.

Define $s(z)$ to be the function that is $+1$ on P, -1 on N, and 0 everywhere else on $\mathbf{C}$. Note that by the harmonic analogue of (2.4) (with $\omega = f$, $f^\star = 0$, and $g^\star = s$), the function s annihilates L^1_h. Therefore

$$0 \;=\; \int_{\mathbf{D}} z^n s(z)dA \;=\; \int_P z^n dA - \int_N z^n dA.$$

But as

$$\delta_{n0} \;=\; \int_{\mathbf{D}} z^n dA \;=\; \int_P z^n dA + \int_N z^n dA,$$

we get that

$$\int_P z^n dA = \int_N z^n dA = \frac{1}{2}\delta_{n0}, \tag{3.6}$$

where δ_{n0} is the Kronecker symbol.

Now, some component of either P or N must intersect the disk of radius $1/\sqrt{2}$. Without loss of generality, we can assume that some component P_0 of P does. Then the boundary of P_0 cannot intersect $\partial\mathbf{D}$ in a set of positive measure. For indeed, the Cauchy transform of N, the function

$$u(z) = \int_{\mathbf{N}} \frac{1}{z-w} dA(w)$$

is continuous and bounded on the entire complex plane and analytic off N; by (3.6), $u(z) = \frac{1}{2z}$ on $\mathbf{C} \setminus \overline{\mathbf{D}}$. Therefore u is analytic on P_0 and equal to $\frac{1}{2z}$ on $\partial P_0 \cap \partial\mathbf{D}$; if this latter set were of positive Lebesgue measure, then $u(z)$ would equal $\frac{1}{2z}$ on all of P_0. As P_0 intersects the disk of radius $1/\sqrt{2}$, we get that the Cauchy transform of N is greater in modulus than $1/\sqrt{2}$ at some point. But as N has area $\pi/2$, this contradicts the Ahlfors-Beurling theorem that says that the maximum value the Cauchy transform of a set of given area can attain is attained when that set is a disk and the point in question is on the boundary (for a proof see *e.g.* [GK]). A calculation shows that therefore the maximum value of the modulus of the Cauchy transform of a set of area $\pi/2$ is $1/\sqrt{2}$.

So ∂P_0 is contained in the zero-set of h union a null set on $\partial\mathbf{D}$ (which is perforce a null-set also with respect to harmonic measure for P_0, by Nevanlinna's majorization principle for harmonic measures). As h is bounded, and vanishes almost everywhere on ∂P_0, it must be identically zero on P_0, and hence on the whole disk. $\triangleleft$

Remarks

(i) We could weaken the hypotheses of the theorem to say that ω cannot have two best harmonic approximants whose difference raised to some power $p > 1$ has a harmonic majorant, because again the vanishing of h almost everywhere on ∂P_0 forces it to be identically zero.

(ii) The theorem is false on other domains. Let G be a null quadrature domain, *i.e.* a domain such that the integral of every $L^1_h(G)$ function is zero (*e.g.* the half-plane -

see [Sh1]). Then if h is any function in $L^1_h(G)$, the function $|h|$ has all the functions $\{ch : -1 \le c \le 1\}$ as best harmonic approximants; if h is also bounded, the theorem fails.

(iii) The previous example can be translated into a remark about weighted approximation on the unit disk, via conformal maps. By considering the conformal map from the disk to the right half plane, we get *e.g.* that, with respect to the measure $\frac{1}{|z-1|^4}dA(z)$ on the unit disk, the function $|(x-1)^3-3(x-1)y^2|$ has many best harmonic approximants that are bounded.

(iv) The problem with generalizing the proof to higher dimensions, using the techniques developed in Sections 5.2 and 6, is that it is not known whether a solution of Poisson's equation with bounded data (so $C^{2-\varepsilon}$), can vanish along with its gradient on a set of positive measure on the sphere. See [BW] where a C^1 example of a non-zero harmonic function that vanishes along with its gradient on a set of positive measure is constructed. Of course, if one knew that $\partial P_0 \cap \partial \mathbf{B}_n$ actually contained an open subset of $\partial \mathbf{B}_n$, there would be no problem.

(v) Finally, we mention that we have not touched here the questions related to the best uniform (Chebyshev) harmonic approximation in $\mathbf{D}$ ($p = \infty$). Some results concerning the best Chebyshev harmonic approximation of continuous functions can be found in [HKL].

4. Hereditary regularity of the best analytic approximation in the disk.

Theorem 4.1. *Let ω belong to the Sobolev space $W^{1,p}(\mathbf{D})$, $p \ge 1$. Then the best approximant $f^\star \in A^p$ to ω is in the Hardy space H^p.*

Proof. Note that by Remark (iii) after Theorem (3.1), the best approximant is unique even for $p = 1$. First, assume ω to be real-analytic in $\overline{\mathbf{D}}$. Let $P_m = \{$ polynomials in z of degree $\le m\}$ and let $\lambda_m := \min_{f \in P_m} \|\omega - f\|_p$. (Since P_m is finite dimensional, the best approximant $f^\star_m \in P_m$ always exists.) Obviously, $\lambda := \min_{f\in A^p} \|\omega - f\| = \lim_{m\to\infty} \lambda_m$, as the polynomials are dense in A^p. Fix m. Since ω is real-analytic, $\omega - f^\star_m \ne 0$ a.e. in $\mathbf{D}$ (except in the trivial case where ω is in P_m) and hence, according to an analogue of

(1.2) and Theorem 2.2, replacing the subspace A^p by P_m (the proof is the same as that of Thm.2.2—cf., e.g., [Kh4–5]) it follows that

$$\frac{1}{\lambda^{p-1}}\frac{|\omega - f_m^\star|^p}{\omega - f_m^\star} \in Ann\,(P_m) \quad in\ L^q. \tag{4.1}$$

Then, using Stokes' formula and (4.1) we obtain

$$\begin{aligned}
\int_{\mathbf{T}} |\omega - f_m^\star|^p \,\frac{d\theta}{2\pi} =& \frac{i}{2\pi}\int_{\mathbf{T}} |\omega - f_m^\star|^p \, z d\overline{z} \\
=& \int_{\mathbf{D}} \frac{\partial}{\partial z}\left(\left(\omega - f_m^\star\right)\right)^{\frac{p}{2}}\left(\overline{\omega} - \overline{f}_m^\star\right)^{\frac{p}{2}}\Big) z dA + \int_{\mathbf{D}} |\omega - f_m^\star|^p \, dA \\
=& \frac{p}{2}\int_{\mathbf{D}} \left[\frac{|\omega - f_m^\star|^p}{\omega - f_m^\star}\left(\frac{\partial\omega}{\partial z} - \left(f_m^\star\right)'\right) + \frac{|\omega - f_m^\star|^p}{\overline{\omega} - \overline{f}_m^\star}\frac{\partial\overline{\omega}}{\partial z}\right] z dA + \lambda_m^p \quad (4.2) \\
=& \frac{p}{2}\int_{\mathbf{D}} \left[\frac{|\omega - f_m^\star|^p}{\omega - f_m^\star}\left(z\frac{\partial\omega}{\partial z}\right) + \frac{|\omega - f_m^\star|^p}{\overline{\omega} - \overline{f}_m^\star} z \overline{\left(\frac{\partial\omega}{\partial\overline{z}}\right)}\right] dA + \lambda_m^p,
\end{aligned}$$

since $z\left(f_m^\star\right)' \in P_m$. Thus, applying Hölder's inequality we obtain

$$\int_{\mathbf{T}} |\omega - f_m^\star|^p \,\frac{d\theta}{2\pi} \leq \begin{cases} \frac{p}{2}\left\|\omega - f_m^\star\right\|_p^{\frac{p}{q}}\left(\left\|\frac{\partial\omega}{\partial z}\right\|_p + \left\|\frac{\partial\omega}{\partial\overline{z}}\right\|_p\right) + \lambda_m^p, & p > 1 \\ \frac{1}{2}\left(\left\|\frac{\partial\omega}{\partial z}\right\|_1 + \left\|\frac{\partial\omega}{\partial\overline{z}_p}\right\|_1\right) + \lambda_m, & p = 1. \end{cases} \tag{4.3}$$

