Complete Nevanlinna-Pick Kernels

Jim Agler *
John E. M^cCarthy †
University of California at San Diego, La Jolla California 92093
Washington University, St. Louis, Missouri 63130

Abstract

We give a new treatment of Quiggin's and M^cCullough's characterization of complete Nevanlinna-Pick kernels. We show that a kernel has the matrix-valued Nevanlinna-Pick property if and only if it has the vector-valued Nevanlinna-Pick property. We give a representation of all complete Nevanlinna-Pick kernels, and show that they are all restrictions of a universal complete Nevanlinna-Pick kernel.

0 Introduction

Let X be an infinite set, and k a positive definite kernel function on X, *i.e.* for any finite collection $x_1, \ldots, x_n$ of distinct points in X, and any complex numbers $\{a_i\}_{i=1}^n$, the sum

$$\sum_{i,j=1}^{n} a_i \bar{a}_j k(x_i, x_j) \geq 0, \tag{0.1}$$

with strict inequality unless all the a_i's are 0. For each element x of X, define the function k_x on X by $k_x(y) := k(x, y)$. Define an inner product on the span of these functions by

$$\langle \sum a_i k_{x_i} , \sum b_j k_{y_j} \rangle \ = \ \sum a_i \bar{b}_j k(x_i, y_j),$$

and let $\mathcal{H} = \mathcal{H}_k$ be the Hilbert space obtained by completing the space of finite linear combinations of k_{x_i}'s with respect to this inner product. The elements of $\mathcal{H}$ can be thought of as functions on X, with the value of f at x given by $\langle f, k_x \rangle$.

A multiplier of $\mathcal{H}$ is a function ϕ on X with the property that if f is in $\mathcal{H}$, so is ϕf. The Nevanlinna-Pick problem is to determine, given a finite set $x_1, \ldots, x_n$ in X, and numbers

*Partially supported by the National Science Foundation
†Partially supported by National Science Foundation grant DMS 9531967.

$\lambda_1, \dots, \lambda_n$, whether there exists a multiplier ϕ of norm at most one that interpolates the data, *i.e.* satisfies $\phi(x_i) = \lambda_i$ for $i = 1, \dots, n$.

If ϕ is a multiplier of $\mathcal{H}$, we shall let T_ϕ denote the operator of multiplication by ϕ. Note that the adjoint of T_ϕ satisfies $T_\phi^* k_x = \overline{\phi(x)} k_x$. So if ϕ interpolates the data (x_i, λ_i), then the n-dimensional space spanned by $\{k_{x_i} : 1 \leq i \leq n\}$ is left invariant by T_ϕ^*, and on this subspace the operator T_ϕ^* is the diagonal

$$\begin{pmatrix} \overline{\lambda_1} & & \\ & \ddots & \\ & & \overline{\lambda_n} \end{pmatrix} \tag{0.2}$$

with respect to the (not necessarily orthonormal) basis $\{k_{x_i}\}$. For a given set of n data points $(x_1, \lambda_1), \dots, (x_n, \lambda_n)$, let $R_{x,\lambda}$ be the operator in (0.2), *i.e.* the operator that sends k_{x_i} to $\overline{\lambda_i} k_{x_i}$. A necessary condition to solve the Nevanlinna-Pick problem is that the norm of $R_{x,\lambda}$ be at most 1; the kernel k is called a *Nevanlinna-Pick kernel* if this necessary condition is also always sufficient.

Notice that $R_{x,\lambda}$ is a contraction on $sp\{k_{x_i} : 1 \leq i \leq n\}$ if and only if $(1 - R_{x,\lambda}^* R_{x,\lambda})$ is positive on that space. As

$$\langle (1 - R_{x,\lambda}^* R_{x,\lambda}) \sum_{i=1}^n a_i k_{x_i}, \sum_{j=1}^n a_j k_{x_j} \rangle \ = \ \sum_{i,j=1}^n a_i \overline{a_j} (1 - \lambda_j \overline{\lambda_i}) \langle k_{x_i}, k_{x_j} \rangle,$$

it follows that the contractivity of $R_{x,\lambda}$ on $sp\{k_{x_i} : 1 \leq i \leq n\}$ is equivalent to the positivity of the n-by-n matrix

$$\Big((1 - \lambda_j \overline{\lambda_i}) \langle k_{x_i}, k_{x_j} \rangle \Big)_{i,j=1}^n . \tag{0.3}$$

The classical Nevanlinna-Pick theorem asserts that the Szegö kernel

$$k(x,y) = \frac{1}{1 - \bar{x}y}$$

on the unit disk is a Nevanlinna-Pick kernel. The condition is normally stated in terms of the positivity of (0.3), but as we see that is equivalent to the contractivity of (0.2).

