BOUNDED POINT EVALUATIONS ON THE BOUNDARIES OF L REGIONS

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Abstract. An *L region* is a region G obtained by deleting from the open unit disc D a sequence of disjoint closed discs $\overline{\Delta(a_n,\ r_n)}$, whose centers a_n lie on the positive real axis and decrease monotonically to zero. We consider under what conditions $\lim_{x\to 0^-} f(x)$ exists for every function f in some space of analytic functions on G. We obtain the following results.
1. Every function in $R^p(|z|^t dm(z))$, where m denotes arc-length measure and $1 < p < \infty$, has a limit from the left at zero if and only if

$$\sum_{n=1}^{\infty} \frac{r_n}{a_n^{q+t(q-1)}} \ < \ \infty, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

2. If $a_n \ = \ a^n$, and ω is harmonic measure for G, then every function in $H^p(G)$, $1 < p < \infty$, has a limit from the left at zero if and only if

$$\sum_{n=1}^{\infty} \frac{r_n^q}{a^{qn}\omega(\Gamma_n)^{q-1}} \ < \ \infty.$$

3. Under certain conditions on a_n, (*e.g.* $a_n = a^n$ or $a_n = \frac{1}{n^k}$, for $k > \frac{p-1}{2}$) every function in $L^p_a(G)$ has a limit from the left at zero if and only if

$$\sum_{n=0}^{\infty} \frac{1}{a_n^2 log\frac{1}{r_n}} \ < \ \infty, \quad p = 2 \quad \text{or} \quad \sum_{n=0}^{\infty} \frac{r_n^{2-q}}{a_n^q} \ < \ \infty, \quad p > 2$$

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0. Introduction

An L *region* is a region G obtained by deleting from the open unit disc D a sequence of disjoint closed discs $\overline{\Delta(a_n,\ r_n)}$, whose centers a_n lie on the positive real axis and decrease monotonically to zero. These regions were first studied by L. Zalcman in [15], where he proved that every bounded analytic function f on G has a left limit at zero (*i.e.* $\lim_{x\to 0^-} f(x)$ exists) if and only if $\sum_{n=1}^{\infty} \frac{r_n}{a_n} < \infty$. In this article we consider under what conditions every element of other spaces of analytic functions on G, such as Bergman and Hardy spaces (see below for definitions), has a left limit at zero. The problem is related to finding finite codimensional invariant subspaces of those spaces (see, for example, [1] and [2]).

For μ a finite positive Borel measure supported on $\overline{G}$, let $R^p(\overline{G},\ \mu)$ denote the $L^p(\mu)$ closure of rational functions with poles off $\overline{G}$. The Hardy space $H^p(G)$ consists of all analytic functions $f:\ G \to \mathbb{C}$ such that $|f|^p$ has a harmonic majorant on G (see [3, p.203], [5, Chapter 3] and [13]). The Bergman space $L^p_a(G)$ is the space of analytic functions on G whose p^{th} powers are integrable with respect to area measure on G.

If X is one of the above spaces, $R^p(\overline{G},\ \mu)$, $H^p(G)$ or $L^p_a(G)$, we shall say zero is a bounded point evaluation (bpe) if the closure of zX does not equal X. Clearly, if every function in X has a left limit at zero, then zero is a bpe; the converse is valid if functions analytic in a neighborhood of zero are dense in X. We shall say zero is a bounded point derivation of order k if $clos(z^{k+1}X) \neq clos(z^k X)$.

R. Olin [11] (also see [12]) showed that there is an L region G and a measure μ on the boundary of G such that $R^p(\overline{G},\ \mu)$ is irreducible and zero is a bpe for $R^p(\overline{G},\ \mu)$. S. Axler and P. Bourdon [2] gave an example of an L region G such that zero is a bpe for the Bergman space $L^p_a(G)$. In both these cases, every function in the space has a left limit at zero.

Throughout this paper, we shall use A to denote area measure, m to denote arc-length measure, q to denote the conjugate index of p ($q = \frac{p}{p-1}$), and Γ_n to denote the boundary of $\Delta(a_n,\ r_n)$. We shall let c denote a positive constant, that may change from one appearance to the next.

The paper will be organized as follows. In section 1, we assume that the measure μ is a weighted arclength measure on the boundary of G, of the form $|z|^t dm(z)$. We prove (Theorem 1.5) that the following are equivalent for $1 < p < \infty$:

(a1) Zero is a bpe for $R^p(\overline{G},\ \mu)$;

(b1) Every function in $R^p(\overline{G},\ \mu)$ has a left limit at zero;

(c1)

$$\int_{\partial G} \frac{dm(z)}{|z|^{q+t(q-1)}} \ < \ \infty$$

(d1)

$$\sum_{n=1}^{\infty} \frac{r_n}{a_n^{q+t(q-1)}} \ < \ \infty.$$

Section 2 obtains some results for the Hardy spaces $H^p(G)$, under the assumption that $a_n = a^n$, for some a between 0 and 1. We prove (Theorem 2.10) that the following conditions are equivalent ($1 < p < \infty$):

(b2) Every function in $H^p(G)$ has a left limit at zero;

(c2)

$$\int_{\partial G} \frac{dm(z)}{|z|^q (\frac{d\omega}{dm(z)})^{q-1}} \ < \ \infty$$

(d2)

$$\sum_{n=1}^{\infty} \frac{r_n^q}{a^{qn}\omega(\Gamma_n)^{q-1}} \ < \ \infty.$$

We discuss the same problem for Bergman spaces. In section 3 we give necessary and sufficient conditions for a point to be a bpe for the Bergman space in terms of the Bessel capacity, which are very similar to the results of Fernström and Polking for $R^p(\overline{G}, A)$ [4]. In Section 4, we look for geometric conditions in terms of a_n and r_n for zero to be a bpe. We must again impose some assumptions on a_n, though they are less strict than those needed in section 2; for $p = 2$, the restriction is that

$$\sum_{j=1}^{\infty} (a_j^2 + a_{j+1}^2) \log \frac{c_j^2 \ - \ c_{j+2}^2}{c_{j+1}^2 \ - \ c_{j+2}^2} < \infty$$

with an analogous condition for $p > 2$. The condition is satisfied if *e.g.* $a_n = a^n$ or $a_n = \frac{1}{n^k}$, for $k > \frac{p-1}{2}$. For these a_n, we are able to prove (Theorems 4.5 and 5.1) that the following statements are equivalent for $p \geq 2$:

(a3) Zero is a bpe for $L^p_a(G)$;

(b3) Every function in $L^p_a(G)$ has a left limit at zero;

(d3) If $p = 2$, then

$$\sum_{n=1}^{\infty} \frac{1}{a_n^2 log \frac{1}{r_n}} < \infty.$$

If $p > 2$, then

$$\sum_{n=1}^{\infty} \frac{r_n^{2-q}}{a_n^q} < \infty.$$

Because $\frac{1}{z}$ is in $L^p_{loc}(A)$ for $p < 2$, it is clear that for these p zero is never a bpe for $L^p_a(G)$.

Our techniques also apply to the question of when the k^{th} derivative of every function in a space has a left limit at zero, and when zero is a bounded point derivation of order k. As the proofs are very similar, we state the theorems to encompass bounded point derivations, though we give proofs only for $k = 0$.

The general condition for bpe's to exist in terms of capacity does not solve the problem in geometric terms. It does simplify a bare hands approach to getting a sufficient condition because of capacity's sub-additivity; but it doesn't seem to help in getting necessary conditions. It would be interesting to know if there were an analogous necessary and sufficient condition for Hardy spaces, which are much harder to localize.