$\left(|\omega - f_m^\star|^{p-1} \in L^q \text{ and its } L^q\text{-norm is } \|\omega - f_m\|_p^{\frac{p}{q}}\right)$. Thus, invoking standard inequalities for Sobolev spaces we obtain from (4.3)

$$\int_{\mathbf{T}} |f_m^\star|^p \, d\theta \leq C\lambda_m^{\frac{p}{q}} \|\omega\|_{W^{1,p}(\mathbf{D})} \leq C_1, \tag{4.4}$$

where C, C_1 are constants. Thus, all H^p norms of $f_m^\star$ are uniformly bounded, so taking a subsequence we can assume $f_m^\star$ converges to some function $f^\star \in H^p$ on compact subsets of $\mathbf{D}$ and so $(\omega - f_m^\star)$ converges to $\omega - f^\star$ pointwise in $\mathbf{D}$. In both cases we have

$$\|\omega - f^\star\|_p \leq \liminf_{m\to\infty} \|\omega - f_m^\star\|_p = \lim_{m\to\infty} \lambda_m = \lambda,$$

so $f^\star$ must be the best approximant to ω in A^p. Since by Fatou's lemma and (4.4)

$$\|f^\star\|_{H^p} \leq \liminf_{m\to\infty} \|f_m^\star\|_{H^p} \leq const\, \|\omega\|_{W^{1,p}(\mathbf{D})}\,, \tag{4.5}$$

it follows that $f^\star \in H^p$. This proves the theorem for real-analytic ω. Since real-analytic functions (even polynomials) are dense in the Sobolev spaces, a standard approximation argument and the fact that the estimate (4.5) depends only on the $W^{1,p}$-norm of ω complete the proof. $\triangleleft$

Remarks.

(i) The idea leading to the calculation in (4.2) goes back to Ryabych [R], where it is applied in the context of another extremal problem—cf. [KS].

(ii) Essentially, the same proof shows that the best *harmonic* approximant to a $W^{1,p}$-function ω in $L^p(\mathbf{D})$ belongs to the class h^p (cf. [D]), i.e. is representable by a Poisson–Lebesgue integral with an $L^p(\mathbf{T})$-density for $p > 1$, or by a Poisson–Stieltjes integral for $p = 1$. The crucial step in the calculation similar to (4.2) is that if $h_m^\star$ is the harmonic polynomial approximant to ω of degree $\le m$, then $z\dfrac{\partial h_m^\star}{\partial z}$, $z\dfrac{\partial \overline{h_m^\star}}{\partial z}$ are both analytic polynomials of degree $\le m$ ($h_m^\star$, $\overline{h_m^\star}$ are harmonic!), while $\dfrac{|\omega - h_m^\star|^p}{\omega - h_m^\star}$ and its conjugate both annihilate A^p.

(iii) If ω in L^1 has $f^\star$ as its best A^1 approximant, and g is any function that is positive a.e., then by looking at the signum one sees in view of Theorem 2.2 that

$$h \;=\; g\,\omega \;+\; (1-g)\,f^\star$$

also has $f^\star$ as its best A^1 approximant. Choosing g to vanish smoothly at an isolated singularity of ω inside $\mathbf{D}$, one can make h smoother, and then apply Theorem 4.1 to h. In particular, one gets that the best analytic approximant to $\omega(z) = \dfrac{1}{z-\lambda}$ is in H^1 for all λ in $\mathbf{D}$.

Another type of regularity can be derived using the ideas from [Sh2]. The key idea is Clarkson's inequality (cf. [HS, p.227]). Let $p : 1 < p \le 2$, $\dfrac{1}{q} + \dfrac{1}{p} = 1$. Then, for any $F, G \in L^p\,(\mathbf{D})$ we have

$$\left\|\frac{F+G}{2}\right\|_p^q + \left\|\frac{F-G}{2}\right\|_p^q \le \left(\frac{1}{2}\|F\|_p^p + \frac{1}{2}\|G\|_p^p\right)^{\frac{q}{p}}. \tag{4.6}$$

For $u \in L^p\,(\mathbf{D})$ and α of modulus 1, denote by $R_\alpha u$ the operator

$$R_\alpha u(z) := u(\alpha z). \tag{4.7}$$

R_α is an isometry of L^p and $R_\alpha A^p = A^p$. Now, let us measure the "smoothness" of the function (the "mean Hölder condition") by saying that $u \in \Lambda^p_\sigma$, $\sigma > 0$, if for $0 \le t \le \pi$,

$$D_t u := \left\| R_{e^{it}} u + R_{e^{-it}} u - 2u \right\|_p = O\left(t^\sigma\right).$$

(For simplicity, we shall just write Λ_σ when the choice of p is understood). Of course, e.g., $u \in C^2 \Rightarrow u \in \Lambda_2$, etc.

Theorem 4.2. *Let $1 < p \le 2$, and let $q = p/(p-1)$. Let $\omega \in \Lambda_\sigma$ for some $\sigma > 0$ and let $f^\star$ be its best approximant in A^p. Then $f^\star \in \Lambda_{\frac{\sigma}{q}}$.*

Proof. By scaling, we can take $\left\|\omega - f^\star\right\|_p = 1$. Define the operator T_t by

$$T_t(F) := \left(R_{e^{it}} F + R_{e^{-it}} F\right)/2. \tag{4.8}$$

Clearly, T_t is a contraction for all $t \in [0, \pi]$, and $\omega \in \Lambda_\sigma$ means that

$$\left\|T_t\omega - \omega\right\|_p \le C t^\sigma.$$

Let $T_t f^\star := g \in A^p$. Now we have, $(\|\cdot\| = \|\cdot\|_p)$:

$$\begin{aligned} \|g - \omega\| &= \|T_t f^\star - \omega\| \\ &\le \|T_t f^\star - T_t\omega\| + \|T_t\omega - \omega\| \\ &\le \|f^\star - \omega\| + Ct^\sigma \\ &= 1 + Ct^\sigma. \end{aligned} \tag{4.9}$$

Applying (4.6) with $F = f^\star - \omega$ and $G = g - \omega$ we get

$$\left\|\frac{f^\star + g}{2} - \omega\right\|^q + \left\|\frac{f^\star - g}{2}\right\|^q \le \left[\frac{1}{2}\|f^\star - \omega\|^p + \frac{1}{2}\|g - \omega\|^p\right]^{\frac{q}{p}}. \tag{4.10}$$

Since $1 = \|f^\star - \omega\|$, the first term on the left in (4.10) is ≥ 1. Using also (4.9) we obtain (C denotes constants that may change from line to line)

$$1 + \left\|\frac{f^\star - g}{2}\right\|^q \le \left[\frac{1}{2} + \frac{1}{2}\left(1 + Ct^\sigma\right)^p\right]^{\frac{q}{p}}$$

$$\leq (1 + Ct^\sigma)^{\frac{q}{p}} \leq 1 + Ct^\sigma, \tag{4.11}$$

where $q \geq p$ and we take t to be smaller than 1. Thus,

$$\|f^\star - g\| \leq Ct^{\frac{\sigma}{q}} \tag{4.12}$$

and recalling that $g = T_t f^\star$ we obtain the result. ◁

Discussion of Theorem 4.2. Letting $\alpha = e^{it}$, the above assertion becomes

$$\left(\int_{\mathbf{D}} |f^\star(\alpha z) + f^\star(\overline{\alpha} z) - 2f^\star(z)|^p \, dA\right)^{\frac{1}{p}} \leq Ct^{\frac{\sigma}{q}}$$

or, writing $f^\star(z) = \sum_0^\infty a_n z^n$,

$$\int_0^1 r dr \int_0^{2\pi} \left| \sum_0^\infty a_n r^n \left(\alpha^{\frac{n}{2}} - \alpha^{-\frac{n}{2}}\right)^2 e^{in\theta} \right|^p d\theta \leq Ct^{\sigma(p-1)}. \tag{4.13}$$

Now, by the Hausdorff–Young inequality ($p \leq 2!$), cf. [D, p.83]:

$$\left(\frac{1}{2\pi}\int_0^{2\pi} \left| \sum_0^\infty a_n r^n \left(\alpha^{\frac{n}{2}} - \alpha^{-\frac{n}{2}}\right)^2 e^{in\theta} \right|^p d\theta\right)$$

$$\geq \left\{ \sum_0^\infty \left(|a_n| \, r^n \left|\alpha^{\frac{n}{2}} - \alpha^{-\frac{n}{2}}\right|^2 \right)^q \right\}^{\frac{p}{q}}.$$

