Cyclic subnormal operators with finite rank self-commutators

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A bounded linear operator S on a Hilbert space $\mathcal{H}$ is called subnormal if it has a normal extension, i.e. there exists a normal operator N on a superspace of $\mathcal{H}$ which leaves $\mathcal{H}$ invariant and such that $N|_{\mathcal{H}} = S$. The self-commutator of S is

$$[S^*, S] = S^*S - SS^*.$$

The operator S is called cyclic if there exists a vector ξ in $\mathcal{H}$ such that $\{p(S)\xi : p \quad \text{a polynomial}\}$ is dense in $\mathcal{H}$. It is called irreducible if it commutes with no non-trivial projection.

In [5], R. Olin et al. classified all cyclic subnormal operators with finite rank self commutators, by first classifying the irreducible cyclic subnormal operators with this property, and then showing when a finite direct sum of such operators is again cyclic (it will clearly always have finite rank self-commutator). In this note, we present an alternative proof of their theorem.

Let σ denote Lebesgue measure on the unit circle, and let H^∞ be the subspace of $L^\infty(\sigma)$ consisting of functions whose negative Fourier coefficients vanish. It can also be thought of as the non-tangential boundary limits of functions that are bounded and analytic in the disk $\mathbb{D}$. The weak-star topology on H^∞ comes from viewing it as a subspace of the dual space of $L^1(\sigma)$; for this, and general facts about subnormal operators, see the book [2]. A function f in H^∞ is called a weak-star generator if the polynomials in f form a weak-star dense subspace of H^∞; such functions have been characterized in [7].

Any cyclic subnormal operator can be represented as multiplication by the coordinate function on $P^2(\mu)$ for some compactly supported measure μ on $\mathbb{C}$, where $P^2(\mu)$ denotes the

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closure of the polynomials in $L^2(\mu)$. This operator will be denoted S_μ, and its minimal normal extension, multiplication by the coordinate function on $L^2(\mu)$, will be denoted N_μ. By Thomson's theorem [8], any space $P^2(\mu)$ can be decomposed into $L^2(\mu_0) \oplus P^2(\mu_1) \oplus P^2(\mu_2) \oplus \ldots$, where each S_{μ_i} is irreducible (commutes with no non-trivial projections) for $i \geq 1$, and each $P^2(\mu_i)$ has a non-empty open set G_i of bounded point evaluations (points λ such that $|p(\lambda)| \leq M\|p\|_{P^2(\mu_i)}$ for all polynomials p).

A domain Ω is called a quadrature domain if there exist points $z_1, \ldots, z_N$ in Ω and constants $a_{m,n}$ such that for every function f analytic in Ω and area-integrable, the identity

$$\int_\Omega f(z)dz \wedge d\bar{z} = \sum_{n=1}^{N} \sum_{m=0}^{N_n} a_{m,n} f^{(m)}(z_n)$$

holds. We need the following facts about quadrature domains:

(i) Let Ω be a bounded open set in $\mathbb{C}$. Ω is a quadrature domain if and only if there is a function R meromorphic in Ω and continuously extendable to each point of $\partial\Omega$ such that $R(z) = \bar{z}$ on $\partial\Omega$ [1].

(ii) A bounded simply connected domain is a quadrature domain if and only if it is the conformal image of the unit disk under a rational function [1].

(iii) The boundary of a quadrature domain is part of an algebraic curve [1].

(iv) The equation $R(z) = \bar{z}$ has at most finitely many solutions inside Ω [4].

Theorem 1 *Let S be an irreducible cyclic subnormal operator. Then S has finite rank self-commutator if and only if there is a rational function r, bounded in $\mathbb{D}$ and a weak-star generator of H^∞, and a measure ν that is the sum of σ and a finite number of point masses in $\mathbb{D}$, such that S is unitarily equivalent to $r(S_\nu)$.*