Before closing this section, we state and prove a result that will be used in this article.

Let X be a Banach space and let T be a bounded linear operator on X. Let Ω be a bounded domain and let π be a representation from $H^\infty(\Omega)$ to the set of all bounded linear operators on X such that $\pi(1) = I$ and $\pi(z) = T$.

Theorem 0. Suppose π is weak-* continuous and $\lambda \in \partial\Omega$ is a peak point for $H^\infty(\Omega)$; then $clos(T - \lambda)X = X$.

Proof: As in the proof of Theorem 3.1 in [7], one finds a sequence of functions $\{f_n\} \subset H^\infty(\Omega)$ such that $(z - \lambda)f_n(z)$ goes to 1 pointwise in Ω and $\|(z - \lambda)f_n(z)\|_\infty \leq M < \infty$.

So $(z - \lambda)f_n(z)$ converges to 1 in the weak-* topology. Thus, $\pi((z - \lambda)f_n)$ converges to I weakly. But

$$\pi((z - \lambda)f_n) = (T - \lambda I)\pi(f_n),$$

so $clos(T - \lambda)X = X$.

In our cases, the operator T is multiplication by z; so we may always assume that zero is not a peak point for $H^\infty(G)$, that is,

$$\sum_{n=1}^{\infty} \frac{r_n}{a_n} < \infty$$

(see [15]). So, throughout the paper we shall always make the following assumption:

Assumption (*): $\quad \dfrac{r_n}{a_n} \to 0$ as $n \to \infty$.

1. Rational approximation with respect to weighted arclength

In this section, we shall assume that the measure μ is supported on the boundary of G, and that $R^p(\overline{G},\ \mu)$ is irreducible, *i.e.* there is no partition of ∂G as $E_1 \cup E_2$ such that

$$R^p(\overline{G},\ \mu) = R^p(\overline{G},\ \mu|_{E_1}) \oplus R^p(\overline{G},\ \mu|_{E_2}).$$

Therefore, the measures μ and m are mutually absolutely continuous except at zero. We assume for convenience that $\mu\{0\} = 0$. Let

$$w(z) = \frac{d\mu}{dm}(z).$$

Theorem 1.1. Suppose that $1 < p < \infty$ and

$$\int \frac{dm(z)}{|z|^q w(z)^{q-1}} < \infty;$$

then every function in $R^p(\overline{G},\ \mu)$ has a left limit at zero, and therefore 0 is a bounded point evaluation for $R^p(\overline{G},\ \mu)$.

Proof: Let r be a rational function with poles off $\overline{G}$, $\lambda < 0$ and $|\lambda| < \frac{1}{2}$. Then using the Cauchy integral formula, we have

$$r(\lambda) \ = \ \frac{1}{2\pi i}\int_{\partial G}\frac{r(z)}{z \ - \ \lambda}dz$$

Hence,

$$\begin{aligned}
|r(\lambda)| \ &\le \ \frac{1}{2\pi}\int_{\partial G}\frac{|r(z)|}{|z \ - \ \lambda|}dm(z) \\
&\le \ \frac{1}{2\pi}\left(\int_{\partial G\setminus\partial D}\frac{|r(z)|}{|z \ - \ \lambda|}dm(z) \ + \ \int_{\partial D}\frac{|r(z)|}{|z \ - \ \lambda|}dm(z)\right) \\
&\le \ \frac{1}{2\pi}\left(\int_{\partial G\setminus\partial D}\frac{|r(z)|}{|z|}dm(z) \ + \ 2\int_{\partial D}|r(z)|dm(z)\right) \\
&\le \ \frac{1}{2\pi}\left((\int\frac{dm(z)}{|z|^q w(z)^{q-1}})^{\frac{1}{q}}\|r\|_p \ + \ 2(\int\frac{dm(z)}{w(z)^{q-1}})^{\frac{1}{q}}\|r\|_p\right) \\
&\le \ c\left(\int\frac{dm(z)}{|z|^q w(z)^{q-1}}\right)^{\frac{1}{q}}\|r\|_p
\end{aligned}$$

Thus, there is a constant $M > 0$ such that for every function $f \in R^p(\overline{G}, \ \mu)$, we have

$$|f(\lambda)| \le M\|f\|_p$$

Therefore the functionals of evaluation at λ remain bounded in norm as λ tends to zero, $\Re(\lambda) < 0$. So some sequence converges weakly to a functional, which can be represented by a vector $k \in L^q(\mu)$. Because the rational functions are dense, any other sequence must have the same weak-limit. Therefore

$$\lim_{\lambda\to 0-} f(\lambda) \ = \ \int f\overline{k}d\mu.$$

Hence, every function has a left limit at zero and zero is a bpe. □

In the rest of this section, we are interested in getting necessary conditions. Let K be a compact subset of $\mathbb{C}$; a function f is called *admissible for* K if it is analytic on the component of $\mathbb{C}_\infty \setminus \mathbb{K}$ containing ∞, has modulus at most one there, and vanishes at ∞. The analytic capacity of K is defined to be:

$$\gamma(K) \ = \ \sup\{|f'(\infty)| : \ f \ \text{ is admissible for } K\}$$

Let L be a compact subset of $[0, 1]$,

$$h^L(z) = \frac{1}{2}\int_L \frac{1}{z - x}dx$$

and

$$f^L(z) = \frac{e^{h^L} - 1}{e^{h^L} + 1}.$$

It is proved in [14, p. 15] that

$$|f^L(z)| \leq 1, \quad \text{for } z \notin L$$

and

$$(f^L)'(\infty) = \frac{1}{4}m(L) = \gamma(L)$$

Lemma 1.2. For $z \neq 0$, we have

$$\lim_{L\to 0} \frac{zf^L(z)}{(f^L)'(\infty)} = 1$$

where $L \to 0$ means that $\max\{y : y \in L\} \to 0$.

Proof: Since $\lim_{L\to 0} h^L(z) = 0$, we have

$$\begin{aligned}\lim_{L\to 0} \frac{zf^L(z)}{(f^L)'(\infty)} &= \lim_{L\to 0} \frac{z}{e^{h^L} + 1} \frac{e^{h^L} - 1}{(f^L)'(\infty)} \\ &= \frac{1}{2}\lim_{L\to 0} \frac{e^{h^L} - 1}{h^L(z)} \frac{2zh^L(z)}{(h^L)'(\infty)} \\ &= 1.\square\end{aligned}$$

Let $0 < a < 1$, and let A_j be the annulus given by:

$$A_j = \{z : a^{j+1} \leq |z| < a^j\}.$$

Let $d\mu(z) = |z|^t dm(z)$, and let

$$C_j = \cup_{a_n \in A_j}\Gamma_n.$$

Lemma 1.3. If $1 < p < \infty$ and the integrals

$$\int_{C_j} \frac{dm(z)}{|z|^q |z|^{t(q-1)}}$$

do not converge to zero as j tends to ∞, then $clos(zR^p(\overline{G},\ \mu)) \ = \ R^p(\overline{G},\ \mu)$.