So, from (4.13) it follows $\left(\alpha = e^{it}\right)$:

$$\int_0^1 r dr \left(\sum_0^\infty \left(|a_n| \, r^n \sin^2 \frac{nt}{2} \right)^q \right)^{p-1} \leq Ct^{\sigma(p-1)}. \tag{4.14}$$

Fix an integer N and replace the integral on the left in (4.14) by that over $\left[1 - \frac{1}{N}, 1\right]$ and $\sum_0^\infty$ by $\sum_1^N$. For that range of r and n, $r^n \geq r^N \geq \left(1 - \frac{1}{N}\right)^N \geq c$, an absolute constant, so (4.14) yields

$$\frac{1}{N} \left(\sum_{n=1}^N \left(|a_n| \sin^2 \frac{nt}{2} \right)^q \right)^{p-1} \leq Ct^{\sigma(p-1)},$$

hence,

$$\sum_{1}^{N} |a_n|^q \sin^{2q} \frac{nt}{2} \le C N^{\frac{1}{p-1}} t^{\sigma}. \tag{4.15}$$

Now, for $\xi < \sigma + 1$, multiply (4.15) by $t^{-\xi}$ and integrate from 0 to 1. We obtain

$$\sum_{1}^{N} |a_n|^q \int_0^1 \left(\sin^2 \frac{nt}{2}\right)^q t^{-\xi} dt \le C(\xi) N^{\frac{1}{p-1}}. \tag{4.16}$$

The integral in (4.16) is, changing variables by $nt = s$,

$$\int_0^n \left(\sin^2 \frac{s}{2}\right)^q \left(\frac{n}{s}\right)^{\xi} \frac{ds}{n} = n^{\xi - 1} \int_0^n \left(\sin^2 \frac{s}{2}\right)^q s^{-\xi} ds \ge C n^{\xi - 1},$$

so

$$\sum_{1}^{N} n^{\xi - 1} |a_n|^q \le C(\xi) N^{\frac{1}{p-1}}$$

and

$$\sum_{[\frac{N}{2}]+1 \le n < N} |a_n|^q \le C(\xi) N^{\frac{1}{p-1} - \xi + 1}. \tag{4.17}$$

(4.17) allows us to extract some particular regularity information about $f^\star$. For example, take $\sigma = 2$. Then, for all $\frac{3}{2} < p \le 2$ it follows from (4.17), letting $\delta = 2 - \frac{1}{p-1}$ and $\tau = 3 - \xi$,

$$\sum_{[\frac{N}{2}]+1 \le n < N} |a_n|^q \le c(\tau) N^{(2-\delta)-(3-\tau)+1} \le C(\tau) N^{-\eta}, \tag{4.18}$$

where $\eta = \delta - \tau$ is positive for τ sufficiently small and positive. Hence, for all $k \ge 1$ we have

$$\sum_{2^{k-1} \le n < 2^k} |a_n|^q < C(\tau) 2^{-\eta k},$$

so by Hölder's inequality

$$\sum_{2^{k-1} \le n < 2^k} |a_n|^2 < C(\tau) [2^{-\eta k}]^{\frac{2}{q}} [2^k]^{\frac{q-2}{q}}. \tag{4.19}$$

The coefficient of k in the exponent on the right-hand side of (4.19) can, by suitable choice of τ, be made negative for $p > \frac{8}{5}$, so we obtain

Corollary 4.3. *For $\dfrac{8}{5} < p \le 2$ the best approximant $f^\star$ in A^p to a function $\omega \in \Lambda_2^p$ belongs to the Hardy space H^2.*

Remarks.

(i) It would be interesting to clarify the relationship between Theorems 4.1 and 4.2.

(ii) Theorems 4.1 and 4.2 certainly leave unanswered most of the natural regularity questions, such as: given ω to be real-analytic in $\overline{\mathbf{D}}$, does it imply that its best approximant in A^p is Hölder continuous or merely continuous in $\overline{\mathbf{D}}$? (Similar results and much more are known to hold for L^p-approximation on the circle–cf., e.g., [CJ], [D, Ch.8], [Ka], [Kh2,3], [RS].) When $p = 1$, much regularity can be lost by harmonic aproximation - see Section 7.

(iii) Here is another set of problems. For the sake of definiteness let us take $p = 1$. Assume $\omega \in C\left(\overline{\mathbf{D}}\right)$, $\|\omega\|_\infty = 1$ and that for some small $\varepsilon > 0$ we can find $f \in A^1$ so that

$$\lambda := \|\omega - f\|_1 \leq \varepsilon. \tag{4.20}$$

Question. What is the distance from ω to the unit ball in H^∞ in the L^1-norm? In other words, estimate (in terms of ε)

$$\mu := \inf\left\{\|\omega - g\|_1 : g \in H^\infty, \|g\|_\infty \leq 1\right\}. \tag{4.21}$$

A similar problem in the context of Hardy spaces was discussed in a recent paper [KP-GS]. There, the authors showed that

(a) $\mu = O\left(\varepsilon \log \dfrac{1}{\varepsilon}\right)$ and

(b) The estimate in (a) cannot be improved to $O(\varepsilon)$.

However, all the major ingredients of the arguments in [KP-GS] fail miserably for Bergman functions. We think that the relationship between quantities (4.20) and (4.21) may be a fruitful topic for future investigations.

5.1 A^p Badly Approximable Functions

We shall call a function $\omega \in L^p(\mathbf{D})$ *badly approximable* with respect to A^p if its best approximant in A^p equals 0.

Example 5.1.

(i) Let $a(r) \in L^p(rdr, [0,1])$, $p \geq 1$, $n \geq 1$. Then the function $\omega := a(r)e^{-in\theta}$ is badly approximable by A^p. Indeed,

$$\frac{|\omega|^p}{\omega} = |a(r)|^{p-1}\, sgn[a(r)]e^{in\theta} \in Ann\,(A^p)$$

in L^q, $\dfrac{1}{p} + \dfrac{1}{q} = 1$, and by Theorem 2.2 the assertion follows.

(ii) Let $p > 2$, $N = \{0, z_1, \ldots, z_n\}$—a finite set and let G be a contractive zero divisor in A^{p-2} (cf. [DKSS]) corresponding to the zero set N. Then, G extends analytically across $\mathbf{T}$. Set $\omega = \overline{G}$. ω is badly approximable. Indeed, one of the characteristic properties of a contractive divisor is that the measure $|G|^{p-2}\, dA$ is a representing measure for bounded analytic functions and since G itself is bounded, for all A^1 functions. Hence, $G\,|G|^{p-2}$ annihilates A^p ($G(0) = 0!$), and by Theorem 2.2 $\omega := \overline{G}$ is badly approximable.

On the other hand, we have

Propsition 5.2. *Let $f(z)$ be analytic and satisfy $|f(z)| \geq c > 0$ in $\mathbf{D}$. Then, $\omega := \overline{f}$ is not badly approximable in L^p, $p \geq 1$.*

Proof. Indeed, otherwise by Theorem 2.2 we would have $\dfrac{|f|^p}{\overline{f}} = f\,|f|^{p-2} \perp A^p$, so in particular (as $\dfrac{1}{f} \in H^\infty$):

$$\int_{\mathbf{D}} \frac{1}{f} f\,|f|^{p-2}\, dA = 0,$$

an obvious contradiction. ◁

It is quite easy, using duality, to characterize all badly approximable functions on the circle (in the context of Hardy spaces). In particular, conjugates of all inner functions vanishing at the origin are badly approximable. In the Bergman space, the situation is more complicated. It can be shown, *e.g.*, that the functions

$$\omega(z) \;=\; \frac{\overline{z}^2(\overline{z}-a)^2}{(1-a\overline{z})^2}, \quad 0<a<1,$$

are not badly approximable in L^1. Contrast the following result with Proposition 5.2.

Proposition 5.3. *The function* $\omega := (\overline{z}-a)^4$, $0<a<1$, *is badly approximable in* L^1.

