Boundary values and Cowen-Douglas curvature

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Abstract

We define a metric in terms of the Cowen-Douglas curvature for an operator T in $\mathcal{B}_1(\Omega)$. Any boundary point of Ω that is a finite distance, with respect to this metric, from the eigenvalues in the interior is itself an eigenvalue of T. If T is represented as the adjoint of multiplication by the coordinate function on some holomorphic Hilbert space on Ω, this gives a condition under which functions in the space have limits along a path going to the boundary of Ω.

0 Introduction

Let Ω be an open connected set in $\mathbb{C}$, and $\mathcal{H}$ be a Hilbert space of functions holomorphic on Ω with the property that evaluation at each point of Ω is a continuous functional on $\mathcal{H}$. It can be shown that there must exist a kernel function $k(z,w)$ on $\Omega \times \Omega$, holomorphic in z and conjugate holomorphic in w, with the property that, for each w, $k(.,w) := k_w(.)$ is in $\mathcal{H}$ and $\langle f, k_w \rangle = f(w)$ for all f in $\mathcal{H}$ [2]. We shall call $\mathcal{H}$ a holomorphic Hilbert space on Ω. Define a metric $\rho(w)$ on Ω by

$$[\rho(w)]^2 = \frac{\partial^2 \log \|k_w\|^2}{\partial w \partial \bar{w}}. \tag{0.1}$$

We address the connection between the growth of ρ near the boundary of Ω and the existence of limits for functions in $\mathcal{H}$ along paths tending to $\partial\Omega$.

The definition (0.1) is motivated by the work of M. Cowen and R. Douglas in [5]. Let $\Omega^* := \{\bar{w} : w \in \Omega\}$; then the Cowen-Douglas class $\mathcal{B}_1(\Omega^*)$ consists of those bounded linear operators T acting on a separable Hilbert space $\mathcal{K}$ that satisfy

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(i) Each point of Ω^* is an eigenvalue of multiplicity one for T.

(ii) $\mathrm{Ran}(T-w) = \mathcal{K}$ for all w in Ω^*.

(iii) $\bigvee_{w\in\Omega^*} \mathrm{Ker}(T-w) = \mathcal{K}$.

Any such operator T can be represented as the adjoint of multiplication by the coordinate function on some holomorphic Hilbert space on Ω (this was pointed out in [5], and a detailed proof can be found in [10]). Many of the commonly studied cyclic operators are the adjoints of operators in this class (for example, weighted shifts, or pure cyclic subnormal operators [15]). Cowen and Douglas proved that the metric $\rho(w)$ is a complete unitary invariant for operators in $\mathcal{B}_1(\Omega^*)$, *i.e.* two such operators are unitarily equivalent if and only if the corresponding metrics are equal everywhere on Ω. Actually they considered $\mathcal{K}_T(w) = -[\rho(w)]^2$, which is the curvature of the Hermitian holomorphic vector bundle over Ω^* whose fibre over each point w is $\mathrm{Ker}(T-w)$, and proved the theorem using techniques of differential geometry; but for our purposes we want to consider ρ as a metric. It can be calculated as in (0.1) for k_w any anti-holomorphic cross-section of the bundle.

Even if Ω is a maximal domain of holomorphy for $\mathcal{H}$, it is possible for all functions in $\mathcal{H}$ to tend to a limit along certain paths approaching $\partial\Omega$ (in operator theoretic terms, this implies that M_z^* also has eigenvalues on $\partial\Omega$). For example, in [8] it is shown that if Ω is the unit disk $\mathbb{D}(0,1)$ minus a sequence of little disks $\mathbb{D}(a^n, r_n)$, for some $0 < a < 1$, and $\mathcal{H}$ is the Bergman space on Ω, *i.e.* all analytic functions f for which $\int_\Omega |f(z)|^2 dz \wedge d\bar{z} < \infty$, then $\lim_{r\to 0^-} f(r)$ exists for all f in $\mathcal{H}$ if and only if $\sum_{n=0}^{\infty} \frac{1}{a^{2n} log \frac{1}{r_n}} < \infty$.

Let $\gamma : [0,1] \to \overline{\Omega}$ be a simple arc, with $\gamma([0,1))$ contained in Ω, and $\gamma(1)$ a point on the boundary of Ω. We are interested in the question of when

$$\lim_{s\to 1^-} f(\gamma(s)) \tag{0.2}$$

exists for every f in $\mathcal{H}$, and how this is related to the metric ρ given by (0.1).

As we are considering $\mathcal{H}$ to be a space of holomorphic functions, evaluating the derivative of a function is also a continuous linear functional. Define the function k_w^1 by $\langle f, k_w^1\rangle = f'(w)$. Then Equation (0.1) becomes, after a routine calculation,

$$[\rho(w)]^2 = \frac{\|k_w\|^2 \|k_w^1\|^2 - |\langle k_w^1, k_w\rangle|^2}{\|k_w\|^4}. \tag{0.3}$$

This is often easier to calculate.

The Hardy space for the unit disk, H^2, is the Hilbert space of analytic functions on the disk for which the norm, given by

$$\| \sum_{n=0}^{\infty} a_n z^n \|_{H^2}^2 = \sum_{n=0}^{\infty} |a_n|^2$$

is finite. The Bergman space $L^2_a(\mathbf{D})$ is the space with norm given by

$$\| \sum_{n=0}^{\infty} a_n z^n \|^2_{L^2_a(\mathbf{D})} = \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1}.$$

A calculation shows that for the Hardy space, the metric is

$$\rho_{H^2}(w) = \frac{1}{(1-|w|^2)},$$

and for the Bergman space it is

$$\rho_{L^2_a(\mathbf{D})}(w) = \frac{\sqrt{2}}{(1-|w|^2)}.$$

It is clear that the limit $\lim_{r\to 1^-} f(r)$ does not exist for every f in either of these spaces, as they both contain unbounded functions, and functions that oscillate infinitely often as the boundary is approached.