The matrix-valued Nevanlinna-Pick problem is as follows. Fix some auxiliary Hilbert space, which for notational convenience we shall assume to be the finite-dimensional space $\mathbb{C}^\nu$. The tensor product $\mathcal{H} \otimes \mathbb{C}^\nu$ can be thought of as a space of vector valued functions on X. A multiplier of $\mathcal{H} \otimes \mathbb{C}^\nu$ is now a ν-by-ν matrix valued function Φ on X with the property that whenever

$$\begin{pmatrix} f_1 \\ \vdots \\ f_\nu \end{pmatrix} \ \in \ \mathcal{H} \otimes \mathbb{C}^\nu$$

then

$$\Phi \begin{pmatrix} f_1 \\ \vdots \\ f_\nu \end{pmatrix} \in \mathcal{H} \otimes \mathbb{C}^\nu .$$

The matrix Nevanlinna-Pick problem is to determine, given points $x_1, \ldots, x_n$ and matrices $\Lambda_1, \ldots, \Lambda_n$, whether there is a multiplier Φ of norm at most one that interpolates: $\Phi(x_i) = \Lambda_i$.

Fix a (not necessarily orthonormal) basis $\{e^\alpha\}_{\alpha=1}^\nu$ for $\mathbb{C}^\nu$. As before,

$$T_\Phi^* k_x \otimes v = k_x \otimes \Phi(x)^* v,$$

so if $\mathcal{M}$ is the span of $\{k_{x_i} \otimes e^\alpha : 1 \leq i \leq n, 1 \leq \alpha \leq \nu\}$, a necessary condition for the Nevanlinna-Pick problem to have a solution is that the $n\nu$-by-$n\nu$ matrix

$$R_{x,\Lambda} : k_{x_i} \otimes e^\alpha \mapsto k_{x_i} \otimes \Lambda_i^* e^\alpha \tag{0.4}$$

be a contraction. We shall call the kernel k a *complete Nevanlinna-Pick kernel* if, for all finite ν and all positive n, the contractivity of $R_{x,\Lambda}$ is also a sufficient condition to extend Φ to a multiplier of all of $\mathcal{H} \times \mathbb{C}^\nu$ of norm at most one.

In Section 1 we give a classification of all complete Nevanlinna-Pick kernels. This was originally done by S. McCullough in [7] in the context of the Carathéodory interpolation problem. The Nevanlinna-Pick problem was studied by P. Quiggin, who in [9] established the sufficiency of the condition in Theorem 1.2. Necessity was proved independently by Quiggin in [10] and by McCullough in [8].

In Section 2 we show that if a kernel has the Nevanlinna-Pick property for row vectors of length ν, then it has the Nevanlinna-Pick property for μ-by-ν matrices for all μ. In particular, having the vector-valued Nevanlinna-Pick property is equivalent to having the complete Nevanlinna-Pick property.

In Section 3, we show that all complete Nevanlinna-Pick kernels have the form

$$k(x,y) = \frac{\overline{\delta(x)}\delta(y)}{1 - F(x,y)}$$

where δ is a nowhere vanishing function and $F : X \times X \to \mathbb{D}$ is a positive semi-definite function.

In Section 4 we introduce the universal complete Nevanlinna-Pick kernels a_m defined on the unit ball $\mathbb{B}_m$ of an m-dimensional Hilbert space (m may be infinite) by

$$a_m(x,y) = \frac{1}{1 - \langle x, y \rangle}.$$

These kernels are universal in the sense that, up to renormalization, every complete Nevanlinna-Pick kernel is just the restriction of an a_m to a subset of $\mathbb{B}_m$.

1 Characterization of Complete Nevanlinna-Pick kernels

To simplify notation, we shall let k_i denote k_{x_i}, and k_{ij} denote $\langle k_i, k_j \rangle = k(x_i, x_j)$. First we want a lemma that says that we can break $\mathcal{H}$ up into summands on each of which k is irreducible, *i.e.* k_{ij} is never 0. For convenience, we shall defer the proof of the lemma until after the proof of the theorem.