Proof. In view of Theorem 2.2 and Khavin's Lemma, we want to find a continuous function v in $\overline{\mathbf{D}}$, $v|_{\mathbf{T}} = 0$ such that

$$\frac{\partial v}{\partial \overline{z}} = \left[\frac{(z-a)}{(\overline{z}-a)}\right]^2 \quad \textit{in } \mathbf{D}. \tag{5.1}$$

Integrating (5.1) we see that it is equivalent to the existence of a holomorphic function h in $\mathbf{D}$ satisfying

$$h(z) = v(z) + \frac{(z-a)^2}{\overline{z}-a}. \tag{5.2}$$

On $\mathbf{T}$ $v=0$, so (5.2) yields that $h(z) = \dfrac{z(z-a)^2}{1-az}$, and so,

$$v(z) = \frac{z(z-a)^2}{1-az} - \frac{(z-a)^2}{\overline{z}-a}$$

has all the desired properties. ◁

5.2 $L^1_h(G)$ Badly Approximable Functions and Harmonic Peak Sets

Many of the ideas in harmonic approximation extend to $\mathbf{R}^n$, so we shall work there. Let G be a domain in $\mathbf{R}^n$, and let us introduce the following two ways of measuring the size of a subset F with respect to harmonic functions.

Definition: *For F a measurable subset of G, define*

$$A(F) \;=\; \sup\{\frac{\int_F |h|}{\int_{G\setminus F}|h|} \;:\; h\in L^1_h(G)\}$$

and

$$B(F) \;=\; \sup\{\frac{|\int_F h|}{\int_{G\setminus F}|h|} \;:\; h\in L^1_h(G)\}.$$

Theorem 5.4. *If $F \subseteq G$ has $B(F) > 1$, and ω in $L^1(G)$ is strictly positive a.e. on F, then ω is not badly approximable.*

Proof. By the harmonic analogue of Theorem 2.2, if ω were badly approximable, then there would be a function g in L^∞ of norm one (*viz.* $sgn(\overline{\omega})$) that annihilated $L^1_h(G)$ and equalled 1 on F. As $B(F) > 1$, there exists h in $L^1_h(G)$ such that $|\int_F h| > \int_{G\setminus F} |h|$. We have

$$\begin{aligned} \left| \int_G h \right| &= \left| \int_G h(1-g) \right| \\ &= \left| \int_{G\setminus F} h - hg \right| \\ &\leq 2 \int_{G\setminus F} |h|. \end{aligned}$$

Now let $\lambda > 0$. Then

$$\begin{aligned} \left| \int_G h \right| &= \left| \int_G h(1+\lambda g) \right| \\ &\geq (\lambda+1) \left| \int_F h \right| - (\lambda+1) \int_{G\setminus F} |h| \end{aligned}$$

so

$$(\lambda+1) \left| \int_F h \right| \leq (\lambda+3) \int_{G\setminus F} |h|.$$

Letting $\lambda \to \infty$ gives

$$\left| \int_F h \right| \leq \int_{G\setminus F} |h|,$$

a contradiction. ◁

Note that Example 3.2 shows, with $F = \mathbf{D}_0$, that $B(F) = 1$ is not a sufficient hypothesis.

A similar argument to the proof of Theorem 5.4 yields the following theorem.

Theorem 5.5. *Suppose $F \subseteq G$ has $A(F) > 1$, so there is some function h in $L^1_h(G)$ such that $\int_F |h| > \int_{G\setminus F} |h|$. If ω in $L^1(G)$ has the property that $\omega\overline{h}$ is strictly positive a.e. on F, then ω is not badly approximable.*

We shall call F a *weak peak set* if $A(F) = \infty$, and a *strong peak set* if $B(F) = \infty$. These sets seem of interest in their own right. A duality argument shows their connection with badly approximable functions and dual interpolation problems.

Proposition 5.6. *The set F is a weak peak set for $L^1_h(G)$ if and only if there is a function g in $L^\infty(F)$ that cannot be extended to a bounded function on G that annihilates $L^1_h(G)$. The set F is a strong peak set for $L^1_h(G)$ if and only if the function that is identically 1 on F cannot be extended to a bounded function on G that annihilates $L^1_h(G)$.*

Proof. F fails to be a weak peak set for $L^1_h(G)$ if and only if there is a constant M such that

$$\int_F |h| dA \ \leq \ M \int_{G\setminus F} |h| dA$$

for all h in $L^1_h(G)$. This implies that if g is any function in $L^\infty(F)$, then there is a function ω_g in $L^\infty(G \setminus F)$ of norm at most $M\|g\|$ such that

$$\int_F hg dA = \int_{G\setminus F} h\omega_g dA.$$

But then $g\chi_F - \omega_g\chi_{G\setminus F}$ is an extension of g that annihilates $L^1_h(G)$. As the reasoning is reversible, this proves the characterization of weak peak sets.

Similarly, F fails to be a strong peak set for $L^1_h(G)$ if and only if there is a constant M such that

$$|\int_F h dA| \ \leq \ M \int_{G\setminus F} |h| dA$$

for all h in $L^1_h(G)$. But this implies that there is a function ω of norm at most M so that

$$\int_F h dA = \int_{G\setminus F} h\omega dA,$$

and so $\chi_F - \omega\chi_{G\setminus F}$ annihilates $L^1_h(G)$. Again the argument is reversible. $\triangleleft$

Now we turn to geometric characterizations of peak sets, motivated by the previous results and Theorem 3.6.

Theorem 5.7. *Suppose G is a bounded domain in $\mathbf{R}^n$ and the boundary of G contains an isolated $(n-1)$-dimensional manifold J which is also in the boundary of $\mathbf{R}^n \setminus \overline{G}$. Then every full neighborhood of a point in J is a weak peak set for $L^1_h(G)$.*

Proof. It is easily shown that there is a point y in G such that a closest point in ∂G to y lies in J. Let z be a point in ∂G that is closest to y. Note that the ball centered at y of radius $|y-z|$ is contained in G.

Let N be the intersection of an open set in $\mathbf{R}^n$ containing z with G. Let u be a harmonic function on $\mathbf{R}^n \setminus \{0\}$ with a non-integrable singularity at 0, such that u is not integrable over any ball with 0 in the boundary (*e.g.* let u be an appropriate partial derivative of the Newton kernel). Let z_j be a sequence in $\mathbf{R}^n \setminus \overline{G}$ that converges to z. Let $u_j(x) := u(x - z_j)$. Then $\int_N |u_j|$ tends to infinity, while $\int_{G \setminus N} |u_j|$ stays bounded. $\triangleleft$

Lemma 5.8. *Let G be a bounded domain in $\mathbf{R}^n$, and suppose F is a weak peak set for $L^1_h(G)$. Then for all $c > 0$, $F_c := F \cap \{x \in G : dist(x, \partial G) < c\}$ is also a weak peak set.*

Proof. In view of the proof of Theorem 5.7, we can assume that $G \setminus F$ has a subset E of positive measure and with $cl(E) \subseteq G$.

As F is weak peak, there is a sequence h_j in $L^1_h(G)$, each function having norm one, and $\int_F |h_j|$ tending to 1 as $j \to \infty$. Passing to a subsequence if necessary, we can assume that h_j converges uniformly on compact subsets of G to a harmonic function h. As $\int_E |h_j| \to 0$, it follows that $h = 0$ on E and therefore on all of G. Therefore h_j tends to zero uniformly on compact subsets of G, and in particular on $F \setminus F_c$. So $\int_{F_c} |h_j| \to 1$, as desired. $\triangleleft$

It is possible for a set F to touch the boundary but not be a weak peak set, provided it is very thin near the boundary.

Theorem 5.9. *Suppose G is a bounded domain in $\mathbf{R}^n$, and $F \subseteq G$ satisfies*

$$\int_F \frac{1}{(dist(z, \partial G))^n} dA < \infty.$$

Then F is not a weak peak set for $L^1_h(G)$.