Let σ denote Lebesgue measure on the unit circle, and P be the orthogonal projection from $L^2(\sigma)$ onto H^2 (so $P(\sum_{-\infty}^{\infty} a_n z^n) = \sum_0^{\infty} a_n z^n$). For a in $L^\infty(\sigma)$, the *Toeplitz operator* $T_a : H^2 \to H^2$ is defined by $T_a(f) = P(af)$. The space $\mathcal{M}(a)$ is defined to be the range of T_a, with an inner product that makes T_a into a partial isometry, i.e. if f_1 and f_2 are orthogonal to the kernel of T_a, then

$$\langle T_a f_1, T_a f_2 \rangle_{\mathcal{M}(a)} = \langle f_1, f_2 \rangle_{H^2}.$$

Now if a is actually analytic, then in $\mathcal{M}(\bar{a})$ the kernel function is

$$k_w^{\mathcal{M}(\bar{a})} = T_{\bar{a}} \frac{a(z)}{1-\bar{w}z},$$

and

$$\|k_w^{\mathcal{M}(\bar{a})}\|^2 = \int \frac{|a(z)|^2}{|1-\bar{w}z|^2} d\sigma(z).$$

Let $a(z) = 1-z$. Then $\lim_{r\to 1^-} f(r)$ exists for every f in $\mathcal{M}(\bar{a})$: indeed the Fourier series of every function in $\mathcal{M}(\bar{a})$ actually converges at 1, not just in the sense of Abel. A calculation yields that

$$[\rho_{\mathcal{M}(\bar{a})}(w)]^2 = \frac{1}{(1-|w|^2)^2} - \frac{1}{(1-|w|^2+|1-w|^2)^2}$$

so as r tends to 1 radially,

$$\rho_{\mathcal{M}(\bar{a})}(r) = \frac{1}{2\sqrt{1-r}} + o(\frac{1}{\sqrt{1-r}}).$$

Notice that instead of growing like the reciprocal of the distance to the boundary, the metric is growing only like the square root of the reciprocal; this leads one to suspect that the existence of limits along some path terminating at 1 may be related to the growth of the metric along that path.

The situation, however, is more complex, for the following reason. Suppose $\mathcal{H}$ is a space for which the limit $\lim_{r\to 1^-} f(r)$ always exists, and such that, for some function g, this limit is non-zero. Let $G(z) = \frac{1}{1-z}$, and consider the space $\mathcal{H}_1 = G\mathcal{H}$, with inner product $\langle Gf_1, Gf_2\rangle_{\mathcal{H}_1} = \langle f_1, f_2\rangle_{\mathcal{H}}$. The metric $\rho_{\mathcal{H}_1}(w) = \rho_{\mathcal{H}}(w)$; but $Gg(r)$ is unbounded as r tends to 1. Notice too that if one considers $\frac{1}{G}H^2$, then every function in the space will have a radial limit at 1, but this limit will always be zero.

The question of interest, then, is when a given holomorphic Hilbert space $\mathcal{H}$ can be rescaled by some holomorphic G so that in $G\mathcal{H}$ limits exist and are not always zero. This is also the right question from the point of view of operator theory, for multiplication by z on $\mathcal{H}$ and $G\mathcal{H}$ are unitarily equivalent (see Section 1 below). If $k_{\gamma(r)}$ converge weakly to a non-zero function k_1, then k_1 will be an eigenvector of T, the adjoint of multiplication by z, of eigenvalue $\overline{\gamma(1)}$. So the existence of the limit (0.2) implies that T has an eigenvalue on the boundary of Ω^*.

Our principal results are the following:

Theorem 2.2 *Let $\mathcal{H}$ be a holomorphic Hilbert space on Ω, with kernel function $k(z, w)$, and such that only a finite number of common zeroes of $\mathcal{H}$ lie on the path $\{\gamma(r) : 0 \leq r < 1\}$. Then $\mathcal{H}$ can be rescaled by a non-vanishing holomorphic function G so that $\Lambda : f \mapsto \lim_{r\to 1^-} f(r)$ is a non-zero bounded linear functional on $G\mathcal{H}$ if and only if, for every a in Ω except for at most a discrete set,*

$$\lim_{r\to 1^-} \lim_{s\to 1^-} \frac{k(\gamma(r), \gamma(s))}{k(a, \gamma(s))k(\gamma(r), a)} \|k_a\|^2$$

exists and is non-zero, and

$$\limsup_{r\to 1^-} \frac{k(\gamma(r), \gamma(r))}{|k(a, \gamma(r))|^2} \|k_a\|^2 < \infty.$$

Theorem 3.1 *Let $\mathcal{H}$ be a reproducing kernel Hilbert space of holomorphic functions on Ω, with only finitely many common zeroes on γ. Suppose that $\int_0^1 \rho_{\mathcal{H}}(\gamma(r))|\gamma'(r)|dr < \infty$. Then there exists a holomorphic function G on Ω such that in the space $G\mathcal{H}$ the limit $\lim_{r\to 1^-} f(\gamma(r))$ always exists and is not always zero.*

Corollary 3.8 *Let T be in the Cowen-Douglas class $\mathcal{B}_1(\Omega)$, and let λ be a point on the boundary of Ω which is a finite distance from a point in the interior with respect to the metric ρ. Then λ is an eigenvalue of T.*

In Section 4, we apply these results to the de Branges-Rovnyak spaces $\mathcal{H}(b)$. The results in this section are not new, as the problem of existence of boundary limits in these spaces has been solved by Ahern and Clark for b inner, and Sarason in general [1], [13] (see Section 4 for a definition of the spaces, and a discussion). The results do show, however, that our general theorems have non-trivial applications.

In Section 5 we show that the condition of Theorem 3.1 is not necessary for the existence of an eigenvalue of $T \in \mathcal{B}_1(\Omega)$ on the boundary of Ω.