Lemma 1.1 *Suppose k is a Nevanlinna-Pick kernel on the set X. Then X can be partitioned into disjoint subsets X_i such that if two points x and y are in the same set X_i, then $k(x,y) \neq 0$; and if x and y are in different sets of the partition, then $k(x,y) = 0$.*

A reducible kernel will have the (complete) Nevanlinna-Pick property if and only if each irreducible piece does, so we shall assume k is irreducible.

Theorem 1.2 *A necessary and sufficient condition for an irreducible kernel k to be a complete Nevanlinna-Pick kernel is that, for any finite set $\{x_1, \ldots, x_n\}$ of n distinct elements of X, the $(n-1)$-by-$(n-1)$ matrix*

$$F_n \;=\; \left(1 - \frac{k_{in}k_{nj}}{k_{ij}k_{nn}}\right)_{i,j=1}^{n-1} \tag{1.3}$$

is positive semi-definite.

Proof: Let $x_1, \ldots, x_{n-1}$ and $\Lambda_1, \ldots, \Lambda_{n-1}$ be chosen, let $\mathcal{M}$ be the span of $\{k_i \otimes e^\alpha : 1 \leq i \leq n-1, 1 \leq \alpha \leq \nu\}$, and define $R_{x,\Lambda}$ on M by (0.4). The operator $R_{x,\Lambda}$ is a contraction if and only if $I - R_{x,\Lambda}^* R_{x,\Lambda} \geq 0$. Calculate

$$\langle (I - R_{x,\Lambda}^* R_{x,\Lambda}) \sum_{i,\alpha} a_i^\alpha k_i \otimes e^\alpha , \sum_{j,\beta} a_j^\beta k_j \otimes e^\beta \rangle \;=\; \sum_{i,\alpha,j,\beta} a_i^\alpha \bar{a}_j^\beta k_{ij} (\langle e^\alpha, e^\beta \rangle - \langle \Lambda_j \Lambda_i^* e^\alpha, e^\beta \rangle) \tag{1.4}$$

A necessary and sufficient condition to be able to find a matrix Λ_n so that the extension $R_{x,\Lambda}^\sim$ of $R_{x,\Lambda}$ that sends $k_{x_n} \otimes e^\alpha$ to $k_{x_n} \otimes \Lambda_n^* e^\alpha$ for each α has the same norm as $R_{x,\Lambda}$ is: whenever $\Lambda_1, \ldots, \Lambda_{n-1}$ are chosen so that

$$I - R_{x,\Lambda}^* R_{x,\Lambda} \;\geq\; 0 \tag{1.5}$$

on $\vee\{k_i \otimes e^\alpha : 1 \leq i \leq n-1, 1 \leq \alpha \leq \nu\}$, then

$$P - (PR^{\sim}_{x,\Lambda}P)^*(PR^{\sim}_{x,\Lambda}P) \;\geq\; 0, \tag{1.6}$$

where P is the orthogonal projection from $\vee\{k_i \otimes e^\alpha : 1 \leq i \leq n, 1 \leq \alpha \leq \nu\}$ onto the orthogonal complement of $\vee\{k_n \otimes e^\alpha : 1 \leq \alpha \leq \nu\}$. (This was first proved in [1] in the scalar case, and a proof of the matrix case is given in [3]. Notice that (1.6) does not depend on the choice of Λ_n. We use $\vee$ to denote the closed linear span of a set of vectors.)

That (1.5) always implies (1.6) for any choice of x and Λ is not only necessary, but also sufficient for k to be a complete Nevanlinna-Pick kernel. Sufficiency is proved by an inductive argument that if one can always extend a multiplier defined on a finite set to any other point without increasing the norm, then one can extend the multiplier to all of X. In the absence of any *a priori* simplifying assumptions about the multiplier algebra of $\mathcal{H}$ being large, the proof of this inductive argument is subtle, and is originally due to Quiggin [9, Lemma 4.3].