Proof. Let c_n be the volume of the unit ball in $\mathbf{R}^n$. For some $c > 0$, the set F_c satisfies

$$\int_{F_c} \frac{1}{(dist(z, \partial G))^n} dA < \frac{c_n}{2}.$$

By Lemma 5.8, it is sufficient to prove that F_c is not a weak peak set. Now suppose h is in $L^1_h(G)$. Then by the mean value property for harmonic functions,

$$|h(z)| \;\le\; \frac{1}{c_n(dist(z,\partial G)^n}\int_G |h|dA$$

for all z in G. Therefore

$$\int_{F_c} |h|dA \;\le\; \frac{1}{2}\int_G |h|dA$$

so

$$\int_{F_c} |h|dA \;\le\; \int_{G\setminus F_c} |h|dA. \qquad \triangleleft$$

Characterizing strong harmonic peak sets is more subtle. To determine whether a subset of the ball is a strong harmonic peak set, the center is of crucial importance. Let $\mathbf{B}$ denote the unit ball in $\mathbf{R}^n$, and recall that c_n is its volume.

Theorem 5.10. *Let $F \subseteq \mathbf{B}$.*

(i) If 0 is not in $\overline{F}$, then F is not a strong peak set for $L^1_h(\mathbf{B})$.

(ii) If 0 is in $\overline{F}$, F is open and connected, and in addition ∂F contains a relatively open subset of $\partial\mathbf{B}$, then F is a strong peak set for $L^1_h(\mathbf{B})$.

Proof. (i) Suppose first that F omits $\mathbf{B}(0,r)$, the ball centered at zero of radius $r > 0$. Then for any integrable harmonic function h

$$\int_F h \;=\; c_n h(0) - \int_{\mathbf{B}\setminus F} h.$$

Therefore

$$\begin{aligned}\left|\int_F h\right| &\le \frac{1}{r^n}\left|\int_{\mathbf{B}(0,r)} h\right| + \left|\int_{\mathbf{B}\setminus F} h\right| \\ &\le \frac{r^n+1}{r^n}\int_{\mathbf{B}\setminus F} |h|,\end{aligned}$$

so F cannot be a strong harmonic peak set.

(ii) If F is a domain that contains an open subset J of the unit sphere in its boundary, and if F is not a strong harmonic peak set, let ψ be a function in $L^\infty(\mathbf{B})$ that annihilates $L^1_h(\mathbf{B})$ and equals 1 on F.

Claim: There is a C^1 function u on $\mathbf{R}^n$ satisfying

$$\Delta u = \psi, \quad u = 0 = \frac{\partial u}{\partial n} \text{ on } J. \tag{5.3}$$

Proof of claim: Let E be the fundamental solution of the Laplacian in $\mathbf{R}^n$, and define u by

$$u \;=\; E * \psi.$$

Then $\Delta u = \psi$ and u is C^1 by elliptic regularity [GT]. Moreover, because for $\xi \in \mathbf{R}^n \setminus \overline{\mathbf{B}}$ the function $z \mapsto E(\xi - z)$ is harmonic on $\mathbf{B}$, it follows from the fact that ψ annihilates $L^1_h(\mathbf{B})$ that $u \equiv 0$ off $\overline{\mathbf{B}}$. As u is C^1, it follows that u and its first order partials vanish on J.

Let v be the modified Schwarz potential of $\partial\mathbf{B}$, *i.e.* the function satisfying

$$\Delta v = 1 \text{ on } U, \quad v = 0 = \frac{\partial v}{\partial n} \text{ on } \partial\mathbf{B}, \tag{5.4}$$

where U is the largest neighbourhood of $\partial\mathbf{B}$ on which (5.4) has a solution (in this case it is $\mathbf{R}^n \setminus \{0\}$). As Δu and Δv agree on $F \setminus \{0\}$ and u and v vanish along with their gradients on J, we must have $u \equiv v$ in F. By direct calculation (or see [Kh1] or [Sh1]), for $n = 2$ we have

$$v(z) \;=\frac{1}{4}(|z|^2 - 1) - \frac{1}{2}\log|z|$$

and for $n \geq 3$ we have

$$v(z) \;= \frac{1}{2n}|z|^2 + \frac{1}{n(n-2)}\frac{1}{|z|^{n-2}} - \frac{1}{2(n-2)}.$$

As v has a non-removable singularity at 0 and u is bounded, 0 cannot be in $\overline{F}$. $\triangleleft$

For $n = 2$, it suffices in *(ii)* for $\partial F \cap \partial\mathbf{D}$ to have positive measure - cf. Remark (iv) after Theorem 3.6.

For an ellipse, the crucial points are the foci. A domain has to join only one of these to an arc on the boundary in order to be a strong harmonic peak set.

Theorem 5.11. *Let* $\mathbf{E}$ *be an ellipse with foci* ± 1*, and let* $F \subset \mathbf{E}$*.*

(i) If there exists a connected open set U containing both foci that is disjoint from F, then F is not a strong peak set for $L^1_h(\mathbf{E})$.

(ii) If F is an open connected set, ∂F contains an arc I of $\partial\mathbf{E}$, and one of the foci of $\mathbf{E}$ is in F, then F is a strong peak set for $L^1_h(\mathbf{E})$.

Proof. (i) By [Sh1, p.21], there is a bounded function w on U such that

$$\int_{\mathbf{E}} hdA \;=\int_U whdA$$

for all h in $L^1_h(\mathbf{E})$. So just as in the proof of the first half of Theorem 5.10, we get

$$|\int_F hdA| \;\leq\; (\|w\|+1)\int_{\mathbf{E}\setminus F} |h|dA.$$

(ii) If F is not a strong peak set for $L^1_h(\mathbf{E})$, as in Theorem 5.10 we can find a function $u \in C^1(\mathbf{R}^2)$ that has $\Delta u = 1$ on F and vanishes along with its gradient on I. Therefore it coincides with the modified Schwarz potential v of $\partial\mathbf{E}$ on F. But $\frac{\partial v}{\partial z}$ has square root type branch points at ± 1 [Sh1,p.21], so v is not C^1 in any neighborhood of a focus. ◁

Remark: The preceding theorem and proof remain valid for ellipsoids in $\mathbf{R}^n$, where the pair of foci are replaced by the $(n-1)$-dimensional focal ellipsoid (or caustic). See [Kh1] and [Sh1].

Interestingly, *any* neighborhood of a rough boundary point is automatically a strong harmonic peak set.

Theorem 5.12. *Let G be a domain in $\mathbf{R}^n$ and F an open subset of G such that $\partial F \cap \partial G \cap \partial \overline{G}^c$ contains an $(n-1)$-dimensional manifold J. If F fails to be a strong $L^1_h(G)$ peak set, then there is a function u, in $C^{2-\varepsilon}(\mathbf{R}^n)$ for all $\varepsilon > 0$, such that u and ∇u vanish on J, but u is not identically zero in a neighborhood of any point on J. Moreover, if $n = 2$, and J is a Jordan arc, then J must actually be an analytic arc.*

Proof. As in the proof of Theorem 5.10, if F is not a strong peak set, there is a function ψ in $L^\infty(G)$ that annihilates $L^1_h(G)$ and equals one on F. Then $u = E * (\psi)$ satisfies equation (5.3).

In the case $n = 2$, it follows from [Sh1,p.39] that the existence of u satisfying (5.3) forces J to be an analytic arc. $\triangleleft$

Example 5.13: If G is a square, it follows from Theorem 5.12 that any neighborhood of a corner is a strong $L^1_h(G)$ peak set. More is true: if F is a ribbon connecting two different sides (though maybe missing the corner), then it is still strong peak. This is because a u satisfying equation (5.3) would actually be uniquely determined by knowing it vanished along with its derivative on an arc of one side of the square - it would have to be the modified Schwarz potential of a half-plane. But it would also have to be the modified Schwarz potential of another half-plane, corresponding to the other side that F touches. These two functions are different, and cannot agree on any open set.

However, if F is a large set that only touches one side, it will not be a strong peak set. For there is a C^∞ function v on $\mathbf{R}^n$, identically 1 on a neighborhood of F, and identically zero on a neighborhood of the three sides that F doesn't touch. Let u be the modified Schwarz potential of the side F does touch. Then it follows from Green's theorem that $f = \Delta(uv)$ annihilates $L^1_h(G)$; moreover f is 1 on F and in L^∞, so F can not be a strong peak set.

Clearly the ideas in Example 5.13 could be extended to other domains.

6. A Proof of the AGHR Theorem

Our methods allow us to give new proofs of the results of Armitage, Gardiner, Haussmann and Rogge [AGHR].