PROOF: (Sufficiency): Suppose $S = r(S_\nu)$, where $r(z) = p(z)/q(z)$ for two polynomials p and q, and $q(z) = \sum_{n=0}^{N} b_n z^n$. Let

$$s(z) = \sum_{n=0}^{N} \overline{b_n} z^{N-n}.$$

Let the atoms of ν be at the points λ_n, and let N_1 be the degree of p. Let

$$t(z) = z^{N_1+1} s(z) \prod (z - \lambda_n).$$

Then the closure of the set of polynomials that have t as a factor, which is of finite codimension, is contained in the kernel of $[S^*, S]$: for indeed, if u is a polynomial,

$$r(S_\nu)^*(tu) = \bar{r}tu \qquad \nu \text{ a.e.}.$$

(Necessity) Let S be S_μ, and let G be the set of bounded point evaluations for $P^2(\mu)$. Then S_μ has the normal extension N_μ on $L^2(\mu)$; let P be the projection from $L^2(\mu)$ onto $\mathcal{H}$. Let $\mathcal{K} = ker[S^*, S]$. It is straightforward to verify that $\mathcal{K} = \{\xi \in \mathcal{H} : N^*\xi \in \mathcal{H}\}$, and hence $\mathcal{K}$ is S-invariant, and of finite codimension by hypothesis. It follows from Thomson's theorem [8] that there is a polynomial p, with zeroes in G, such that $\mathcal{K} = pP^2(\mu)$.

As p is in $\mathcal{K}$, $S^*p = PN^*p = N^*p$, so $\bar{z}p(z)$ is in $P^2(\mu)$, and clearly it is continuous on the support of μ. By [6, Thm. 4.2], any such function must agree μ-a.e. with some function f, analytic on G and continuous on $\bar{G}$:

$$\bar{z}p(z) = f(z) \qquad \mu \text{ a.e.}$$

Therefore

$$H(z) := \frac{f(z)}{p(z)}$$

is meromorphic in G, extends continuously to ∂G, and equals $\bar{z}$ on ∂G (since the support of μ contains ∂G). Therefore G is a quadrature domain, and as $H(z)$ can only equal $\bar{z}$ finitely often in G, $\mu|_G$ can only be supported on a finite set.

Let r be the Riemann map from $\mathbb{D}$ onto G, which is rational as G is a quadrature domain. Let ν_1 be the pull-back of μ to $\bar{\mathbb{D}}$ by r. It follows from [6] and Szegö's theorem (see [2]) that $\nu_1|_{\partial\mathbb{D}} = |h|^2\sigma$ for some outer function h; $\nu_1|_{\mathbb{D}}$ is a finite sum of point masses, $\sum_{n=1}^N a_n\delta_{\lambda_n}$ say.

Let $\nu = \sigma + \sum_{n=1}^N a_n|h(\lambda_n)|^2\delta_{\lambda_n}$. By construction, S is unitarily equivalent to $r(S_{\nu_1})$ restricted to the closed linear span of $\{r^n\}$ in $P^2(\nu_1)$; by [6, Thm. 2.4] this is all of $P^2(\nu_1)$. As multiplication by h implements a unitary equivalence between S_{ν_1} and S_ν, all that remains to show is that r is a weak-star generator of H^∞.

By [7], it is sufficient to prove that G is the interior of the polynomially convex hull of its closure. As ∂G is an algebraic curve with at most finitely many double points [4], G cannot have any slits, so we must show it also has no holes.

Let U be the interior of the polynomially convex hull of $\bar{G}$. Let ω_U denote harmonic measure for U, and ω_G denote harmonic measure for G restricted to ∂U (for the definition and basic properties of harmonic measure, see e.g. [2]; for convenience we will take our defining point to be $r(0)$). We claim that $\log \frac{d\omega_G}{d\omega_U}$ is integrable with respect to ω_U. Given the claim, it follows that $P^2(\mu)$ has all of U as bounded point evaluations (by the Szegö theorem - see e.g. [3, p.136]), so $U = G$ as desired.