Using the fact that

$$P(k_i \otimes e^\alpha) \;=\; (k_i - \frac{k_{in}}{k_{nn}} k_n) \otimes e^\alpha,$$

we can calculate that

$$\langle (P - (PR^{\sim}_{x,\Lambda}P)^*(PR^{\sim}_{x,\Lambda}P) \sum_{i,\alpha} a_i^\alpha k_i \otimes e^\alpha \,,\, \sum_{j,\beta} a_j^\beta k_j \otimes e^\beta \rangle$$

equals

$$\sum_{i,\alpha,j,\beta} a_i^\alpha \bar{a}_j^\beta k_{ij} \left(1 - \frac{k_{in}k_{nj}}{k_{ij}k_{nn}}\right) \left[\langle e^\alpha, e^\beta\rangle - \langle \Lambda_j \Lambda_i^* e^\alpha, e^\beta\rangle\right] \tag{1.7}$$

Comparing (1.4) and (1.7), we see that we want that whenever the matrix whose $(i,\alpha)^{th}$ column and $(j,\beta)^{th}$ row is given by

$$k_{ij}(\langle e^\alpha, e^\beta\rangle - \langle \Lambda_j \Lambda_i^* e^\alpha, e^\beta\rangle) \tag{1.8}$$

is positive, then the Schur product of this matrix with $F_n \otimes J$ is positive, where J is the ν-by-ν matrix all of whose entries are 1. As the Schur product of two positive matrices is positive, the positivity of (1.3) is immediately seen to be a sufficient condition for k to be a complete Nevanlinna-Pick kernel.

We shall prove necessity by induction on n. The case $n = 2$ holds by the Cauchy-Schwarz inequality. So assume that F_{n-1} is positive, and we shall prove that F_n is positive.

Note first the sort of matrices that can occur in (1.8). For each i and α, one can choose the vector $\Lambda_i^* e^\alpha$ arbitrarily. In particular, let G be any positive $(n-1)$-by-$(n-1)$ matrix, let

$\varepsilon > 0$, and choose $\{e^\alpha\}$ so that $\langle e^\alpha, e^\beta \rangle = \varepsilon \delta_{\alpha,\beta} + 1$. Choose vectors v_i so that $\langle v_i, v_j \rangle = G_{ij}$. Let $\nu = n - 1$, and choose Λ_i^* to be the rank one matrix that sends each e^α to v_i. Then (1.8) becomes

$$k_{ij}(\varepsilon \delta_{\alpha,\beta} + 1 - G_{ij}). \tag{1.9}$$

We know that F_n has the property that if G is a positive matrix and the $(n-1)\nu$-by-$(n-1)\nu$ matrix (1.9) is positive, then the Schur product of $F_n \otimes J$ with (1.9) is also positive. Denote by K the $(n-1)$-by-$(n-1)$ matrix whose (i,j) entry is k_{ij}, and let $\cdot$ denote Schur product. By letting ε tend to zero, we get that whenever $G \geq 0$ and

$$[K \cdot (J - G)] \otimes J \ \geq \ 0,$$

then

$$[F_n \otimes J] \cdot ([K \cdot (J - G)] \otimes J) \ \geq \ 0,$$

which is the same as saying

$$K \cdot (J - G) \geq 0 \quad \Rightarrow \quad F_n \cdot K \cdot (J - G) \geq 0. \tag{1.10}$$

Let L be the rank one positive $(n-1)$-by-$(n-1)$ matrix given by

$$L_{ij} \ = \ \frac{k_{i(n-1)} k_{(n-1)j}}{k_{(n-1)(n-1)}},$$

and let G be the matrix given by

$$G_{ij} \ = \ 1 - \frac{L_{ij}}{k_{ij}}.$$

Then G is the matrix that agrees with F_{n-1} in the first $(n-2)$ rows and columns, and all the entries in the $(n-1)^{st}$ row and column are zero. Therefore G is positive by the inductive hypothesis. Moreover, $K \cdot (J - G) = L$ and so is positive. Therefore $F_n \cdot L$ is positive. But L is rank one, so $1/L$ (the matrix of reciprocals) is also positive, and therefore

$$F_n \cdot L \cdot 1/L \ = \ F_n \geq 0,$$

as desired. $\square$

Proof of Lemma 1.1: Let $X_x = \{y : k(x,y) \neq 0\}$. We need to show that for any two points x and y, the sets X_x and X_y are either equal or disjoint. This is equivalent to proving that if $k(x,z) \neq 0$ and $k(y,z) \neq 0$, then $k(x,y) \neq 0$.