Let $\rho = \rho_n = 2^{-1/n}$, and let $\mathbf{B}_0$ be the open ball centered at zero of radius ρ (so it has exactly half the volume of $\mathbf{B}$). Let σ be the function that is -1 on $\mathbf{B}_0$, $+1$ on $\mathbf{B} \setminus \mathbf{B}_0$, and 0 off $\mathbf{B}$.

For $n \geq 2$, let $\mathcal{L}$ be the differential operator on $\mathbf{R}^n$ given by

$$\mathcal{L}(f) \;=\; \sum_{j=1}^{n} x_j \frac{\partial f}{\partial x_j} + \frac{n-2}{2} f.$$

First we prove the following Lemma.

Lemma 6.1. *Suppose g is in $L^\infty(\mathbf{B})$ and $\|g\| \leq 1$. If $n = 2$, suppose also that $\int_{\mathbf{D}} g = 0$. Then for all y in $\mathbf{B}$ with $|y| = \rho$, we have*

$$|\mathcal{L}_y[E * g(y)]| \leq |\mathcal{L}_y[E * \sigma(y)]|,$$

with strict inequality unless g is, almost everywhere, a unimodular constant times σ.

Proof. First assume $n \geq 3$. Then a calculation yields that

$$\mathcal{L}_y |x-y|^{2-n} \;=\; \left(\frac{n-2}{2}\right) \frac{|x|^2 - |y|^2}{|x-y|^n}.$$

Therefore, as $E(x-y) = c|x-y|^{2-n}$ for the appropriate constant $c = c(n)$, we get that

$$\mathcal{L}_y \; E * g(y) \;=\; \frac{n-2}{2}\, c \int_B \frac{|x|^2 - |y|^2}{|x-y|^n} g(x)dx. \tag{6.1}$$

For $|y| = \rho$, the right-hand side of (6.1) is maximized if

$$g(x) \;=\; sgn\left(\frac{|x|^2 - |y|^2}{|x-y|^n}\right) \;=\; \sigma(x).$$

Moreover, there will be cancellation in the integral in (6.1) unless g is a unimodular constant times σ.

Now consider the case $n = 2$. A calculation gives

$$\mathcal{L}_y[\log|y-x|^2] - 1 \;=\; \frac{|y|^2 - |x|^2}{|y-x|^2}.$$

Therefore

$$\begin{aligned}
\mathcal{L}_y \; E * g(y) \;&=\; c \int_{\mathbf{D}} \mathcal{L}_y \log|y-x|^2 g(x)dx \\
&=\; c \int_{\mathbf{D}} \left[\mathcal{L}_y \log|y-x|^2 - 1\right] g(x)dx \\
&=\; c \int_{\mathbf{D}} \left[\frac{|y|^2 - |x|^2}{|y-x|^2}\right] g(x)dx
\end{aligned}$$

As before, this will be maximized when $|y| = \rho$ by $g(x) = \sigma(x)$. $\triangleleft$

Now we can prove Proposition 2 from [AGHR].

Theorem 6.2. *Let $F \subseteq \mathbf{B}$, and assume F is open and connected, and ∂F contains a relatively open subset of $\partial\mathbf{B}$. Suppose g annihilates $L^1_h(\mathbf{B})$, $\|g\|_\infty = 1$ and $g \equiv 1$ on F. Then F has empty intersection with $\mathbf{B}_0$.*

Proof. Let $\Omega = F \cap (\mathbf{B} \setminus \overline{\mathbf{B}_0})$. Then both $E * g$ and $E * \sigma$ have Laplacian 1 on Ω, and vanish along with their gradients on $\partial\Omega \cap \partial\mathbf{B}$ (since they both vanish identically off $\mathbf{B}$). Therefore they agree on Ω, and in particular $\mathcal{L}(E * g) = \mathcal{L}(E * \sigma)$ on Ω. If $\partial\Omega \cap \partial\mathbf{B}_0$ is non-empty, then Lemma 6.1 forces g to equal σ. $\triangleleft$

Note that in dimension 2, one only needs $F \cap \partial\mathbf{D}$ to have positive measure.

In the terminology of Section 5.2, Theorem 6.2 says that if F is a domain containing a full neighborhood of a point of ∂B, then $B(F) \leq 1$ if and only if $F \cap \mathbf{B}_0$ is empty.

We need the following result for the case that G is the ball and K the center point. As we think it may be useful in other cases, we give it in greater generality. Note that hypothesis (6.3) will be satisfied if, for example, the capacity of K is zero and μ is positive.

Proposition 6.3. *Suppose G is a domain in $\mathbf{R}^n$, with piecewise smooth boundary, that satisfies a quadrature identity*

$$\int_G h(x)dx \;=\; \int_K h(x)d\mu(x) \tag{6.2}$$

for all h in $L^1_h(G)$, where μ is a signed measure supported on K, and K has $(n-1)$-dimensional Hausdorff measure zero. Let $U^\mu = E * \mu$ be the Newtonian potential of μ, and assume

$$\limsup_{\mathbf{R}^n \setminus K \,\ni\, x \to y} \left[|U^\mu(x)| + |\nabla U^\mu(x)| \right] \;=\; \infty \qquad \forall y \in K. \tag{6.3}$$

Let ω be continuous on $\overline{G}$ and subharmonic on G, and assume that it is badly approximable in $L^1_h(G)$. Then if ω is non-negative on K, it is non-negative on G.

Proof. As ω is badly approximable, there is a function g in the ball of $L^\infty(G)$ that agrees with $sgn(\omega)$ when $\omega \neq 0$ and that annihilates $L^1_h(G)$; let us extend this function to be 0 off G, and denote the new function also by g. Let

$$u \;= E * (\chi_G - \mu)$$

$$v \;= E * g$$

Then v is C^1 on $\mathbf{R}^n$, u is C^1 on $\mathbf{R}^n \setminus K$, and both vanish identically off G.

Note first that if P_0 is any component of $P := \{\omega > 0\}$, then by subharmonicity and continuity of ω we must have that ∂P_0 contains a relatively open subset of ∂G. As u and v agree outside G, it follows from Holmgren's theorem (which asserts that if a harmonic function and its gradient both vanish on an $(n-1)$-dimensional manifold, then the function must be identically zero) and the fact that K has $(n-1)$-dimensional Hausdorff measure 0 that the function $u - v$, which is harmonic on $P_0 \setminus K$, must vanish identically on $P_0 \setminus K$. As v and $u - U^\mu$ are C^1, it follows from (6.3) that K must be disjoint from $\overline{P}$.

Now let N be a component of $\{\omega < 0\}$. By hypothesis, $K \cap N = \emptyset$. Moreover, as ω is subharmonic, $\partial N \cap G \subseteq \partial P$.

Claim: K is disjoint from $\overline{N}$.

(i) If ∂N contains a relatively open subset of ∂G, then as before the fact that $u + v$ is harmonic on N and zero off G forces it to be zero on N. Therefore (6.3) implies $K \cap \overline{N} = \emptyset$.

(ii) If ∂N does not contain a relatively open subset of ∂G, then $\partial N \cap G$ is dense in ∂N, so $\partial N \subseteq \partial P$, and therefore K is disjoint from $\overline{N}$.

Now consider $\nabla(u - v)$. This is a harmonic vector field on N, continuous on $\overline{N}$. Moreover, it is zero on ∂N (because it is zero on ∂P and ∂G). Therefore on N, the function $u - v$ is constant. As $\Delta(u - v) = 2$, this forces N to be empty. $\triangleleft$

The main result of [AGHR] now follows from Theorem 6.2 and Proposition 6.3.

Corollary 6.4. *Suppose ω is continuous on $\overline{\mathbf{B}}$ and subharmonic on $\mathbf{B}$, and that h is continuous on $\overline{\mathbf{B}}$ and harmonic on $\mathbf{B}$. Then h is a best L^1-approximant to ω if and only if*

(i) $h = \omega$ on $\partial \mathbf{B}_0$, and

(ii) $h \leq \omega$ on $\overline{\mathbf{B}} \setminus \mathbf{B}_0$.

Proof. (Sufficiency) If hypotheses (i) and (ii) hold, then $sgn(\omega - h) = \sigma$ whenever $\omega - h$ is non-zero. As σ annihilates $L^1_h(\mathbf{B})$, it follows that h is a best harmonic approximant of ω.