Proof: For simplicity, assume $t \geq 0$ (for $t < 0$, the proof is almost identical). By passing to a subsequence, if necessary, we may assume that there is an $\epsilon > 0$ such that

$$\int_{C_j} \frac{dm(z)}{|z|^q |z|^{t(q-1)}} \ \geq \ \epsilon.$$

Hence,

$$\frac{m(C_j)}{a^{q(j+1)} a^{t(q-1)(j+1)}} \ \geq \ \epsilon.$$

Let $L_j \ = \ \mathbb{R} \cap (\cup_{\Game_{\ltimes} \in \mathbb{A}_{\daleth}} \Delta(\Game_{\ltimes},\ \diagdown_{\ltimes}))$ and

$$f_j(z) \ = \ f^{L_j}(z).$$

We shall show that $\dfrac{zf_j(z)}{f_j'(\infty)}$ remain bounded in norm, so have a weak-cluster point, which by Lemma 1.2 is the function 1. Therefore 1 is in

$$clos(zR^p(\overline{G},\ \mu)),$$

so $clos(zR^p(\overline{G},\ \mu)) \ = \ R^p(\overline{G},\ \mu)$, and 0 is not a bpe.

We first calculate the integral of $|\dfrac{zf_j(z)}{f_j'(\infty)}|^p$ over those C_n such that $|n-j| > 1$.

Note that if $f(z)$ is an admissible function for L, then

$$|f(z)| \leq \frac{\gamma(L)}{d(z,\ L)}$$

for $z \in L^c$ (see [6, p. 201]). Therefore, if $n \neq j-1, j, j+1$, we get that

$$\begin{aligned} &\int_{C_n} \left| \frac{zf_j(z)}{f_j'(\infty)} \right|^p |z|^t |dz| \\ &\leq \frac{1}{\gamma(L_j)^p} a^{pn} \int_{C_n} \frac{\gamma(L_j)^p}{d(z,\ A_j)^p} |z|^t dm(z) \\ &\leq \frac{2a^{pn}}{a^{p(n+1)}(1-a)^p} \mu(C_n), \end{aligned}$$

where we used the assumption $(*)$ that $\dfrac{r_m}{a_m}$ is small. Hence,

$$\int_{\cup_{n\neq j-1,j,j+1}C_n} \left|\frac{zf_j(z)}{f_j'(\infty)}\right|^p |z|^t dm(z) \ \leq \ \frac{2}{a^p(1-a)^p}\mu(\partial G) \ < \ \infty. \tag{I}$$

Next, we calculate the integral over C_j:

$$\begin{aligned}
\int_{C_j} \left|\frac{zf_j(z)}{f_j'(\infty)}\right|^p |z|^t dm(z) &\leq \frac{1}{\gamma(L_j)^p} a^{pj} a^{tj} \int_{C_j} dm(z) \\
&\leq \frac{c}{m(C_j)^p} a^{pj} a^{tj} m(C_j) \\
&\leq c \left(\frac{a^{qj} a^{t(q-1)j}}{m(C_j)}\right)^{p-1} \\
&\leq c \left(\frac{1}{a^q a^{t(q-1)\epsilon}}\right)^{p-1} \\
&\leq c. \qquad\qquad (II)
\end{aligned}$$

To calculate the integral over C_{j+1}, let

$$B \ = \ \{z : \ d(z, \ L_j) < \gamma(L_j)\}$$

and

$$B_k \ = \ \{z : \ 2^k\gamma(L_j) \leq d(z, \ L_j) < 2^{k+1}\gamma(L_j)\},$$

where $k = 0, 1, 2, ...$

$$\begin{aligned}
&\int_{C_{j+1}} \left| \frac{zf_j(z)}{f_j'(\infty)} \right|^p |z|^t dm(z) \\
&= \int_{C_{j+1}\cap B} \left| \frac{zf_j(z)}{f_j'(\infty)} \right|^p |z|^t dm(z) \ + \ \sum_{k=0}^{\infty} \int_{C_{j+1}\cap B_k} \left| \frac{zf_j(z)}{f_j'(\infty)} \right|^p |z|^t dm(z) \\
&\leq \frac{a^{p(j+1)}}{\gamma(L_j)^p} a^{t(j+1)} 2\pi\gamma(L_j) \ + \ \sum_{k=0}^{\infty} a^{p(j+1)} a^{t(j+1)} \int_{C_{j+1}\cap B_k} \left(\frac{1}{d(z,\ L_j)} \right)^p dm(z) \\
&\leq c \frac{a^{p(j+1)} a^{t(j+1)}}{\gamma(L_j)^{p-1}} \ + \ c \sum_{k=1}^{\infty} a^{p(j+1)} a^{t(j+1)} \left(\frac{1}{2^k \gamma(L_j)} \right)^p 2^k \gamma(L_j) \\
&\leq c \frac{a^{p(j+1)} a^{t(j+1)}}{\gamma(L_j)^{p-1}} \\
&\leq c \left(\frac{a^{q(j+1)} a^{t(q-1)(j+1)}}{m(C_j)} \right)^{p-1} \\
&\leq c(\frac{1}{\epsilon})^{p-1} \\
&< \infty. \qquad (III)
\end{aligned}$$

Similarly, we get that

$$\int_{C_{j-1}} \left| \frac{zf_j(z)}{f_j'(\infty)} \right|^p |z|^t dm(z) \ \leq \ c \ < \ \infty. \qquad (IV)$$

Adding I - IV, we get

$$\int_{\partial G} \left| \frac{zf_j(z)}{f_j'(\infty)} \right|^p d\mu \leq c \ < \ \infty,$$

where c does not depend on j; so the functions

$$\frac{zf_j(z)}{f_j'(\infty)}$$

are uniformly bounded, as desired. □

Theorem 1.4. For $1 < p < \infty$,if

$$\int_{\partial G} \frac{dm(z)}{|z|^q |z|^{t(q-1)}} \ = \ \infty$$

then $clos(zR^p(\overline{G},\ \mu)) \ = \ R^p(\overline{G},\ \mu)$.

Proof: We keep the notation of Lemma 1.3, and by its result we may assume that

$$\int_{C_n} \frac{dm(z)}{|z|^q|z|^{t(q-1)}} \to 0.$$

So for each k, choose N_k such that

$$1 \ \leq \ \sum_{n=k}^{N_k} \int_{C_n} \frac{dm(z)}{|z|^q|z|^{t(q-1)}} \ \leq \ 2$$

Let

$$B_j \ = \ A_{j-1} \cup A_j \cup A_{j+1}$$

and

$$g_k(z) \ = \ \sum_{n=k}^{N_k} \int_{C_n} \frac{dm(\zeta)}{|\zeta|^q|\zeta|^{t(q-1)}} \frac{zf_n(z)}{f_n'(\infty)}$$

Then

$$\begin{aligned}
&\int_{C_j} |g_k(z)|^p d\mu(z) \\
&\leq 2^p \int_{C_j} \left| \sum_{a_n \notin B_j} \int_{C_n} \frac{dm(\zeta)}{|\zeta|^q|\zeta|^{t(q-1)}} \frac{zf_n(z)}{f_n'(\infty)} \right|^p d\mu(z) \\
&+ 2^p \int_{C_j} \left| \sum_{n=j-1}^{j+1} \int_{C_n} \frac{dm(\zeta)}{|\zeta|^q|\zeta|^{t(q-1)}} \frac{zf_n(z)}{f_n'(\infty)} \right|^p d\mu(z).
\end{aligned}$$