Assume this fails. Consider the 2-by-2 matrix T^* defined on the linear span of k_x and k_y by

$$\begin{aligned} T^* k_x &= k_x \\ T^* k_y &= -k_y \end{aligned}$$

This has norm one, because $k(x,y) = 0$. By the hypothesis that k is a Nevanlinna-Pick kernel, T^* can be extended to the space spanned by k_x, k_y and k_z so that the new operator has the same norm and has k_z as an eigenvector. But for this to hold, from equation (1.7) we would need

$$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} k_{xx} - \frac{|k_{xz}|^2}{k_{zz}} & k_{xy} - \frac{k_{xz}k_{zy}}{k_{zz}} \\ k_{yx} - \frac{k_{yz}k_{zx}}{k_{zz}} & k_{yy} - \frac{|k_{yz}|^2}{k_{zz}} \end{pmatrix} \geq 0. \tag{1.11}$$

But the Schur product of the two matrices in (1.11) is zero on the diagonal, non-zero off the diagonal, and therefore cannot be positive. □

By the same argument as in the theorem, an irreducible kernel will have the (scalar) Nevanlinna-Pick property if and only if whenever G is positive and rank one, (1.10) holds. We do not know how to classify such kernels in the sense of Theorem 1.2.

The positivity of F_n can be expressed in other ways. The proof that F_n being positive is equivalent to $1/K$ having only one positive eigenvalue below is due to Quiggin [9].

Corollary 1.12 *A necessary and sufficient condition for the irreducible kernel k to have the complete Nevanlinna-Pick property is that for any finite set $x_1, \ldots, x_n$, the matrix*

$$H_n := \left(\frac{1}{k_{ij}} \right)_{i,j=1}^{n}$$

has exactly one positive eigenvalue (counting multiplicity).

Proof: As all the diagonal entries of H_n are positive, H_n must have at least one positive eigenvalue.

The condition that F_{n+1} be positive is equivalent to saying

$$M_n := \left(\frac{k_{n+1,n+1}}{k_{i,n+1}k_{n+1,j}} - \frac{1}{k_{ij}} \right)_{i,j=1}^{n} \geq 0, \tag{1.13}$$

because $k_{i,n+1}k_{n+1,j}$ is rank one so its reciprocal is positive. But (1.13) says that H_n is less than or equal to a rank one positive operator, so has at most one positive eigenvalue.

Conversely, any symmetric matrix

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$$

with C invertible is congruent to

$$\begin{pmatrix} A - BC^{-1}B^* & 0 \\ 0 & C \end{pmatrix}.$$

(The top left entry is called the Schur complement of C.) Applying this to H_n with C the (n,n) entry, we get that H_n is congruent to

$$\begin{pmatrix} -M_{n-1} & 0 \\ 0 & \frac{1}{k_{nn}} \end{pmatrix}.$$

So if H_n has only one positive eigenvalue, $-M_{n-1}$ must be negative semi-definite, and therefore F_n must be positive semi-definite. □

As an application of the Corollary, consider the Dirichlet space of holomorphic functions on the unit disk with reproducing kernel $k(w,z) = \frac{1}{\bar{w}z} \log \frac{1}{1-\bar{w}z}$. It is shown in [1] that this is a Nevanlinna-Pick kernel, and in the course of the proof it is established that $1 - 1/k$ is positive semi-definite (because all the coefficients in the power series are positive). It then follows at once from Corollary 1.12 that the Dirichlet kernel is actually a complete Nevanlinna-Pick kernel.

2 Vector-valued Nevanlinna-Pick kernels

Let $\mathcal{M}_{\mu,\nu}$ denote the μ-by-ν matrices. Let us say that a kernel k has the n-point $\mathcal{M}_{\mu,\nu}$ Nevanlinna-Pick property if, for any points $x_1, \ldots, x_{n+1}$, and any matrices $\Lambda_1, \ldots, \Lambda_n$ in $\mathcal{M}_{\mu,\nu}$, the operator

$$\begin{aligned} T^* : sp\{k_{x_i} : 1 \leq i \leq n\} \otimes \mathbb{C}^\mu &\to sp\{k_{x_i} : 1 \leq i \leq n\} \otimes \mathbb{C}^\nu \\ k_{x_i} \otimes v &\mapsto k_{x_i} \otimes \Lambda_i^* v \qquad \forall\, v \in \mathbb{C}^\mu \end{aligned}$$

has an extension $\tilde{T}^*$ from $sp\{k_{x_i} : 1 \leq i \leq n+1\} \otimes \mathbb{C}^\mu$ to $sp\{k_{x_i} : 1 \leq i \leq n+1\} \otimes \mathbb{C}^\nu$ that sends $k_{x_{n+1}} \otimes v$ to $k_{x_{n+1}} \otimes \Lambda_{n+1}^* v$ for some Λ_{n+1}, and such that $\tilde{T}^*$ has the same norm as T^*. We shall say that k is a vector-valued Nevanlinna-Pick kernel if k has the n point $\mathcal{M}_{1,\nu}$ Nevanlinna-Pick property for all n and ν.