1 Unitary Equivalence

In [5], it is proved that operators in $\mathcal{B}_1(\Omega^*)$ are unitarily equivalent if and only if their curvature functions are equal. This result was generalised to m-tuples of operators on domains in $\mathbb{C}^n$ by R. Curto and N. Salinas, and they gave a more analytic proof; they also described what form the unitary intertwining operators could take [6]. The following result is therefore not new; that (i) and (iv) are equivalent is contained in [5], and that (i) and (iii) are equivalent is contained in [6]. We think it is worthwhile, however, for expository reasons, to include a simple proof that illustrates our ideas.

Theorem 1.1 *Let $\mathcal{H}$ and $\mathcal{F}$ be holomorphic Hilbert spaces on a planar domain Ω, in neither of which all functions vanish at some point. Suppose that both $\mathcal{H}$ and $\mathcal{F}$ are invariant under multiplication by the coordinate function, denoted by $S_{\mathcal{H}}$ (respectively $S_{\mathcal{F}}$). Suppose, moreover, that $\mathcal{H}$ (or $\mathcal{F}$) has the property that if a function $f(z)$ in the space vanishes at some point w in Ω, then $(f(z) - f(w))/(z - w)$ is also in the space.*

Let $k(z, w)$ be the kernel function for $\mathcal{H}$, and $j(z, w)$ be the kernel function for $\mathcal{F}$. Then the following are equivalent:
(i) There exists a unitary operator $U : \mathcal{H} \to \mathcal{F}$ such that $US_{\mathcal{H}}U^* = S_{\mathcal{F}}$.
(ii) There exists a non-vanishing holomorphic function G on Ω such that $U(f) := Gf$ for all f in $\mathcal{H}$ defines a unitary operator $U : \mathcal{H} \to \mathcal{F}$ such that $US_{\mathcal{H}}U^* = S_{\mathcal{F}}$.
(iii) There exists a non-vanishing holomorphic function G on Ω such that

$$j(z, w) = \overline{G(w)}G(z)k(z, w).$$

(iv) $\rho_{\mathcal{H}}(w) = \rho_{\mathcal{F}}(w)$ for all w in Ω.

PROOF: (i) $\Rightarrow$ (ii): By hypothesis, any function in $\mathcal{H}$ orthogonal to k_w must be in the range of $S_{\mathcal{H}} - w$, so the kernel of $(S_{\mathcal{H}} - w)^*$ is precisely the one-dimensional space spanned by k_w.

Define G by

$$Uk_w = \frac{1}{\overline{G}(w)} j_w.$$

As U must map the kernel of $(S-w)^*_{\mathcal{H}}$ onto the kernel of $(S-w)^*_{\mathcal{F}}$, k_w must get mapped to a non-zero multiple of j_w, so G is well-defined. Moreover, as

$$\langle f, k_w\rangle_{\mathcal{H}} = \frac{1}{G(w)}\langle Uf, j_w\rangle_{\mathcal{F}} = \frac{(U(f))(w)}{G(w)},$$

it follows that $U(f) = Gf$ for all f in $\mathcal{H}$, and hence G is holomorphic.

(ii) $\Rightarrow$ (iii):

$$f(w) = \langle f(z), k_w(z)\rangle_{\mathcal{H}} = \langle G(z)f(z), G(z)k_w(z)\rangle_{\mathcal{F}}$$

and

$$f(w) = \langle G(z)f(z), \frac{1}{\overline{G(w)}} j_w(z)\rangle_{\mathcal{F}},$$

so $j(z,w) = \overline{G(w)}G(z)k(z,w)$ as required.

(iii) $\Rightarrow$ (iv): As G does not vanish, $\log|G(w)|$ is harmonic, so (0.1) will be the same for both spaces.

(iv) $\Rightarrow$ (i): There is a harmonic function h such that

$$\|j_w\|^2 = e^{2h(w)}\|k_w\|^2.$$

We would like to write $e^{h(w)}$ as the modulus of a holomorphic function, but there might be a problem if Ω is not simply connected, so we shall do it locally. Let $\mathbb{D}$ be a disk contained in Ω, centred at 0. On $\mathbb{D}$, h has a harmonic conjugate ${}^*h$; let $G = e^{h+i{}^*h}$. Define $U : \mathcal{H} \to \mathcal{F}$ by

$$U(\sum_{n=1}^{N} a_n k_{w_n}) = \sum_{n=1}^{N} a_n \frac{1}{\overline{G}(w_n)} j_{w_n} \tag{1.2}$$

for $w_1, \ldots, w_N$ in $\mathbb{D}$. As Ω is connected, U is densely defined; moreover for w, z in $\mathbb{D}$

$$\langle Uk_w, Uk_z\rangle_{\mathcal{F}} = \langle k_w, k_z\rangle_{\mathcal{H}}. \tag{1.3}$$

(This holds because both $j(z,w)$ and $\overline{G(w)}G(z)k(z,w)$ are holomorphic in z and anti-holomorphic in w; as they agree on the diagonal of $\mathbb{D}\times\mathbb{D}$ by (1.2), they must be identically equal, which implies (1.3)). Therefore U extends by continuity to a unitary operator; and, as $\langle (S_{\mathcal{F}}U - US_{\mathcal{H}})k_w, j_z\rangle = 0$, U intertwines $S_{\mathcal{F}}$ and $S_{\mathcal{H}}$ as desired. $\square$

Remark 1: One can also consider spaces that have common zeroes, *e.g.* zH^2. In this case one must modify the definition (0.1) by subtracting off the point masses that one would get at the zeroes, and work with the absolutely continuous part of the Laplacian. One must then modify the preceding theorem to allow G to have zeroes at the common zeroes of $\mathcal{F}$, and poles at the common zeroes of $\mathcal{H}$. (By a *common zero of* $\mathcal{H}$ we mean a point a in Ω such that every function in $\mathcal{H}$ vanishes at a (so $k_a = 0$).)