(Necessity) Conversely, if h is a best harmonic approximant of ω, let $f = \omega - h$. As f is badly approximable, there is a function g of norm 1 in $L^\infty(\mathbf{B})$ that annihilates $L^1_h(\mathbf{B})$ and agrees with $sgn(f)$ whenever f is non-zero.

As f is subharmonic and continuous on $\overline{\mathbf{B}}$, it cannot be strictly positive at any point of $\mathbf{B}_0$ without being positive on a set F which satisfies the hypotheses of Theorem 6.2. So by that theorem, we must have that $f \leq 0$ on $\mathbf{B}_0$.

By the sub-mean value property of subharmonic functions, we must also have

$$\partial\{f < 0\} \cap \mathbf{B} \ \subseteq \ \partial\{f > 0\}.$$

Therefore we must either have that $f < 0$ on $\mathbf{B}_0$, or $f \equiv 0$ on $\mathbf{B}_0$. In the first case, g must equal σ a.e., and (i) and (ii) follow. In the second case, (i) is immediate, and (ii) follows from Proposition 6.3, as $f(0) \geq 0$ forces f to be non-negative on all of $\overline{B}$. ◁

Another consequence of Lemma 6.1 is the following "equigravitational" result, which was suggested to us by Björn Gustafsson.

Corollary 6.5. *Let $K \subseteq \overline{\mathbf{B}}$ be a closed set with volume equal to the volume of $\mathbf{B}_0$, and such that its potential $U_K := E * \chi_K$ agrees outside $\mathbf{B}$ with $U_{\mathbf{B}_0}$. If $K \neq \overline{\mathbf{B}_0}$, then no boundary point y of $\mathbf{B}_0$ can be joined to $\partial\mathbf{B}$ by an arc Γ that is disjoint from $K \setminus \{y\}$.*

Proof. Define g to be -1 on K and $+1$ on $\mathbf{B} \setminus K$. If there were such an arc Γ, it could be thickened to give an open set F which does not meet K except possibly at y. As in the proof of Theorem 6.2, we have $E * g = E * \sigma$ in F, and Lemma 6.1 gives a contradiction. ◁

7. Smooth functions with unbounded best approximants

First we characterize the best harmonic approximant to the Newton kernel with pole in the ball of radius ρ_n^2, where as before $\rho_n = 2^{-1/n}$. For any point y in $\mathbf{R}^n$, let y' be the Kelvin reflection in the sphere $\partial\mathbf{B}_0$, *i.e.* y' is on the same ray through the origin as y and $|y||y'| = \rho_n^2$. We shall continue to use σ to denote the function that is -1 on $\mathbf{B}_0$ and $+1$ on $\mathbf{B} \setminus \mathbf{B}_0$.

Theorem 7.1. *For $n \geq 3$, the best harmonic approximant in $L^1(\mathbf{B}_n)$ of the function $f(x) = \dfrac{1}{|x-y|^{n-2}}$ when $|y| \leq \rho_n^2$ is the function*

$$h(x) \ = \ \left(\frac{\rho_n}{|y|}\right)^{n-2} \frac{1}{|x - y'|^{n-2}}.$$

For $n = 2$, the best $L^1_h(\mathbf{D})$ approximant to $f(x) = \log|x-y|$ for $|y| \le \frac{1}{2}$ is the function

$$h(x) \;=\; \log \sqrt{2}|y||x-y'|$$

When $y = 0$, the best approximants are the constant functions $\dfrac{1}{\rho_n}$ *and* $\log \dfrac{1}{\sqrt{2}}$ *respectively.*

Proof. Let $|y| \le \rho_n^2$. By direct computation,

$$|x-y| \;<\; \frac{|y|}{\rho_n}|x-y'|$$

if and only if x is in $\mathbf{B}_0$. So $sgn(f-h) = -\sigma$ and annihilates $L^1_h(\mathbf{B})$, and therefore h is the best harmonic approximant. $\triangleleft$

Notice that if f is replaced by $\min(f, M)$ for some large constant M, or even by a C^∞ smoothing, the function $sgn(h-f)$ will still be σ, so h will still be the best approximant (if $n = 2$, take the cut-off from below). Letting $|y| = \rho_n^2$, therefore, we get:

Corollary 7.2. *There exists a C^∞-function that is real-analytic in a neighborhood of $\partial\mathbf{B}$ and whose best harmonic approximant is unbounded on $\mathbf{B}$.*

This is in marked contrast with the behaviour in L^2:

Theorem 7.3. *If G is a domain in $\mathbf{R}^n$ with smooth boundary that is real-analytic near the boundary point x_0, and f in $L^2(G)$ extends real-analytically across x_0, then its best approximant in $L^2_h(G)$ also extends real-analytically across x_0.*

Proof. Let u be the orthogonal projection of f onto $L^2_h(G)$, so

$$f \;=\; u + g$$

where g is in $L^2(G)$ and annihilates $L^2_h(G)$. By the harmonic analogue of Khavin's Lemma, there is v in $W^{2,2}_0(G)$ with $\Delta v = g$ in G.

As f extends real-analytically across x_0, and denoting the extension also by f, there is, in some small ball B centered at x_0, a solution to the Cauchy problem

$$\Delta w = f, \quad w = 0 = \nabla w \text{ on } \partial G \cap B.$$

Let $\Omega = G \cap B$. Then on Ω, we have $\Delta w = u + \Delta v$, so $\Delta\Delta(w - v) = 0$. Thus, $w - v$ satisfies the biharmonic equation in Ω, and vanishes along with its gradient on $\partial\Omega \cap \partial G$ (*i.e.* a trace of the function in $W^{2,2}(\Omega)$ does).

As $\partial\Omega \cap \partial G$ is is real-analytic near x_0, by "regularity up to the boundary" theorems for elliptic operators [F, p.205] we get that $w - v$ extends real-analytically across x_0, and so therefore does v. Thus we get that $u = f - \Delta v$ extends real-analytically across x_0. $\triangleleft$

Another corollary to Theorem 7.1 is the following:

Corollary 7.4. *If $\|g\|_\infty \leq 1$ and g annihilates $L^1_h(G)$, then*

$$|E * g(y)| \leq |E * \sigma(y)|, \quad |y| \leq \rho_n^2,$$

with strict inequality unless g equals a.e. a unimodular constant times σ. Moreover, ρ_n^2 is the largest number for which this is true.

Proof. For simplicity, we give the proof in the case $n \geq 3$; the case $n = 2$ is similar. Let h_y be the best harmonic approximant to $\dfrac{1}{|x-y|^{2-n}}$. For $|y| \leq \rho_n^2$, we have

$$\begin{aligned}
|E * g(y)| &= c\left|\int_{\mathbf{B}} \left[\frac{1}{|x-y|^{2-n}} - h_y(x)\right] g(x)dx\right| \\
&\leq c\int_{\mathbf{B}} \left|\frac{1}{|x-y|^{2-n}} - h_y(x)\right| dx \\
&= c\int_{\mathbf{B}} \left[h_y(x) - \frac{1}{|x-y|^{2-n}}\right] \sigma(x)dx \\
&= |E * \sigma(y)|
\end{aligned}$$

Clearly equality requires g to be a constant times σ.

Now, if $|y| > \rho_n^2$, we cannot have

$$sgn\left([h_y(x) - \frac{1}{|x-y|^{2-n}}] \cdot \sigma(x)\right)$$

constant a.e. For this would force $h_y(x) - \dfrac{1}{|x-y|^{2-n}}$ to vanish on $\partial\mathbf{B}_0$. If $\rho_n^2 < |y| \leq \rho_n$, this would force h_y to have a pole at y' which is inside $\mathbf{B}$; and if $\rho_n < |y| < 1$, this would force h_y to have a pole at y.