Denote the first integral by F_j and the second integral by S_j. Using Minkowski's inequality, we have

$$\begin{aligned}
F_j \ &\leq \ \left(\sum_{a_n \notin B_j} \int_{C_n} \frac{dm(\zeta)}{|\zeta|^q|\zeta|^{t(q-1)}} (\int_{C_j} \left| \frac{zf_n(z)}{f_n'(\infty)} \right|^p d\mu)^{\frac{1}{p}} \right)^p \\
&\leq \ \left(\sum_{a_n \notin B_j} \int_{C_n} \frac{dm(\zeta)}{|\zeta|^q|\zeta|^{t(q-1)}} (\int_{C_j} \left| \frac{z}{d(z,\ L_n)} \right|^p d\mu)^{\frac{1}{p}} \right)^p \\
&\leq \ \left(\sum_{a_n \notin B_j} \int_{C_n} \frac{dm(\zeta)}{|\zeta|^q|\zeta|^{t(q-1)}} \right)^p \frac{1}{(1\ -\ a)^p} \mu(C_j) \\
&\leq \ 2^p \frac{1}{(1\ -\ a)^p} \mu(C_j)
\end{aligned}$$

To calculate S_j:

$$\begin{aligned}
&\left(\int_{C_{j+1}} \frac{dm(\zeta)}{|\zeta|^q|\zeta|^{t(q-1)}}\right)^p \int_{C_j} \left|\frac{zf_{j+1}(z)}{f'_{j+1}(\infty)}\right|^p d\mu \\
&\leq c\frac{\gamma(L_{j+1})^p}{a^{pq(j+2)}a^{t(q-1)p(j+2)}}\frac{a^{pj}a^{tj}}{\gamma(L_{j+1})^p}\gamma(L_j) \\
&\leq c\frac{1}{a^{qj}a^{t(q-1)j}}\gamma(L_j) \\
&\leq c\int_{C_j} \frac{dm(z)}{|z|^q|z|^{t(q-1)}}
\end{aligned}$$

A similar computation yields that

$$\left(\int_{C_n} \frac{dm(\zeta)}{|\zeta|^q|\zeta|^{t(q-1)}}\right)^p \int_{C_j} \left|\frac{zf_n(z)}{f'_n(\infty)}\right|^p d\mu \leq c\int_{C_j} \frac{dm(z)}{|z|^q|z|^{t(q-1)}}$$

for $n = j, \ j-1$. Hence,

$$S_j \ \leq \ 3\times 3^p c\int_{C_j} \frac{dm(z)}{|z|^q|z|^{t(q-1)}},$$

and so $\int_{\partial G} |g_k(z)|^p d\mu$ is bounded by some constant independent of k. It follows from Lemma 1.2 that there exists a subsequence $\{g_{k_n}(z)\}$ such that

$$g_{k_n}(z) \to d$$

pointwise except at 0, where d is a constant between 1 and 2. Therefore

$$g_{k_n}(z) \to d$$

weakly, and as $g_{k_n}(z)$ is divisible by z in $R^p(\overline{G}, \ \mu)$, 1 is in $clos(zR^p(\overline{G}, \ \mu))$. Therefore 0 is not a bpe. □

All the above proofs work *mutatis mutandis* if one uses the Cauchy integral formula for $r^{(k)}(\lambda)$; doing this, and combining Theorems 1.1 and 1.4, we get:

Theorem 1.5. Let $d\mu(z) = |z|^t dm(z)$, and $1 < p < \infty$. The following statements are equivalent:

(a1) Zero is a bounded point derivation of order k for $R^p(\overline{G},\ \mu)$.

(b1) The k^{th} derivative of every function in $R^p(\overline{G},\ \mu)$ has a left limit at zero.

(c1)

$$\int \frac{dm(z)}{|z|^{(k+1)q}|z|^{t(q-1)}} < \infty.$$

(d1)

$$\sum_{n=1}^{\infty} \frac{r_n}{a_n^{(k+1)q+(q-1)t}} < \infty.$$

Example 1.6. It follows that, for any $k \leq \infty$, one can choose a_n and r_n so that $closz^{j+1}R^p(\overline{G},\ \mu) \subsetneq closz^j R^p(\overline{G},\ \mu)$ for $j < k$, and, if $\infty > j \geq k$, $closz^j R^p(\overline{G},\ \mu) = closz^k R^p(\overline{G},\ \mu)$.

Example 1.7. Suppose that

$$\sum \frac{r_n}{a_n} < \infty$$

and

$$\sum \frac{r_n}{a_n^2} = \infty.$$

Then 0 is a bpe for $R^2(\overline{G},\ \frac{dm(z)}{|z|})$ and 0 is not a bpe for $R^2(\overline{G},\ dm(z))$. Hence, the operator of multiplication by z on $R^2(\overline{G},\ \frac{dm(z)}{|z|})$ is not quasisimilar to the operator of multiplication by z on $R^2(\overline{G},\ dm(z))$. Therefore, in [9], the hypo-Dirichlet assumption cannot be weakened even to such relatively nice infinitely connected domains as these.

2. Rational approximation with respect to harmonic measure and $H^p(G)$

In this section, we assume that $a_n = a^n$, $n = 1, 2, 3, ...$ for some a between 0 and 1.

Let us first note that there is an equivalent way of defining the Hardy spaces $H^p(G)$ (see [3, p. 205]). Fix a point $b \in G$, let $\omega(G,\ .,\ b)$ be the harmonic measure for G evaluated at b and let $\{G_n\}$ be a sequence of regions contained in G such that $b \in G_1$, $clos(G_n) \subset G_{n+1}$ and $G = \cup_{n=1}^{\infty} G_n$. Let $\omega_n = \omega(G_n,\ .,\ b)$. The Hardy space $H^p(G)$ consists of all analytic functions $f: G \to \mathbb{C}$ such that

$$\sup_n \int_{\partial G_n} |f|^p d\omega_n < \infty.$$

Since $\int_{\partial G_n} |f|^p d\omega_n$ is an increasing sequence, the norm can be defined by

$$\|f\| = \sup_n \int_{\partial G_n} |f|^p d\omega_n = \lim_n \int_{\partial G_n} |f|^p d\omega_n.$$

Let $A(a;\ R_1,\ R_2)$ be the annulus centered at a with radii R_1 and R_2.

Lemma 2.1. Let $\Gamma_1 = \{z: |z| = 1\}$, $\Gamma_2 = \{z: |z| = R_1\}$ and $\Gamma_3 = \{z: |z| = R_2\}$, where $1 < R_1 < R_2$. Let $\Omega = A(0; 1, R_2)$. Then there are two constants $M_0,\ M_1 > 0$ (depending only on $R_1,\ R_2$) such that, for any arc I on Γ_1, and any point z in Γ_2

$$M_0 m(I) \leq \omega(\Omega, I, z) \leq M_1 m(I).$$

Proof: By Harnack's inequality, there are two constants $M_0',\ M_1' > 0$ such that

$$M_0' \omega(\Omega,\ I,\ z') \leq \omega(\Omega,\ I,\ z) \leq M_1' \omega(\Omega,\ I,\ z')$$

for all $z, z' \in \Gamma_2$ (see [5, p. 19]). Let I and J be two arcs of equal length on Γ_1. By symmetry, one sees that

$$M_0' \omega(\Omega,\ J,\ z) \leq \omega(\Omega,\ I,\ z) \leq M_1' \omega(\Omega,\ J,\ z)$$

for all $z \in \Gamma_2$. Therefore, there are $M_0'',\ M_1'' > 0$ such that

$$M_0'' m(I) \omega(\Omega,\ \Gamma_1,\ z) \leq \omega(\Omega,\ I,\ z) \leq M_1'' m(I) \omega(\Omega,\ \Gamma_1,\ z)$$

We know that

$$\omega(\Omega, \Gamma_1, z) = \frac{log \frac{R_2}{R_1}}{log R_2}$$

since $log \frac{R_2}{|z|}$ is a harmonic function on Ω. Let $M_0 = M_0'' \frac{log \frac{R_2}{R_1}}{log R_2}$ and $M_1 = M_1'' \frac{log \frac{R_2}{R_1}}{log R_2}$. The lemma is proved. $\square$

The proof of the following lemma was shown to us by Albert Baernstein II.