Theorem 2.1 *Let $\nu \geq n-1$. Then k has the n-point $\mathcal{M}_{\mu,\nu}$ Nevanlinna-Pick property for some positive integer μ if and only if it has the property for all positive integers μ.*

PROOF: It is clear that the n-point $\mathcal{M}_{\mu,\nu}$ Nevanlinna-Pick property implies the n-point $\mathcal{M}_{\pi,\nu}$ Nevanlinna-Pick property for all π smaller than μ. So it is sufficient to prove that the

n-point $\mathcal{M}_{1,\nu}$ Nevanlinna-Pick property implies the n-point $\mathcal{M}_{\mu,\nu}$ Nevanlinna-Pick property for all μ.

As in the proof of Theorem 1.2, the kernel k has the n-point $\mathcal{M}_{\mu,\nu}$ Nevanlinna-Pick property if and only if the positivity of the matrix

$$\left[k_{ij}(\langle e^\alpha, e^\beta\rangle_{\mathbb{C}^\mu} - \langle \Lambda_j \Lambda_i^* e^\alpha, e^\beta\rangle_{\mathbb{C}^\nu})\right]_{i,j=1;\alpha,\beta=1}^{i,j=n;\alpha,\beta=\mu} \tag{2.2}$$

implies the positivity of the Schur product of (2.2) with $F_{n+1} \otimes J_\mu$. Again, as in the proof of Theorem 1.2, this implies that whenever $K \cdot (J_n - G)$ is positive, then so is $F_{n+1} \cdot K \cdot (J_n - G)$, for G any positive n-by-n matrix of rank less than or equal to $\max(\nu, n)$.

So, if k has the n-point $\mathcal{M}_{1,\nu}$ Nevanlinna-Pick property, then we can choose G to be the rank $(n-1)$ matrix used in the proof of Theorem 1.2, and conclude that F_{n+1} has to be positive. But the positivity of F_{n+1} clearly implies that k has the n-point $\mathcal{M}_{\mu,\nu}$ Nevanlinna-Pick property for all values of μ and ν. $\square$

Corollary 2.3 *The kernel k is a complete Nevanlinna-Pick kernel if and only if it is a vector-valued Nevanlinna-Pick kernel.*

See [3] for another approach to describing $\mathcal{M}_{\nu,\nu}$ Nevanlinna-Pick kernels when there is a distinguished operator (or tuple of operators) acting on $\mathcal{H}$ for which all the k_x's are eigenvectors.

3 Representation of Complete Nevanlinna-Pick kernels

It is a consequence of Theorem 1.2 that all complete Nevanlinna-Pick kernels have a very specific form.

Theorem 3.1 *The irreducible kernel k on X is a complete Nevanlinna-Pick kernel if and only if there is a positive semi-definite function $F : X \times X \to \mathbb{D}$ and a nowhere vanishing function δ on X so that*

$$k(x, y) = \frac{\overline{\delta(x)}\delta(y)}{1 - F(x, y)}. \tag{3.2}$$

PROOF: (Sufficiency): If k has the form of (3.2), then $1/k$ is a rank-one operator minus a positive operator, so has exactly one positive eigenvalue, and the result follows from Corollary 1.12.