Remark 2. In general, a holomorphic Hilbert space need not be invariant under multiplication by z. If not, (iii) and (iv) of Theorem 1.1 are still equivalent, as is (ii) if it is required merely that multiplication by G is a unitary operator from $\mathcal{H}$ onto $\mathcal{F}$. These equivalent conditions seem to be the right notion of equivalence for general holomorphic Hilbert spaces.

2 Rescaling

A space can always be rescaled (*i.e.* multiplied by a holomorphic function G as in Theorem 1.1) so that the limit (0.2) always exists, by choosing G to vanish sufficiently rapidly. In this section we consider when it can be rescaled so that the limit is non-zero.

We shall use the following theorem of A. Nersejan on approximation by holomorphic functions [9]; if the curve γ is a line segment, this result goes back to T. Carleman [3].

Theorem 2.1 [9] *Let D be a proper subdomain of the extended complex plane, and let γ be a continuous injective map from $[0,1]$ into $\bar{D}$, with $\gamma([0,1))$ contained in D and $\gamma(1) \in \partial D$. Let $\varepsilon(r)$ be a positive continuous function on $[0,1)$, and let $f(r)$ be an arbitrary continuous function on $[0,1)$. Then there exists a function g holomorphic on D such that $|f(r) - g(\gamma(r))| < \varepsilon(r)$ for all r in $[0,1)$.*

We can give necessary and sufficient for non-zero boundary limits to exist in a rescaled space:

Theorem 2.2 *Let $\mathcal{H}$ be a holomorphic Hilbert space on Ω, with kernel function $k(z,w)$, and such that only a finite number of common zeroes of H lie on the path $\{\gamma(r) : 0 \le r < 1\}$. Then $\mathcal{H}$ can be rescaled by a non-vanishing holomorphic function G so that $\Lambda : f \mapsto \lim_{r\to1^-} f(\gamma(r))$ is a non-zero bounded linear functional on $G\mathcal{H}$ if and only if, for every a in Ω except for at most a discrete set,*

$$\lim_{r\to1^-} \lim_{s\to1^-} \frac{k(\gamma(r),\gamma(s))}{k(a,\gamma(s))k(\gamma(r),a)} \|k_a\|^2 \tag{2.3}$$

exists and is non-zero, and

$$\limsup_{r\to 1^-} \frac{k(\gamma(r),\gamma(r))}{|k(a,\gamma(r))|^2}\|k_a\|^2 < \infty. \tag{2.4}$$

PROOF: (Necessity) Suppose there does exist a holomorphic function G such that in the space $G\mathcal{H}$ the kernel functions $j_{\gamma(r)}$ converge weakly to a non-zero function, j_1. Choose a so that $\langle j_a, j_1\rangle \neq 0$ (this function of a can vanish on at most a discrete set). Then

$$\begin{aligned}
\lim_{r\to 1^-}\lim_{s\to 1^-} \frac{k(\gamma(r),\gamma(s))}{k(a,\gamma(s))k(\gamma(r),a)}\|k_a\|^2 &= \lim_{r\to 1^-}\lim_{s\to 1^-} \frac{j(\gamma(r),\gamma(s))}{j(a,\gamma(s))j(\gamma(r),a)}\|j_a\|^2 \\
&= \frac{\|j_1\|^2}{|\langle j_a, j_1\rangle|^2}\|j_a\|^2 \\
&\neq 0.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\limsup_{r\to 1^-} \frac{k(\gamma(r),\gamma(r))}{|k(a,\gamma(r))|^2}\|k_a\|^2 &= \limsup_{r\to 1^-} \frac{j(\gamma(r),\gamma(r))}{|j(a,\gamma(r))|^2}\|j_a\|^2 \\
&= \frac{\|j_a\|^2}{|\langle j_a, j_1\rangle|^2}\limsup_{r\to 1^-}\|j_{\gamma(r)}\|^2 \\
&< \infty.
\end{aligned}$$

(Sufficiency) Define $G_1(z) = \frac{\|k_a\|}{k(z,a)}$. Then

$$\|j_{\gamma(r)}\|^2 = \frac{\|k_a\|^2 k(\gamma(r),\gamma(r))}{|k(a,\gamma(r))|^2},$$

so from (2.4) it follows that $\{\|j_{\gamma(r)}\|^2\}$ is bounded. Let j_1 be the weak-limit of some sequence $\{j_{\gamma(s_n)}\}$ with s_n increasing to 1 (such a j_1 exists because the unit ball of $G_1\mathcal{H}$ is weakly compact). Now, j_1 is well-defined, because it follows from (2.3) that, for r sufficiently close to 1,

$$\lim_{s\to 1^-}\langle j_{\gamma(r)}, j_{\gamma(s)}\rangle$$

exists and so

$$\langle j_{\gamma(r)}, j_1\rangle$$

does not depend on the choice of $\{s_n\}$, and linear combinations of the $j_{\gamma(r)}$ are dense in $G_1\mathcal{H}$. Moreover, the fact that (2.3) is non-zero means that j_1 is non-zero.

The problem is that G_1 may not be holomorphic, because $k(z,a)$ can have zeroes in Ω. Let H_1 be a holomorphic function on Ω which has zeroes only at the poles of G_1, and with the same multiplicities, so that G_1H_1 is a non-vanishing holomorphic function on Ω (such an H_1 exists by Weierstrass' theorem). The existence of the limit (2.3) means that there exists some $\delta > 0$ such that for $1-\delta < r < 1$, $k(\gamma(r), a) \neq 0$ (here we are using the fact that there are no common zeroes of H on the tail-end of γ). Therefore H_1 is non-zero on the end of the path, so we can apply Theorem 2.1 to find a holomorphic function H_2 with $|H_1(\gamma(r))H_2(\gamma(r))-1| < 1-r$ for r close to 1. Define G to be $G_1H_1H_2$. This is holomorphic on Ω, and as $G(\gamma(r))$ is close to $G_1(\gamma(r))$, the same argument that proved limits exist in $G_1\mathcal{H}$ applies to $G\mathcal{H}$. $\square$

Remark 1. If there is a point a such that $\lim_{r\to 1^-} k(\gamma(r), a)$ exists and is not zero, then rescaling has no effect: either limits already exist before rescaling, or non-zero limits can never be attained by rescaling.