Lemma 2.2. Let G be an L region. Let $\Gamma_n^1 = \Gamma_n$, $\Gamma_n^2 = \{z : |z - a_n| = 2r_n\}$ and $\Gamma_n^3 = \{z : |z - a_n| = 4r_n\}$. Suppose $A(a_n; r_n, 4r_n) \subset G$. Fix a point $b \in G \setminus \cup A(a_n; r_n, 4r_n)$. Then there are two constants $M_0,\ M_1 > 0$ such that

$$M_0 \ \le\ \frac{\omega(G,\ I,\ b)}{\omega(G,\ J,\ b)} \ \le\ M_1$$

whenever I and J are arcs of equal length on Γ_n^1.

Proof: Let $\phi(z) = \frac{z-a_n}{r_n}$. Fix a point $z \in \Gamma_n^2$. Using Lemma 2.1 and the fact that harmonic measures are conformally invariant, we conclude that

$$\begin{aligned}
\omega(G,\ I,\ z) &= \omega(\phi(G),\ \phi(I),\ \phi(z)) \\
&\ge \omega(A(0;1,4),\ \phi(I),\ \phi(z)) \\
&\ge M_0 m(\phi(I)) \\
&\ge M_0 \frac{m(I)}{r_n}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\omega(\phi(G),\ \phi(I),\ \phi(z)) &\le \omega((\hat{\Gamma}_n^1)^c, \phi(I),\ \phi(z)) \\
&\le M_1 m(\phi(I)) \\
&= M_1 \frac{m(I)}{r_n}
\end{aligned}$$

where $\hat{K}$ denote the polynomial convex hull of K. Hence, there are two constants $M_0,\ M_1 > 0$ such that

$$M_0 \ \le\ \frac{\omega(G,\ I,\ z)}{\omega(G,\ J,\ z)} \ \le\ M_1$$

for all $z \in \Gamma_n^2$. However,

$$\frac{\omega(G,\ I,\ b)}{\omega(G,\ J,\ b)} = \frac{\int_{\Gamma_n^2} \omega(G,\ I,\ z) d\omega(G \setminus \hat{\Gamma}_n^2,\ .,\ b)}{\int_{\Gamma_n^2} \omega(G,\ J,\ z) d\omega(G \setminus \hat{\Gamma}_n^2,\ .,\ b)}$$

Therefore,

$$M_0 \ \le\ \frac{\omega(G,\ I,\ b)}{\omega(G,\ J,\ b)} \ \le\ M_1$$

□

Theorem 2.3. Let $b \in G$ and $\omega = \omega(G, \ . b)$. Suppose that

$$\int \frac{dm(z)}{|z|^q (\frac{d\omega}{dm(z)})^{q-1}} = \infty$$

Then

$$clos(zR^p(\overline{G}, \omega)) = R^p(\overline{G}, \omega)$$

Proof: From Lemma 2.2, it is easy to show that

$$\int \frac{dm(z)}{|z|^q (\frac{d\omega}{dm(z)})^{q-1}} = \infty$$

if and only if

$$\sum_{k=0}^{\infty} \frac{r_k^q}{a^{qk} \omega(\Gamma_k)^{q-1}} = \infty$$

Let the function $h_j(z)$ be $\frac{z}{z - a^j}$ instead of $\frac{z f_j(z)}{f_j'(\infty)}$ and mimic the proofs of Lemma 1.3 and Theorem 1.4 to conclude that either h_j or some linear combination tends to a non-zero constant, which is therefore in $clos(zR^p(\overline{G}, \omega))$ (the calculations are much less messy in this case, because of the separation of the centers a^k). □

Corollary 2.4. The following statements are equivalent for $1 < p < \infty$:

(a) Zero is a bounded point evaluation for $R^p(\overline{G}, \omega)$;

(b) Every function in $R^p(\overline{G}, \omega)$ has a left limit at zero;

(c)

$$\int \frac{dm(z)}{|z|^q (\frac{d\omega}{dm(z)})^{q-1}} < \infty$$

(d)

$$\sum_{k=0}^{\infty} \frac{r_k^q}{a^{qk} \omega(\Gamma_k)^{q-1}} < \infty$$

Corollary 2.5. Suppose that $1 < p < \infty$ and

$$\int \frac{dm(z)}{|z|^q (\frac{d\omega}{dm(z)})^{q-1}} = \infty$$

Then not every function in $H^p(G)$ has a left limit at zero.

Proof: This follows from the inclusion $R^p(\overline{G},\ \omega) \subseteq H^p(G)$. $\square$

Remark: We would like to prove that 0 is a bpe for $H^p(G)$ if and only if the preceding integral is finite. We cannot do this, however, because do not know whether the functions analytic at zero are dense in $H^p(G)$. The remainder of this section is devoted to proving that if the integral in 2.5 is finite, every function in the space $H^p(G)$ does have a left limit at zero.

Let

$$\Gamma_k^l \;=\; \{z :\ |z \;-\; a^k| \;<\; (1 + \frac{1}{l})r_k\}$$

and

$$G_l \;=\; D \setminus \left(\{z :\ |z| \leq \frac{1+a}{2} a^l\} \cup (\cup_{k=0}^l \hat{\Gamma}_k^l) \right)$$

For definiteness, fix $-\frac{1}{2}$ as the norming point for ω.

Lemma 2.6. Let C_k in G be small circles that surround Γ_k and do not intersect, and let C_0 be a circle centered at 0 with radius slightly less than one. Suppose that

$$\int \frac{dm(z)}{|z|^q (\frac{d\omega}{dm(z)})^{q-1}} \;<\; \infty$$

Then for $-\frac{1}{2} \leq \lambda \;\leq\; 0$,

$$L_\lambda(f) \;=\; \frac{1}{2\pi i} \sum_{k=0}^{\infty} \int_{C_k} \frac{f(z)}{z \;-\; \lambda} dz$$

is a bounded linear functional on $H^p(G)$. Moreover,

$$\lim_{\lambda \to 0-} L_\lambda(f) \;=\; L_0(f)$$

for every $f \in H^p(G)$.

Proof: Let $\omega_l = \omega(G_l, ., -\frac{1}{2})$. We have the following computation:

$$\begin{aligned}&\left|\sum_{k=n}^{m}\frac{1}{2\pi i}\int_{C_k}\frac{f(z)}{z-\lambda}dz\right|\\ \le\ &\sum_{k=n}^{m}\frac{1}{2\pi}\int_{\Gamma_k^l}\left|\frac{f(z)}{z-\lambda}\right|dm(z)\\ \le\ &\frac{1}{2\pi}\left(\sum_{k=n}^{m}\int_{\Gamma_k^l}\frac{dm(z)}{|z|^q(\frac{d\omega_l}{dm(z)})^{q-1}}\right)^{\frac{1}{q}}\|f\|\end{aligned}$$

We know that $\omega_l(\Gamma_k^l) \to \omega(\Gamma_k)$, as $l \to \infty$. Using Lemma 2.2, we see that there exists an absolute constant $c > 0$ such that for l large enough

$$\int_{\Gamma_k^l}\frac{dm(z)}{|z|^q(\frac{d\omega_l}{dm(z)})^{q-1}} \le c\int_{\Gamma_k}\frac{dm(z)}{|z|^q(\frac{d\omega}{dm(z)})^{q-1}}$$

So

$$\left|\sum_{k=n}^{m}\frac{1}{2\pi i}\int_{C_k}\frac{f(z)}{z-\lambda}dz\right| \le c\left(\sum_{k=n}^{m}\int_{\Gamma_k}\frac{dm(z)}{|z|^q(\frac{d\omega}{dm(z)})^{q-1}}\right)^{\frac{1}{q}}\|f\|$$

Therefore, L_λ is a bounded linear functional for $\lambda \le 0$.