(Necessity): Suppose k is a complete Nevanlinna-Pick kernel. Fix any point x_0 in X. Then the kernel

$$F(x,y) \;=\; 1 - \frac{k(x,x_0)k(x_0,y)}{k(x,y)k(x_0,x_0)} \tag{3.3}$$

is positive semi-definite by Theorem 1.2. Let

$$\delta(x) = \frac{k(x_0,x)}{\sqrt{k(x_0,x_0)}}.$$

It is immediate that equation (3.2) is satisfied. As $k(x,x)$ is positive and finite for all x, $F(x,x)$ must always lie in $[0,1)$; as $F(x,y)$ is a positive semi-definite kernel, it follows that $|F(x,y)| < 1$ for all x, y. $\square$

Any positive definite kernel $k(x,y)$ can be rescaled by multiplying by a nowhere-vanishing rank-one kernel $\overline{\delta(x)}\delta(y)$. Let $j(x,y) = \overline{\delta(x)}\delta(y)k(x,y)$. Then the Hilbert space $\mathcal{H}_j$ is just a rescaled copy of $\mathcal{H}_k$: a function f is in $\mathcal{H}_k$ if and only if δf is in $\mathcal{H}_j$, so $\mathcal{H}_j = \delta\mathcal{H}_k$. The multipliers of $\mathcal{H}_k$ and $\mathcal{H}_j$ are the same, and one space has the complete Nevanlinna-Pick property if and only if the other one does (the matrices F_n are identical, as the scaling factors cancel). We shall say that the kernel k is *normalized at* x_0 if $k(x_0,x) = 1$ for all x; this is equivalent to scaling the kernel by $\dfrac{\sqrt{k(x_0,x_0)}}{k(x_0,x)}$, and means that in (4.1) δ can be chosen to be one, and $F(x,y)$ becomes $1 - \dfrac{1}{k(x,y)}$.

4 The Universal Complete Nevanlinna-Pick Kernels

It follows from Theorem 3.1 that there is a *universal* complete Nevanlinna-Pick kernel (actually a family of them, indexed by the cardinal numbers). Let l^2_m be m-dimensional Hilbert space, where m is any cardinal bigger than or equal to 1. Let $\mathbb{B}_m$ be the unit ball in l^2_m, and define a kernel a_m on $\mathbb{B}_m$ by

$$a_m(x,y) \;=\; \frac{1}{1 - \langle x,y\rangle} \tag{4.1}$$

Let H^2_m be the completion of the linear span of the functions $\{a_m(\cdot,y) : y \in \mathbb{B}_m\}$, with inner product defined by $\langle a_m(\cdot,y), a_m(\cdot,x)\rangle = a_m(x,y)$. We shall show that the spaces H^2_m are universal complete Nevanlinna-Pick spaces.

Theorem 4.2 *Let k be an irreducible kernel on X. Let m be the rank of the Hermitian form F defined by (3.3). Then k is a complete Nevanlinna-Pick kernel if and only if there is an injective function $f : X \to \mathbb{B}_m$ and a nowhere vanishing function δ on X such that*

$$k(x, y) \;=\; \overline{\delta(x)}\delta(y)\, a_m(f(x), f(y)). \tag{4.3}$$

Moreover if this happens, then the map $k_x \mapsto \overline{\delta(x)}(a_m)_{f(x)}$ extends to an isometric linear embedding of $\mathcal{H}_k$ into δH^2_m.

If in addition there is a topology on X so that k is continuous on $X \times X$, then the map f will be a continuous embedding of X into $\mathbb{B}_m$.

Proof: (Sufficiency): Any kernel of the form (4.3) is of the form (3.2).

(Necessity): Suppose k is a complete Nevanlinna-Pick kernel. As F is positive semi-definite, there exists a Hilbert space of dimension m (which we shall take to be l^2_m) and a map $f : X \to l^2_m$ so that $F(x, y) = \langle f(x), f(y)\rangle$. Moreover, as F takes value in $\mathbb{D}$, f actuallly maps into $\mathbb{B}_m$. It now follows from Theorem 3.1 that k has the form (4.3).

The linear map that sends k_x to the function $\dfrac{\overline{\delta(x)}}{1 - \langle f(x), \cdot\rangle}$ is an isometry on $\vee\{k_x : x \in X\}$ by (4.3) and gives the desired embedding.

If $f(x) = f(y)$ then $k_x = k_y$; as k is positive definite, this implies $x = y$.

Finally, f can be realised as the composition of the four maps

$$\begin{aligned} x &\mapsto k_x \\ k_x &\mapsto \overline{\delta(x)} a_m(f(x), \cdot) \\ \overline{\delta(y)} a_m(y, \cdot) &\mapsto a_m(y, \cdot) \\ a_m(y, \cdot) &\mapsto y \end{aligned}$$

The fourth map is continuous by direct calculation, the second is an isometry by the theorem, and the first and third maps are continuous if k is continuous. □

Note that if one first normalizes k at some point, δ can be taken to be 1 in Theorem 4.2.