Remark 2. The quantities in (2.3) and (2.4) are invariant under the equivalence relation of Theorem 1.1, and therefore are expressible in terms of the metric $\rho_{\mathcal{H}}$. It would be nice to have a version of Theorem 2.2 in which necessary and sufficient conditions are given explicitly in terms of the metric.

Remark 3. One might think that condition (2.3) should imply (2.4), for j_1 is defined by its inner product with $j_{\gamma(r)}$ for r close to 1, linear combinations of which are dense; and then $\|j_1\|^2$ is

$$\lim_{r\to 1^-} \lim_{s\to 1^-} j(\gamma(r), \gamma(s))$$

which is the value of (2.3). This presupposes, however, that the norm of j_1 as a linear functional is finite, and it need not be, as the following example shows.

Let

$$k(z,w) = \frac{(1-\bar{w})^{\frac{1}{3}}(1-z)^{\frac{1}{3}}}{1-\bar{w}z} + 1.$$

This is positive definite, so is the reproducing kernel for some holomorphic Hilbert space on the unit disk [2]. Letting $a = 0$, $\gamma(r) = r$, then (2.3) becomes

$$\begin{aligned}
\lim_{r\to 1^-} \lim_{s\to 1^-} \quad & 2\left(\frac{(1-r)^{\frac{1}{3}}(1-s)^{\frac{1}{3}}}{1-rs} + 1\right)\left(\frac{1}{(1-s)^{\frac{1}{3}}+1}\right)\left(\frac{1}{(1-r)^{\frac{1}{3}}+1}\right) \\
&= \lim_{r\to 1^-} 2\left(\frac{1}{(1-r)^{\frac{1}{3}}+1}\right) \\
&= 2\,;
\end{aligned}$$

whereas (2.4) becomes

$$\lim_{r\to 1^-} 2\left(\frac{(1-r)^{\frac{2}{3}}}{1-r^2} + 1\right)\left(\frac{1}{(1-r)^{\frac{1}{3}}+1}\right)^2 = \infty\,.$$

3 Main Theorem

Here is our principal result.

Theorem 3.1 *Let $\mathcal{H}$ be a reproducing kernel Hilbert space of holomorphic functions on Ω, and assume there are only a finite number of common zeroes of $\mathcal{H}$ on the path γ. If $\int_0^1 \rho(\gamma(r))|\gamma'(r)|dr < \infty$, then there exists a holomorphic function G on Ω such that in the space $G\mathcal{H}$ the limit $\lim_{r\to 1^-} f(\gamma(r))$ always exists and is not always zero.*

PROOF: (i) Suppose first that there exists $c > 0$ such that the angle between $k_{\gamma(r)}$ and $k^1_{\gamma(r)}$ is greater than c, for $r \in [0,1)$. This extra hypothesis means that $(\|k^1_{\gamma(r)}\|/\|k_{\gamma(r)}\|)|\gamma'(r)|$ is integrable, and then we can prove the theorem without rescaling.

For simplicity, assume γ is parametrized by arc-length, so $|\gamma'(r)|$ is identically 1. Choose $a < 1$ so that

$$\int_a^1 \frac{\|k^1_{\gamma(r)}\|}{\|k_{\gamma(r)}\|} dr \leq \frac{1}{2}.$$

Then

$$k_{\gamma(r)}(\zeta) = \int_{\gamma(a)}^{\zeta} \langle k_{\gamma(r)}, k^1_w \rangle dw + k_{\gamma(r)}(\gamma(a)),$$

so

$$k_{\gamma(r)}(\gamma(r)) \leq \|k_{\gamma(r)}\| (\int_{\gamma(a)}^{\gamma(r)} \|k^1_w\| dw + \frac{|k_{\gamma(r)}(\gamma(a))|}{\|k_{\gamma(r)}\|}),$$

and

$$1 \leq \int_a^r \frac{\|k^1_{\gamma(s)}\|}{\|k_{\gamma(s)}\|} \frac{\|k_{\gamma(s)}\|}{\|k_{\gamma(r)}\|} ds + \frac{|\langle k_{\gamma(r)}, k_{\gamma(a)} \rangle|}{\|k_{\gamma(r)}\|^2}. \tag{3.2}$$

Suppose that $\{\|k_{\gamma(r)}\|\}$ is not bounded. Choose r_n tending to 1 so that $\|k_{\gamma(r_n)}\| \geq \sup\{\|k_{\gamma(w)}\| : 0 \leq w < r_n\}$. It follows from (3.2) that

$$\begin{aligned} 1 &\leq \int_a^{r_n} \frac{\|k^1_{\gamma(s)}\|}{\|k_{\gamma(s)}\|} ds + \frac{\|k_{\gamma(a)}\|}{\|k_{\gamma(r_n)}\|} \\ &\leq \frac{1}{2} + \frac{\|k_{\gamma(a)}\|}{\|k_{\gamma(r_n)}\|}, \end{aligned}$$

and hence that

$$\|k_{\gamma(r_n)}\| \leq 2\|k_{\gamma(a)}\|.$$

As a is fixed, this yields a contradiction, so $\{\|k_{\gamma(r)}\| : r \in [0,1)\}$ is bounded.