For $\epsilon > 0$, there is an N such that

$$\left(\sum_{k=N+1}^{\infty}\int_{\Gamma_k}\frac{dm(z)}{|z|^q(\frac{d\omega}{dm(z)})^{q-1}}\right)^{\frac{1}{q}} < \epsilon$$

On the other hand,

$$\begin{aligned}&\left|\sum_{k=1}^{m}\frac{1}{2\pi i}\int_{C_k}\frac{f(z)}{z-\lambda}dz - \sum_{k=1}^{m}\frac{1}{2\pi i}\int_{C_k}\frac{f(z)}{z}dz\right|\\ \le\ &2c\left(\sum_{k=N+1}^{m}\int_{\Gamma_k}\frac{dm(z)}{|z|^q(\frac{d\omega}{dm(z)})^{q-1}}\right)^{\frac{1}{q}}\|f\|\\ &+\left|\sum_{k=1}^{N}\frac{1}{2\pi i}\int_{C_k}\frac{f(z)}{z-\lambda}dz - \sum_{k=1}^{N}\frac{1}{2\pi i}\int_{C_k}\frac{f(z)}{z}dz\right|\end{aligned}$$

Hence,

$$|L_\lambda(f) - L_0(f)| \le c\epsilon\|f\| + \left|\sum_{k=1}^{N}\frac{1}{2\pi i}\int_{C_k}\frac{f(z)}{z-\lambda}dz - \sum_{k=1}^{N}\frac{1}{2\pi i}\int_{C_k}\frac{f(z)}{z}dz\right|$$

Thus we have that

$$L_\lambda(f) \to L_0(f) \qquad \text{as} \qquad \lambda \to 0$$

for all $f \in H^p(G)$. $\square$

We must show that $L_\lambda(f)$ really is $f(\lambda)$ for all f in $H^p(G)$; first we need a simple estimate of harmonic measure.

Lemma 2.7. There is a constant $c > 0$ such that

$$\omega(G_n,\ \Gamma'_n,\ -\frac{1}{2}) \ \geq\ c\frac{1+a}{2}a^n$$

where $\Gamma'_n = \{z:\ |z| \leq \frac{1+a}{2}a^n\}$.

Proof: Denote

$$\Omega_n \ =\ \{z \in D:\ Rez < 0\} \cap G_n$$

Then

$$\begin{aligned}
&\omega(G_n,\ \Gamma'_n \cap \partial\Omega_n,\ -\frac{1}{2}) \\
&\quad \geq \omega(\Omega_n,\ \Gamma'_n \cap \partial\Omega_n,\ -\frac{1}{2}) \\
&\quad \geq \omega(\{z:\ |z| < 1,\ Rez < 0\},\ \hat{\Gamma}'_n \cap \{z:\ Rez = 0\},\ -\frac{1}{2}) \\
&\quad \geq cm(\Gamma'_n \cap \{z : Rez = 0\}) \\
&\quad \geq c\frac{1+a}{2}a^n
\end{aligned}$$

Therefore,

$$\omega(G_n,\ \Gamma'_n,\ -\frac{1}{2}) \ \geq\ c\frac{1+a}{2}a^n.\square$$

Lemma 2.8. Suppose that $1 < p < \infty$ and

$$\int \frac{dm(z)}{|z|^q(\frac{d\omega}{dm(z)})^{q-1}} \ <\ \infty$$

Then for $\lambda < 0$, we have that

$$L_\lambda(f) \ =\ f(\lambda)$$

for all $f \in H^p(G)$.

Proof: Let

$$n \geq \frac{1 - log|\lambda|}{log|\frac{1}{a}|}.$$

Then

$$f(\lambda) \;=\; \frac{1}{2\pi i}\int_{\Gamma'_n} \frac{f(z)}{z \;-\; \lambda}dz \;+\; \frac{1}{2\pi i}\sum_{k=0}^{n-1}\int_{C_k} \frac{f(z)}{z \;-\; \lambda}dz$$

and

$$\begin{aligned} &\left| f(\lambda) \;-\; \frac{1}{2\pi i}\sum_{k=0}^{n-1}\int_{C_k} \frac{f(z)}{z \;-\; \lambda}dz \right| \\ \leq\; &\frac{1}{2\pi}\int_{\Gamma'_n} \frac{|f(z)|}{|z \;-\; \lambda|}dm(z) \\ \leq\; &\frac{c}{|\lambda|}\left(\int_{\Gamma'_n} \frac{|dz|}{(\frac{d\omega(G_n,\;.,\;-\frac{1}{2})}{dm(z)})^{q-1}}\right)^{\frac{1}{q}} \|f\| \end{aligned}$$

Using Lemma 2.2 and Lemma 2.7, we see that on Γ'_n

$$\frac{d\omega(G_n,\;.,\;-\frac{1}{2})}{dm(z)} \;\geq\; c\frac{\omega(G_n,\;\Gamma'_n,\;-\frac{1}{2})}{m(\Gamma'_n)} \;\geq\; c,$$

so

$$\begin{aligned} &\left| f(\lambda) \;-\; \frac{1}{2\pi i}\sum_{k=0}^{n-1}\int_{C_k} \frac{f(z)}{z \;-\; \lambda}dz \right| \\ \leq\; &c\frac{1}{|\lambda|}\left(\frac{1+a}{2}a^n\right)^{\frac{1}{q}} \end{aligned}$$

Letting n tend to ∞, we get that $f(\lambda) \;=\; L_\lambda(f)$ for all $f \in H^p(G)$. $\square$

Combining the last three lemmata, we get:

Theorem 2.9. If $1 < p < \infty$ and

$$\int \frac{dm(z)}{|z|^q(\frac{d\omega}{dm(z)})^{q-1}} \;<\; \infty,$$

then every function in $H^p(G)$ has a left limit at zero.

Again, derivatives are handled just as well by the Cauchy integral formula, so we get:

Theorem 2.10. The following statements are equivalent for $1 < p < \infty$:

(b2) The k^{th} derivative of every function in $H^p(G)$ has a left limit at zero.

(c2)

$$\int \frac{dm(z)}{|z|^{(k+1)q}(\frac{d\omega}{dm(z)})^{q-1}} \ < \ \infty.$$

(d2)

$$\sum_{n=0}^{\infty} \frac{r_n^{(k+1)q}}{a^{qn(k+1)}\omega(\Gamma_n)^{q-1}} \ < \ \infty.$$

3. Bergman spaces : Conditions for bpe's in terms of Bessel capacities

Let $G(x, y)$ be the Bessel kernel, which is most easily defined as the inverse Fourier transform of $(1 \ + \ x^2 \ + \ y^2)^{-\frac{1}{2}}$. For each set E in $\mathbb{C}$, the q-Bessel capacity C_q is defined by

$$C_q(E) \ = \ \inf_f \int |f(x, y)|^q dxdy$$

where $f \in L^q(\mathbb{R}^2)$, $f(x, y) \geq 0$ and $\int G(t - x, s - y)f(t, s)dtds \geq 1$ for $(x, y) \in E$.