So if $s(x) = sgn(h_y(x) - \dfrac{1}{|x-y|^{2-n}})$, then $|E * s(y)|$ will be strictly larger than $|E * \sigma(y)|$. $\triangleleft$

Similarly we have

Corollary 7.5. *Let $K \subseteq \overline{\mathbf{B}}$ be a closed set with volume equal to the volume of $\mathbf{B}_0$, and such that its potential $U_K := E * \chi_K$ agrees outside $\mathbf{B}$ with $U_{\mathbf{B}_0}$. If $K \neq \overline{\mathbf{B}_0}$, then $|U_K(y)| < |U_{\mathbf{B}_0}(y)|$ for $|y| \leq \rho_n^2$.*

Let us mention one last consequence of these ideas. Let $y_0 \in \mathbf{B}$, thought of as close to the boundary. Let $h(x)$ be the best harmonic approximant of $E(y_0 - x)$ and $s(x)$ be $sgn[E(y_0 - x) - h(x)]$. Let F be an open connected set such that ∂F contains a relatively open subset of ∂B, and with y_0 in $\overline{F}$. Then if g is in the closed unit ball of $L^\infty(\mathbf{B})$, annihilates $L^1_h(\mathbf{B})$ and equals s on F, then g must equal s a.e. on $\mathbf{B}$. For indeed, $E * g = E * s$ on F, so

$$E * g(y_0) \;=\; \int [E(y_0 - x) - h(x)]g(x) \;=\; \int [E(y_0 - x) - h(x)]s(x).$$

Therefore there is no cancellation in the first integral, and so g must equal s a.e.

In other words, knowledge of g on the (small) set F, along with the fact that g annihilates $L^1_h(\mathbf{B})$ and is of norm 1, uniquely determines it.

In the analytic case, we can construct a continuous function with unbounded best approximant, but have not been able to make ω any smoother:

Proposition 7.6. *There is a function ω that is continuous on the closed disk and whose best analytic approximant in $L^1(\mathbf{D})$ is unbounded near every point of $\partial\mathbf{D}$.*

Proof. Let $f = u + iv$ be a holomorphic function on the unit disk, whose imaginary part is continuous on $\partial\mathbf{D}$ and whose real part is positive and unbounded on $\partial\mathbf{D}$ (*e.g.* the Riemann map onto the set $\{x + iy : x > 1, 0 < y < \frac{1}{x}\}$). By taking a suitable convex combination of rotates of f, we can moreover assume that u is unbounded near every point of $\partial\mathbf{D}$, and that f is in A^1.

Let

$$\omega(z) \;=\; 2(1 - |z|^2)u(z) + iv(z).$$

Then ω is continuous on $\overline{\mathbf{D}}$, because $u(z) = o(\log|1 - z|)$. Moreover,

$$\omega(z) - f(z) \;=\; [1 - 2|z|^2]u(z),$$

which is positive in $\mathbf{D}_0$ and negative outside $\mathbf{D}_0$. Therefore f is the best analytic aproximant to ω. ◁

Question If ω is Hölder continuous on $\overline{\mathbf{D}}$ must its best A^1 approximant be continuous on $\overline{\mathbf{D}}$?

References

[**Ak**] **:** N.I. Akhiezer, *Theory of Approximation* , (Translated by C. Hyman) Unger, New York, 1956.

[**AGHR**] **:** D.H. Armitage, S.J. Gardiner, W. Haussmann, and L. Rogge, Characterization of best harmonic and superharmonic L^1-approximants, *J. Reine Angew. Math.* **478**(1996), 1-15.

[**BW**] **:** J. Bourgain and T. Wolff, Note on gradients of harmonic functions in dimension ≥ 3, *Colloq. Math.* **60/61** 1 (1990), 253-260.

[**CJ**] **:** L. Carleson and S. Jacobs, Best uniform approximation by analytic functions, *Ark. Mat.* **10**(1972), 219–229.

[**D**] **:** P. Duren, *The Theory of H^p-Spaces*, Academic Press, New York, 1980.

[**DKSS**] **:** P. Duren, D. Khavinson, H.S. Shapiro, and C. Sundberg, Contractive zero divisors in Bergman spaces, *Pacific J. Math.*, **157**(1993), 37–56.

[**EG**] **:** L.C. Evans and R.F. Gariepy, *Measure theory and fine properties of functions*, CRC Press, Boca Raton, 1992.

[**F**] **:** A. Friedman *Partial Differential Equations*, Holt, Rinehrat and Winston, New York, 1969.

[**GK**] **:** T. Gamelin and D. Khavinson, The isoperimetric inequality and rational approximation, *Amer. Math. Monthly*, **96** (1989) No. 1, 18–30.

[**GHJ**] **:** M. Goldstein, W. Haussmann, and K. Jetter, Best harmonic L^1-approximation to subharmonic functions, *J. London Math. Soc.* (2), **30**(1984), 257–264.

[GT] : D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order,* Springer, Berlin, 1977.

[HKL] : W.K. Hayman, D. Kershaw, and T.J. Lyons, The best harmonic approximation to a continuous function, Proceedings of the Conference on Functional Analysis and Approximation, Oberwolfach 1983, *International Series on Numerical Mathematics* **65**(1984), (eds. P.L. Butzer, B.Sz.-Nagy, and R. Stens) Birkhäuser, Basel, 317–327.

[HS] : E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, Springer–Verlag, New York, 1965.

[Ka] : J.-P. Kahane, Best approximation in $L^1(\mathbf{T})$, *Bull. Amer. Math. Soc.*, **80**(1974), 788–804.

[Kh1] : D. Khavinson *Holomorphic Partial Differential Equations and Classical Potential Theory,* Universidad de la Laguna, 1996.

[Kh2] : S.Ya. Khavinson, Foundations of the theory of extremal problems for bounded analytic functions and various generalizations of them, *Amer. Math. Soc. Transl.* (2), Vol.**129**(1986), 1–56.

[Kh3] : S.Ya. Khavinson, Foundations of the theory of extremal problems for bounded analytic functions with additional conditions, *Amer. Math. Soc. Transl.* (2), Vol. **129**(1986), 64–114.

[Kh4] : S.Ya. Khavinson, Duality relations and criteria for elements giving the best approximation, *Lecture Notes*, Moscow Institute of Civil Engineering, Moscow, 1976, 1–46 (in Russian).

[Kh5] : S.Ya. Khavinson, Chebyshev's ideas in the theory of best approximation, II, III, *Lecture Notes*, Moscow Institute of Civil Engineering, Moscow, 1977, 1–20 and 1–21 (in Russian), (ed. by E.Sh. Chatskaya).
(Part II: Completeness of Systems.)
(Part III: Uniqueness and non-uniqueness of best approximations.)

[Kh6] : S.Ya. Khavinson, On uniqueness of the function giving the best approximation in the metric of the space L^1, *Izv. Akad. Nauk SSSR*, ser. matem. **22**(1958), 243–270 (in Russian).

[KP-GS] : D. Khavinson, F. Peréz-González, and H.S. Shapiro, Approximation in L^1-norm by elements of a uniform algebra, *Constructive Approximation,* to appear

[KS] : D. Khavinson and M. Stessin, Certain linear extremal problems in Bergman spaces of analytic functions, *Indiana Math. J.,* **46** (1997), 933–973

[RS] : W.W. Rogosinski and H.S. Shapiro, On certain extremum problems for analytic functions, *Acta Math.* **90**(1953), 287–318.

[R] : V.G. Ryabych, Extremal problems for summable analytic functions, *Siberian Math. J.* **XXVIII**(1986), No.3, 212–217, (in Russian).

[Sh1] : H.S. Shapiro, *The Schwarz Function and its Generalizations to Higher Dimensions*, Wiley, New York, 1991.

[Sh2] : H.S. Shapiro, Regularity properties of the element of closest approximation, *Trans. Amer. Math. Soc.*, **181**(1973), 127–142.

[V] : D. Vukotić, Linear extremal problems for Bergman spaces, *Exposition. Math.*, **14** (1996) No. 4, 313–352.

[W] : J. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, American Mathematical Society, Providence, Rhode Island, 1960.

D. Khavinson
Department of Mathematics
University of Arkansas
Fayetteville, Arkansas 72701
U.S.A.
e-mail: dmitry@comp.uark.edu

J.E. McCarthy
Department of Mathematics
Washington University
Saint Louis, Missouri 63130
U.S.A.
e-mail: mccarthy@math.wustl.edu

H.S. Shapiro
Department of Mathematics
Royal Institute of Technology
Stockholm, S-10044
SWEDEN
e-mail: shapiro@math.kth.se