It follows that for any sequence of points $\{r_n\}$ there is a subsequence $\{r_{n_k}\}$ such that the functions $k_{\gamma(r_{n_k})}$ converge weakly; we must show that if for two different sequences $\{r_n\}$ and $\{s_n\}$ both converging to 1, the kernel functions $k_{\gamma(r_n)}$ and $k_{\gamma(s_n)}$ both converge weakly, then they converge to the same limit. Deny. Then there is some function f such that, for n large, $|f(\gamma(r_n)) - f(\gamma(s_n))| \geq 1$. For convenience, assume $r_n < s_n$; then

$$1 \leq |\int_{r_n}^{s_n} \langle f, k^1_{\gamma(r)} \rangle dr| \leq \|f\| \int_{r_n}^{s_n} \|k^1_{\gamma(r)}\| dr. \tag{3.3}$$

But because $\{\|k_{\gamma(r)}\|\}$ is bounded, $\|k^1_{\gamma(r)}\|$ is integrable, so the right-hand side of (3.3) tends to zero as r_n tends to 1, yielding a contradiction.

It remains to show that $k_{\gamma(r)}$ does not tend weakly to zero. Suppose it does. Then

$$0 = \int_r^1 \langle k_{\gamma(r)}, k^1_{\gamma(s)} \rangle ds + \|k_{\gamma(r)}\|^2, \tag{3.4}$$

so

$$\|k_{\gamma(r)}\| \leq \int_r^1 \|k^1_{\gamma(s)}\| ds.$$

As $\|k^1_{\gamma(r)}\|$ is integrable, it follows that $k_{\gamma(r)}$ tends to zero in norm.

Now, choose r_n tending to 1 so that $\|k_{\gamma(r_n)}\| \geq \sup\{\|k_{\gamma(w)}\| : r_n \leq w < 1\}$. Then (3.4) applied to r_n gives

$$\begin{aligned} 1 \quad &\leq \quad \int_{r_n}^1 \frac{\|k^1_{\gamma(s)}\|}{\|k_{\gamma(s)}\|} \frac{\|k_{\gamma(s)}\|}{\|k_{\gamma(r_n)}\|} ds \\ &\leq \quad \int_{r_n}^1 \frac{\|k^1_{\gamma(s)}\|}{\|k_{\gamma(s)}\|} ds, \end{aligned}$$

yielding a contradiction. Therefore the weak-limit of $k_{\gamma(r)}$ is non-zero, as desired.

(ii) If one rescales by G, so $j(z,w) = \overline{G(w)} G(z) k(z,w)$, one does not change the metric, but one can change the angle between j^1_w and j_w. Letting $\xi = \frac{\overline{G'(w)}}{\overline{G(w)}}$, a calculation yields

$$\frac{|\langle j^1_w, j_w \rangle|^2}{\|j^1_w\|^2 \|j_w\|^2} = \frac{\|k_w\|^2 |\xi|^2 + 2\Re(\xi \langle k_w, k^1_w \rangle) + |\langle k^1_w, \frac{k_w}{\|k_w\|} \rangle|^2}{\|k_w\|^2 |\xi|^2 + 2\Re(\xi \langle k_w, k^1_w \rangle) + \|k^1_w\|^2}. \tag{3.5}$$

Let $F(\xi)$ denote the right-hand side of (3.5). Then F is continuous, and vanishes at

$$\xi_w = -\frac{\langle k^1_w, k_w \rangle}{\|k_w\|^2}.$$

It follows that there is a positive continuous function $\varepsilon(r)$ such that if $|\xi - \xi_{\gamma(r)}| < \varepsilon(r)$ then, for $w = \gamma(r)$, $F(\xi) < \frac{1}{2}$. By Theorem 2.1, there is a function g holomorphic on $\mathbb{C}_\infty \setminus \{1\}$ such that

$$|g(r) + \frac{\langle k_{\gamma(r)}, k^1_{\gamma(r)} \rangle}{\|k_{\gamma(r)}\|^2}| < \varepsilon(r)$$

on $[0,1)$. Define $G(w) = exp \int_0^w g(z)dz$. Then in $G\mathcal{H}$, the angle between $j_{\gamma(r)}$ and $j^1_{\gamma(r)}$ is at least $\frac{\pi}{4}$, so Case (i) can be applied. □

Remark 1: Equation 0.1 is the standard way of defining the square of the metric associated with a reproducing kernel (see *e.g.* [7]). The hypothesis that ρ be in L^1 then says that the path $[0,1)$ has finite length. The rescaling makes functions Lipschitz with respect to the metric along $[0,1)$, and the limit of a Lipschitz function along a path of finite length must exist. Then the fact that the functions k_r are themselves Lipschitz allows us to prove that they converge to a non-zero function.

Remark 2: If $\mathcal{H}$ does have an infinite number of common zeroes on γ accumulating at $\gamma(1)$, one must rescale by a meromorphic function.

Actually, the kernel functions converge not only weakly, but also in norm:

Corollary 3.6 *If $\int_0^1 \rho_{\mathcal{H}}(\gamma(r))|\gamma'(r)|dr < \infty$, then there exists a holomorphic function G on Ω such that in the space $G\mathcal{H}$ the kernel functions $j_{\gamma(r)}$ converge in norm to a non-zero function.*