The following properties of C_q will be used in this section.

Lemma 3.1. (a) If $q < 2$, there are constants F_1 and F_2 such that

$$F_1\delta^{2-q} \ \leq \ C_q(D(\lambda, \ \delta)) \ \leq \ F_2\delta^{2-q}$$

(b) If $q = 2$, there are constants F_3 and F_4 such that

$$F_3\{log(\frac{1}{\delta})\}^{-1} \ \leq \ C_q(D(\lambda, \ \delta)) \ \leq \ F_4\{log(\frac{1}{\delta})\}^{-1}$$

(c) For each set K,

$$C_q(K) \ = \ \inf_{K \subset U} C_q(U)$$

where U stands for an open subset.

(d) Let K_1 and K_2 be two subsets of $\mathbb{C}$, then

$$C_q(K_1 \cup K_2) \ \leq \ C_q(K_1) \ + \ C_q(K_2).$$

For proofs, see Meyers [10].

The following theorem was proved by Fernstrom and Polking [4] for K a compact set in the plane.

Theorem 3.2. Assume that $p \ \geq \ 2$. Then λ is a bounded point derivation of order k for $R^p(K, A)$ if and only if

$$\sum_{j=0}^{\infty} \frac{C_q(A_j(\lambda) \setminus K)}{a^{(k+1)jq}} \ < \ \infty$$

where $0 < a < 1$ and $A_j(\lambda) \ = \ A(\lambda, a^{j+1}, a^j) \ = \ \{z : \ a^{j+1} < |z - \lambda| \leq a^j\}$.

Applying the above results, we are able to prove a similar result for Bergman spaces.

Theorem 3.3. Let G be a bounded open subset of $\mathbb{C}$. Suppose $p \geq 2$ and $\lambda \in \partial G$, then λ is a bounded point derivation of order k for $L^p_a(G)$ if and only if

$$\sum_{j=0}^{\infty} \frac{C_q(A_j(\lambda) \setminus G)}{a^{(k+1)jq}} \ < \ \infty$$

Proof: The proof of necessity is exactly the same as that for $R^p(K)$ (see [4] and [8]). It suffices to show the sufficiency. Suppose

$$\sum_{j=0}^{\infty} \frac{C_q(A_j(\lambda) \setminus G)}{a^{(k+1)jq}} \ < \ \infty.$$

Let $\Gamma_j \ = \ \{z : \ |z - \lambda| = a^j\}$. As

$$\sum_{j=0}^{\infty} \frac{C_q(\Gamma_{j+1} \setminus G)}{a^{(k+1)jq}} \leq \sum_{j=0}^{\infty} \frac{C_q(A_{j+1}(\lambda) \setminus G)}{a^{(k+1)jq}} \ < \ \infty,$$

Lemma 3.1 (d) yields that

$$\sum_{j=0}^{\infty} \frac{C_q(\overline{A_j(\lambda)} \setminus G)}{a^{(k+1)jq}} \ < \ \infty.$$

Using Lemma 3.1 (c), we can find an open subset U_j such that

$$\overline{A_j(\lambda)} \setminus G \subset U_j$$

and

$$\sum_{j=0}^{\infty} \frac{C_q(U_j)}{a^{(k+1)jq}} < \infty$$

Let $K_j = \overline{A_j(\lambda)} \setminus U_j$, and $K = \cup_{j=1}^{\infty} K_j \cup \{\lambda\}$. Then K is a compact subset,

$$\sum_{j=0}^{\infty} \frac{C_q(A_j(\lambda) \setminus K)}{a^{(k+1)jq}} < \infty$$

and

$$\begin{aligned} K \cap A_j(\lambda) &= (K_j \cap A_j(\lambda)) \cup (K_{j-1} \cap \Gamma_j) \\ &\subset (G \cap A_j(\lambda)) \cup (\overline{A_{j-1}(\lambda)} \cap G \cap \Gamma_j) \\ &= A_j(\lambda) \cap G \end{aligned}$$

So

$$K \setminus \{\lambda\} \subset G.$$

It follows from Theorem 3.1 that λ is a bounded point derivation for $R^p(K, A)$. Suppose $f \in L^p(G)$ is analytic in a neighborhood of λ; then f is in $R^p(K, A)$ and

$$|f^{(k)}(\lambda)| \le c(\int_K |f|^p dA)^{\frac{1}{p}} \le c(\int_G |f|^p dA)^{\frac{1}{p}}$$

where c is a constant. Hence, λ is a bounded point derivation for $L^p_a(G)$. □

Corollary 3.4. Let G be an L-region.

(a) If $p = 2$ and

$$\sum_{n=0}^{\infty} \frac{1}{a_n^{2(k+1)} log \frac{1}{r_n}} < \infty$$

then zero is a bounded point derivation of order k for $L^2_a(G)$.

(b) If $p > 2$ and

$$\sum_{n=0}^{\infty} \frac{r_n^{2-q}}{a_n^q (k+1)} < \infty$$

then zero is a bounded point derivation of order k for $L^p_a(G)$.

Proof of (a): Using Lemma 3.1 (d), we see that

$$\begin{aligned} C_2(A_j(0)\setminus G) &\leq \sum_{a_n\in A_j(0)} C_2(D(a_n,\ r_n)) \\ &\leq F_4 \sum_{a_n\in A_j(0)} \frac{1}{log\frac{1}{r_n}} \quad \text{(Lemma 1 (b))} \end{aligned}$$

It follows from Theorem 3.3 that zero is a bounded point derivation of order k.

The proof of (b) is similar. □

4. Bergman L^2_a spaces on L regions : Geometric conditions for bpe's

Because of possible interference between adjacent Δ_n, we cannot give necessary and sufficient conditions, in terms of a_n and r_n, that 0 be a bpe for the Bergman space of an arbitrary L region. We must impose the requirement that there is a sequence $\{c_j\}$ decreasing monotonically to zero, such that the number of a_n's in $A_j := \{z :\ c_{j+1} \leq |z| < c_j\}$ is less than some constant N, and with the following property:

(P) $\sum_{j=1}^{\infty} c_j^2 d_j \ < \ \infty$, where

$$d_j \ = \ \max\left(log\frac{c_j^2 \ - \ c_{j+2}^2}{c_{j+1}^2 \ - \ c_{j+2}^2},\ log\frac{c_{j-1}^2 \ - \ c_{j+1}^2}{c_j^2 \ - \ c_{j+1}^2}\right).$$

Example: If $c_j \ = \ a^j$ and the number of a_n's in any A_j remains bounded as j tends to ∞, condition (P) is satisfied.

If $c_j \ = \ (\frac{1}{j})^k$, where $k > \frac{1}{2}$, and the number of a_n's in A_j is no more than N, then condition (P) is satisfied.

We now show that if the series in Corollary 3.4 diverges, and G satisfies condition (P), then 0 can not be a bpe. First, we show that this happens if the terms do not tend to zero, then prove it in general.

Lemma 4.1: If

$$\liminf_{n\to\infty} a_n^2 log\frac{1}{r_n} \ < \ \infty$$

then zero is not a bpe.