For $m = 1$, the space H^2_m is the regular Hardy space on the unit disk. For larger m, it is a Hilbert space of analytic functions on the ball $\mathbb{B}_m$. Thus every reproducing kernel Hilbert space with the complete Nevanlinna-Pick property is a restriction of a space of analytic functions.

It was shown in [2] that the Sobolov space $W^{1,2}[0, 1]$, the functions g on the unit interval for which $\int_0^1 |g|^2 + |g'|^2 dx$ is finite, has the Nevanlinna-Pick property. It follows from [9, Cor. 6.5] that the condition of Corollary 1.12 is satisfied, so the Sobolov space has the complete Nevanlinna-Pick property. We can normalize $W^{1,2}[0,1]$ at 1 say, by calculating that $k_1(t) = \text{cosinh}(1)\cosh(t)$, and hence $\delta(t) = \sqrt{\sinh(1)\cosh(1)}\text{cosech}(t)$. Therefore there is a continuous embedding $f : [0,1] \to \mathbb{B}_{\aleph_0}$ so that if g is any function in $W^{1,2}[0,1]$, then $(\delta.g) \circ f^{-1}$ extends off the curve $f([0,1])$ to be analytic on all of $\mathbb{B}_{\aleph_0}$ - even though $\delta.g$ need not be analytic in any neighborhood of the unit interval on which it is originally defined.

After normalization, every separable reproducing kernel Hilbert space with the complete Nevanlinna-Pick property is the restriction of the space $H^2_{\aleph_0}$ to a subspace spanned by a set of kernel functions, which is why we call this space universal. The kernel k is just the restriction of $a_{\aleph_0}$ to a subset of $\mathbb{B}_{\aleph_0}$.

Let $\mathcal{A}$ be a normed algebra of functions on a set X with the complete Nevanlinna-Pick property, *i.e.* there exists a positive definite function k on $X \times X$ such that there is a function f in $\mathcal{A} \otimes \mathcal{M}_k$ of norm at most one and with $f(x_i) = \Lambda_i$ if and only if the nk-by-nk matrix

$$k(x_i, x_j) \otimes [I_k - \Lambda_i^* \Lambda_j]$$

is positive. It is then immediate that $\mathcal{A}$ is the multiplier algebra of $\mathcal{H}_k$, and k is a complete Nevanlinna-Pick kernel. If $\mathcal{H}_k$ is separable, k is therefore the restriction of $a_{\aleph_0}$ to some subset of $\mathbb{B}_{\aleph_0}$. By the Pick property, every function in $\mathcal{A}$ extends to an element of the multiplier algebra of $H^2_{\aleph_0}$ *without increasing the norm.* So every separably acting algebra with the complete Nevanlinna-Pick property embeds isometrically in the multiplier algebra of $H^2_{\aleph_0}$.

It is probably the universality of the kernel a_m which is responsible for the recent surge of interest in it - see *e.g.* [3, 4, 5, 6].

References

[1] J. Agler. Some interpolation theorems of Nevanlinna-Pick type. Preprint, 1988.

[2] J. Agler. Nevanlinna-Pick interpolation on Sobolov space. *Proc. Amer. Math. Soc.*, 108:341–351, 1990.

[3] J. Agler and J.E. McCarthy. Nevanlinna-Pick kernels and localization. To appear.

[4] A. Arias and G. Popescu. Noncommutative interpolation and Poisson transforms. To appear.

[5] W.B. Arveson. Subalgebras of C*-algebras III: Multivariable operator theory. To appear.

[6] K.R. Davidson and D.R. Pitts. Nevanlinna-Pick interpolation for non-commutative analytic Toeplitz algebras. To appear.

[7] S. McCullough. Carathéodory interpolation kernels. *Integral Equations and Operator Theory*, 15(1):43–71, 1992.

[8] S. McCullough. The local de Branges-Rovnyak construction and complete Nevanlinna-Pick kernels. In *Algebraic methods in Operator theory*, pages 15–24. Birkhäuser, 1994.

[9] P. Quiggin. For which reproducing kernel Hilbert spaces is Pick's theorem true? *Integral Equations and Operator Theory*, 16(2):244–266, 1993.

[10] P. Quiggin. *Generalisations of Pick's Theorem to Reproducing Kernel Hilbert Spaces.* PhD thesis, Lancaster University, 1994.