PROOF: We can assume, as in the preceding theorem, that the space has been rescaled so that the angle between $j_{\gamma(r)}$ and $j^1_{\gamma(r)}$ is bounded away from zero, and hence that $\|j_{\gamma(r)}\|$ is bounded and $\|j^1_{\gamma(r)}\|$ is integrable. Let j_1 be the weak limit of $j_{\gamma(r)}$; to prove it is also the norm limit, it suffices to prove that $\lim_{r\to 1^-} \|j_{\gamma(r)}\| = \|j_1\|$. But

$$\begin{aligned} \lim_{r\to 1^-} |\|j_{\gamma(r)}\|^2 - \|j_1\|^2| &= \lim_{r\to 1^-} |\int_r^1 \langle j_{\gamma(r)}, j^1_{\gamma(w)}\rangle dw| \\ &\leq \lim_{r\to 1^-} \|j_{\gamma(r)}\| \int_r^1 \|j^1_{\gamma(w)}\| dw, \end{aligned} \tag{3.6}$$

and (3.6) tends to zero, as desired. □

It is possible, in general, that kernel functions can converge weakly but not in norm. Let

$$k(z,w) = \frac{\sqrt{1-\bar{w}}\sqrt{1-z}}{1-\bar{w}z} + 1. \tag{3.7}$$

This is a kernel function for some holomorphic Hilbert space on the disk. By Theorem 2.2, with $a = 0$, k_r converges weakly to k_1 as r increases to 1. But

$$\|k_1\|^2 = \lim_{r\to 1^-} \lim_{s\to 1^-} \frac{\sqrt{1-r}\sqrt{1-s}}{1-rs} + 1 = 1,$$

and

$$\lim_{r\to 1^-} \|k_r\|^2 = \lim_{r\to 1^-} \frac{|1-r|}{1-|r|^2} + 1 = \frac{3}{2},$$

so k_r does not converge in norm.

This example also shows that the integrability of the metric is not a necessary condition for the kernel functions to converge weakly to a non-zero function. However the space corresponding to the kernel function (3.7) is not z-invariant (it corresponds to the space $\{\sqrt{1-z}f + \lambda \,:\, f \in H^2, \lambda \in \mathbb{C}\}$). In Section 5, an example is given of a z-invariant space that does have a bounded point evaluation on the boundary that is not a finite distance from the interior. (By z-invariant we mean invariant under multiplication by the coordinate function, so it is a model space for some operator in the Cowen-Douglas class).

Theorem 3.1 can be reformulated in the terminology of operator theory. The metric ρ enables one to define the distance between any two points a, b in $\bar{\Omega}$ as the infimum, over all paths $\gamma : [0,1] \to \bar{\Omega}$ which map 0 to a, 1 to b, and $(0,1)$ into Ω, of

$$\int_0^1 \rho(\gamma(r))|\gamma'(r)|dr.$$

Then Theorem 3.1 gives:

Corollary 3.8 *Let T be in the Cowen-Douglas class $\mathcal{B}_1(\Omega)$, and let λ be a point on the boundary of Ω which is a finite distance from a point in the interior with respect to the metric ρ. Then λ is an eigenvalue of T.*

PROOF: Represent T as the adjoint of multiplication by the coordinate function on some holomorphic Hilbert space $\mathcal{H}$, and choose a path γ of finite length (with respect to ρ) that maps $[0,1)$ into Ω and maps 1 to λ. By Corollary 3.6, in some space $G\mathcal{H}$, the functions $j_{\gamma(r)}$ converge in norm to a non-zero function j_1. As $(S_{G\mathcal{H}} - \gamma(r))^* j_{\gamma(r)} = 0$, it follows that $(S_{G\mathcal{H}} - \lambda)^* j_1 = 0$. Then $(1/G)j_1$ is an eigenvector of T with eigenvalue λ. □

This can be contrasted with the result of Cowen and Douglas (in the proof of Proposition 1.29 in [5]) that if $\partial\Omega$ is C^1 and $\overline{\Omega}$ is a spectral set for T, then

$$\rho(w) \geq \frac{C}{dist(w, \partial\Omega)}.$$

Note that if $\overline{\Omega}$ is a spectral set for T, then no boundary point of Ω can be an eigenvalue; this is explained in [4].

4 An application: de Branges-Rovnyak Spaces

For b a function that is holomorphic in the unit disk and with modulus bounded by 1, the de Branges-Rovnyak space $\mathcal{H}(b)$ is defined as $Ran(1 - T_b T_b^*)^{1/2}$, with an inner product which makes $(1 - T_b T_b^*)^{1/2}$ an isometry from H^2 onto $\mathcal{H}(b)$. For more information about these spaces, see the book by D. Sarason [14], or his papers [12, 11]. When the limit $\lim_{r\to 1^-} f(r)$ exists for every function f in $\mathcal{H}(b)$ was investigated by P. Ahern and D. Clark, in the case that b is inner ($|b| = 1 \quad \sigma - a.e.$, so $\mathcal{H}(b)$ is in this case just $H^2 \ominus bH^2$, a backward shift invariant subspace of H^2), and by Sarason in general [1], [13]. The necessary and sufficient condition they found is that b have an angular derivative in the sense of Carathéodory at the point 1, which is equivalent to requiring that

$$\lim_{r\to 1^-} \frac{1 - |b(r)|}{1 - r} = c < \infty. \tag{4.1}$$

In $\mathcal{H}(b)$ the kernel function

$$k_{\mathcal{H}(b)}(z, w) = \frac{1 - \overline{b(w)}b(z)}{1 - \overline{w}z}.$$

Let us apply Theorem 2.2, taking $a = 0$, so $k_0(z) = 1 - \overline{b(0)}b(z)$. As $|b(0)| < 1$ except in the trivial case, and condition (4.1) implies that $\lim_{r\to 1^-} b(r)$ exists and has unit modulus [14, p. 48], rescaling has no effect; so from Theorem 2.2 limits exist if and only if

$$\lim_{r\to 1^-} \lim_{s\to 1^-} \frac{1 - \overline{b(r)}b(s)}{1 - \bar{r}s}$$

exists, and

$$\limsup_{r\to 1^-} \frac{1 - |b(r)|^2}{1 - r^2} < \infty.$$

The existence of these limits can be fairly easily shown to be equivalent to the existence of the limit (4.1).