To establish the claim, let I be some fixed small interval on ∂U, and let $E = r^{-1}(I)$. Then $\omega_G(I) = \sigma(E)$. But the length of I is at most

$$\sup_{z\in\mathbb{D}} |r'(z)|\sigma(E).$$

Therefore the Radon-Nikodym derivative of ω_G with respect to arc-length is bounded away from zero. Now, let ϕ be the Riemann map from the disk to U. As ∂U is rectifiable, ϕ' is in H^1 [2, Thm. 9.8], and therefore $\log |\phi'|$ is in $L^1(\sigma)$. But transferring to U, this says that the Radon-Nikodym derivative of arc-length with respect to ω_U is log-integrable, and *a fortiori* the same holds for ω_G, as claimed. $\square$

Theorem 2 *Let S be a cyclic subnormal operator. Then S has finite rank self-commutator if and only if S is of the form $S_1 \oplus S_2 \oplus \dots S_n \oplus N$, where*
(i) Each $S_i = S_{\mu_i}$ is an irreducible cyclic subnormal operator with finite rank self-commutator.
(ii) If G_i denotes the set of bounded point evaluations of $P^2(\mu_i)$, then $\cup_{i=1}^n \overline{G_i}$ is polynomially convex.
(iii) For $i \neq j$, the intersection of $\overline{G_i}$ and $\overline{G_j}$ contains at most one point.
(iv) $N = N_{\nu_0}$ is a cyclic normal operator, $\nu_0(G_i) = 0$ for all i, and ν_0 is singular with respect to harmonic measure for each G_i.

PROOF: The only thing necessary to prove is that conditions (ii) through (iv) are the correct conditions for cyclicity. Both (ii) and (iii) hold because, by the proof of the previous theorem, each μ_i is boundedly absolutely continuous with respect to arc-length, so any point in the interior of the polynomially convex hull of $\cup_{i=1}^n \overline{G_i}$ is a bounded point evaluation for $P^2(\mu_1 + \dots \mu_n)$, and by hypothesis these are $\cup_{i=1}^n G_i$. Condition (iv) is necessary by [6]. The sufficiency of (ii) through (iv) to guarantee cyclicity of S follows from [8] and [6]. $\square$

In [5, Question 13], the authors ask whether every irreducible pure subnormal operator with finite rank self-commutator is of the form described in Theorem 1, but with r no longer required to be a weak-star generator of H^∞. We give an example to show that this is not so.

Example Let $\mathcal{H} = P^2(\sigma) \oplus P^2(\sigma + \delta_0)$, and with respect to this decomposition let $S : \mathcal{H} \to \mathcal{H}$ be given by

$$S = \begin{pmatrix} X & Y \\ Z & 0 \end{pmatrix},$$

where Y and Z are both multiplication by z, and X is multiplication by $4z$. If $f(z) = \sum_{n=0}^\infty \hat{f}(n) z^n$ is in $P^2(\sigma)$, and $g(z) = \sum_{n=0}^\infty \hat{g}(n) z^n$ is in $P^2(\sigma + \delta_0)$, then

$$(S^*S - SS^*) \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\hat{f}(1)z + 17\hat{f}(0) + 4\hat{g}(0) \\ 2\hat{f}(0) + \frac{1}{2}\hat{g}(0) \end{pmatrix},$$

so S is a subnormal operator with rank 3 self-commutator. Moreover, it is not of the form $r(S_\nu)$, for some ν as in Theorem 1, because any such operator has a connected essential spectrum (its essential spectrum being the image of the circle under r), whereas the essential spectrum of S is two concentric circles: for S is similar to the operator

$$T = \begin{pmatrix} X & \frac{1}{4}X \\ \frac{1}{4}X & 0 \end{pmatrix}$$

on $P^2(\sigma) \oplus P^2(\sigma)$, and T is unitarily equivalent to $\frac{1}{4}(2+\sqrt{5})X \oplus \frac{1}{4}(2-\sqrt{5})X$.

To show S is irreducible, let M be a reducing subspace. By passing to $M^{\perp}$ if necessary, we can assume M contains a vector (f,g) with $\hat{f}(1) \neq 0$. Applying $S^*(S^*S - SS^*)$ to (f,g) gives $(\frac{1}{2}\hat{f}(1), (\frac{1}{4}\hat{f}(1))$, so $(2,1)$ is in M. Applying S^*S to this we get $(11, \frac{9}{2})$, so both $(1,0)$ and $(0,1)$ are in M. It is now routine to see that for every n, both $(z^n, 0)$ and $(0, z^n)$ are in M, so M is all of $\mathcal{H}$. $\square$

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