Proof: Let

$$f_n(z) \;=\; \frac{z}{z-a_n}$$

Then

$$\begin{aligned}\|f_n\|^2 \;&=\; \int_G |f_n(z)|^2 dA(z)\\ &\leq \int_{\{z:\; r_n\leq|z-a_n|<1+a_n\}} |\frac{z}{z-a_n}|^2 dA(z)\\ &\leq 4\pi a_n^2 log\frac{1}{r_n}+4\pi.\end{aligned}$$

By hypothesis, there is a subsequence of $\{f_n\}$ whose norms remain bounded; therefore some subsequence , $\{f_{n_k}\}$ say, converges weakly. But as f_n converges pointwise to 1, this must be the weak-limit of f_{n_k}, so 1 is in $clos(zL_a^2(G))$. Because functions analytic in a neighborhood of 0 are dense [1], it follows that

$$clos(zL_a^2(G)) \;=\; L_a^2(G).\square$$

Theorem 4.2. Suppose $\{a_n, r_n\}$ satisfies the condition (P) and

$$\sum_{n=1}^{\infty}\frac{1}{a_n^2 log\frac{1}{r_n}} \;=\; \infty. \tag{4.3}$$

Then zero is not a bpe for $L_a^2(G)$.

Proof: Let

$$b_n \;=\; \frac{1}{a_n^2 log\frac{1}{r_n}}.$$

From Proposition 3.5, we can assume that $\lim_{n\to\infty} b_n \;=\; 0$, so for each m, we can find N_m such that

$$1\leq\sum_{n=m}^{N_m} b_n\leq 2.$$

Let

$$f_n(z) \;=\; \sum_{n=m}^{N_m} b_n\frac{z}{z-a_n}$$

and

$$B_j \;=\; A_{j-1} \cup A_j \cup A_{j+1}$$

We want to show $\|f_m\|$ stays bounded. We have

$$\int_{A_j \cap G} |f_m(z)|^2 dA(z)$$
$$\leq 2 \int_{A_j \cap G} \left(\sum_{a_n \notin B_j} b_n |\frac{z}{z-a_n}| \right)^2 dA(z) \;+\; 2 \int_{A_j \cap G} \left(\sum_{a_n \in B_j} b_n |\frac{z}{z-a_n}| \right)^2 dA(z)$$

Denote the first integral by F_j and the second integral by S_j. By Minkowski's inequality

$$F_j \;\leq\; 2[\sum_{a_n \notin B_j} b_n (\int_{A_j \cap G} |\frac{z}{z-a_n}|^2 dA(z))^{\frac{1}{2}}]^2.$$

But

$$\begin{aligned}
\int_{A_j} |\frac{z}{z-a_n}|^2 dA(z) \leq c_j^2 \int_{A_j} |\frac{1}{z-a_n}|^2 dA(z) \\
&= c_j^2 \begin{cases} \pi log \dfrac{c_j^2 - a_n^2}{c_{j+1}^2 - a_n^2} & a_n < c_{j+2} \\ \pi log \dfrac{a_n^2 - c_{j+1}^2}{a_n^2 - c_j^2} & a_n > c_{j-1} \end{cases} \\
&\leq c_j^2 \begin{cases} \pi log \dfrac{c_j^2 - c_{j+2}^2}{c_{j+1}^2 - c_{j+2}^2} & a_n < c_{j+2} \\ \pi log \dfrac{c_{j-1}^2 - c_{j+1}^2}{c_{j-1}^2 - c_j^2} & a_n > c_{j-1} \end{cases} \\
&\leq \pi c_j^2 d_j,
\end{aligned}$$

so

$$F_j \;\leq\; 2\pi (\sum_{a_n \notin B_j} b_n)^2 c_j^2 d_j \;\leq\; 8\pi c_j^2 d_j.$$

For the second integral, we have the following computation:

$$\begin{aligned}
S_j &\leq 2 \int_{A_j \cap G} (\sum_{a_n \in B_j} b_n^2 |\frac{z}{z-a_n}|^2)(3N) dA(z) \\
&\leq 6N \sum_{a_n \in B_j} b_n^2 (4\pi b_n^{-1} + 4\pi) \\
&\leq 100N \sum_{a_n \in B_j} b_n.
\end{aligned}$$

So we have

$$\int_{A_j \cap G} |f_m(z)|^2 dA(z) \le c(c_j^2 d_j + \sum_{a_n \in B_j} b_n),$$

and therefore

$$\int |f_m(z)|^2 dA(z) \le c(\sum_{j=1}^{\infty} c_j^2 d_j + \sum_{n=m}^{N_m} b_n).$$

So the norms of f_m are uniformly bounded, and as before we conclude that $1 \in clos(zL_a^2(G))$, and hence

$$clos(zL_a^2(G)) \;\; = \;\; L_a^2(G).\square$$

The divergence of (4.3), without assumption (1), is not enough to prevent 0 from being a bpe, as the following example shows.

Example 4.4. Let $a_n = 2^{-n}$, $r_n = exp(-8^n)$. Let $G = D \setminus \cup_{n=1}^{\infty} \overline{\Delta(a_n, r_n)}$. Then G satisfies the hypotheses of Theorem 3.3, and 0 is a bpe for $L_a^2(G)$. Now, replace each disc $\Delta(a_n, r_n)$ by 2^n small discs, each of radius $\frac{r_n}{2^{n+1}}$ equally spaced along the diameter of $\Delta(a_n, r_n)$. Let G' be the larger region obtained by removing these discs from D. For G', the sum (3.4) is greater than

$$\frac{1}{2} \sum_{n=1}^{\infty} \sum_{j=1}^{2^n} \frac{1}{a_n^2 log \frac{2^{n+1}}{r_n}} = \frac{1}{2} \sum \frac{2^n}{n+1} \frac{1}{a^{2n} log \frac{2}{r_n}} = \infty.$$

But 0 is still a bpe for $L_a^2(G')$ because $G \subset G'$. $\square$

Combining Corollary 3.4 and Theorem 4.2, with the obvious modifications to its proof to handle derivatives, we get:

Theorem 4.5. Suppose that G satisfies condition (P). Then the following statements are equivalent.

(a3) Zero is a bounded point derivation of order k for $L_a^2(G)$.

(b3) The k^{th} derivative of every function in $L_a^2(G)$ has a left limit at zero.

(d3)

$$\sum_{n=1}^{\infty} \frac{1}{a_n^{2(k+1)} log \frac{1}{r_n}} \;<\; \infty$$

5. Bergman spaces on L regions : $L^p_a, p > 2$

For $p > 2$, we can use similar techniques. We shall just state the results obtained.

Condition (P) on the sequence c_j is replaced by: there is a sequence $\{c_j\}$ decreasing monotonically to zero, such that the number of a_n's in $A_j := \{z : c_{j+1} \le |z| < c_j\}$ is less than some constant N, and with the following property:

$$(P') \qquad \sum_{n=1}^{\infty} c_j^p d_j < \infty,$$

where

$$d_j = \max\left(\int_{A_j} \left| \frac{1}{z - c_{j+2}} \right|^p dA(z), \int_{A_j} \left| \frac{1}{z - c_{j-1}} \right|^p dA(z) \right).$$

Theorem 4.5 becomes:

Theorem 5.1. If $p > 2$ and G satisfies the condition (P'), the following statements are equivalent.

(a4) Zero is a bounded point derivation of order k for $L^p_a(G)$.

(b4) The k^{th} derivative of every function in $L^p_a(G)$ has a left limit at zero.

(c4)

$$\sum_{n=1}^{\infty} \frac{r_n^{2-q}}{a_n^{q(k+1)}} < \infty$$

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