Now consider the Bergman space version of $\mathcal{H}(b)$. Let A be area measure on the disk; the Bergman space $L_a^2(\mathbf{D})$ is the set of holomorphic f for which $\int |f|^2 dA < \infty$. Let P' be the orthogonal projection from $L^2(A)$ onto $L_a^2(\mathbf{D})$, and define $T'_m : L_a^2(\mathbf{D}) \to L_a^2(\mathbf{D})$ by $T'_m(f) = P'(mf)$. Let $\mathcal{H}'(b)$ be $Ran(1 - T'_b T_b'^*)^{1/2}$, with an inner product which makes $(1 - T'_b T_b'^*)^{1/2}$ an isometry. The kernel for $\mathcal{H}'(b)$ is

$$k(z, w) = \frac{1 - \overline{b(w)}b(z)}{(1 - \overline{w}z)^2}.$$

Again letting $a = 0$, we see by Schwarz' Lemma that the space can never be rescaled to allow non-vanishing boundary limits.

Now consider Theorem 3.1 applied to the spaces $\mathcal{H}(b)$. The metric for $\mathcal{H}(b)$ is given by

$$[\rho_{\mathcal{H}(b)}(w)]^2 = \frac{1}{1 - |w|^2} \frac{1}{1 - |b(w)|^2} \{ \frac{1 - |b(w)|^2}{1 - |w|^2} - \frac{1 - |w|^2}{1 - |b(w)|^2} |b'(w)|^2 \} \tag{4.2}$$

If b has an angular derivative in the sense of Carathéodory at 1, then

$$\lim_{w \to 1^-} |b'(w)| = \lim_{w \to 1^-} \frac{1 - |b(w)|^2}{1 - |w|^2},$$

so the term in braces in (4.2) tends to zero. If b is in $C^2[0,1]$, the term in braces vanishes to at least first order, so

$$\rho_{\mathcal{H}(b)}(w) = O(\frac{1}{\sqrt{1 - |w|^2}})$$

and is clearly integrable. (Without some extra assumption on b, such as being C^2 on the radius, it is hard to estimate the growth of ρ.)

5 An Example

We give an example of a naturally occurring z-invariant space in which $\lim_{r \to 1^-} f(r)$ exists for every f, but for which $\int_{\frac{1}{2}}^{1} \rho(r) dr = \infty$.

Example 5.1

Let Ω be the disk $\mathbb{D}(1,1)$ minus the disks $\mathbb{D}(a_n, r_n)$, where $a_n = 1 + \frac{1}{2^n}$, and $r_n = exp(-(n^2 4^n))$. Let $\mathcal{H}$ be the Bergman space for Ω. It follows from [8] that the norm of the functional $f \mapsto \lim_{r \to 1^-} f(r)$ is comparable to

$$\sum_{n=1}^{\infty} \frac{1}{(1 - a_n)^2 \log \frac{1}{r_n}} = \sum_{n=1}^{\infty} \frac{1}{4^{-n} 4^n n^2} < \infty,$$

and so

$$\limsup_{x \to 0^+} \|k_{(1-x)}\| < \infty \ . \tag{5.2}$$

Lemma 5.3 *For x small and positive,*

$$\|k^1_{(1-x)}\| \geq C \frac{1}{x \log \frac{1}{x}}.$$

PROOF: Choose a positive integer m so that x is in the interval $[2^{-(m+1)}, 2^{-m}]$. Let $f(z) = \frac{1}{z - a_m}$. Then

$$\|f\| \approx \sqrt{\log \frac{1}{r_m}} = m 2^m.$$

Therefore

$$\begin{aligned} \|k^1_{(1-x)}\| &\geq \frac{|f'(1-x)|}{\|f\|} \\ &\geq C \frac{1}{(2^{-m}+x)^2} \frac{1}{m2^m} \\ &\geq C \frac{1}{x \log \frac{1}{x}}, \end{aligned}$$

as desired. □

Now, we claim that $\rho(r)$ is not integrable on $[\frac{1}{2}, 1]$. Indeed we claim more:

$$\liminf_{x \to 0^+} [x \log \frac{1}{x}]\, \rho(1-x) > 0. \tag{5.4}$$

For suppose (5.4) is false, and there is a sequence x_n decreasing to zero for which the left-hand side of (5.4) tends to zero. For each x, let η_x be the number with modulus

$$|\eta_x| = \frac{|\langle k_{(1-x)}, k^1_{(1-x)} \rangle|}{\|k^1_{(1-x)}\|^2}$$

and argument chosen so that $\langle k_{(1-x)}, \eta_x k^1_{(1-x)} \rangle$ is positive. Let l_x be the vector $k_{(1-x)} - \eta_x k^1_{(1-x)}$, so

$$\|l_x\| = \frac{\|k_{(1-x)}\|^2}{\|k^1_{(1-x)}\|} \rho(1-x) \leq C x \log \frac{1}{x} \rho(1-x)$$

by (5.2) and (5.4). Take the inner product of l_{x_n} with $\frac{1}{z-a_m}$. Then

$$\begin{aligned} \left|\frac{1}{x_n + 2^{-m}} - \eta_{x_n} \frac{1}{(x_n + 2^{-m})^2}\right| &= \left|\langle \frac{1}{z - a_m}, l_{x_n} \rangle\right| \\ &\leq C(m2^m) x_n \log \frac{1}{x_n} \rho(1 - x_n) \end{aligned} \tag{5.5}$$

by Lemma 5.3. Letting n tend to infinity in (5.5), the right-hand side tends to zero, so

$$\lim_{n \to \infty} \eta_{x_n} = 2^{-m}.$$

Now do the same thing with a different m to get a contradiction. □

We do not know if $k_{(1-x)}$ converges in norm in this example. Note that if r_n is reduced to $r_n = exp(-(n^{2+\varepsilon} 4^n))$ for any positive ϵ, then ρ is integrable.

Question 5.6 *Let $\mathcal{H}$ be a z-invariant holomorphic Hilbert space on Ω. Suppose $\rho(1-x) \geq \frac{c}{x}$. Can $k_{(1-x)}$ converge in norm to a non-zero k_1?*

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