GEOMETRIC INTERPOLATION BETWEEN HILBERT SPACES

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Abstract. We prove that there is a unique way to construct a geometric scale of Hilbert spaces interpolating between two given spaces. We investigate what properties of operators, other than boundedness, are preserved by interpolation. We show that self-adjointness is, but subnormality and Krein subnormality are not. On the way to this last result, we establish a representation theorem for cyclic Krein subnormal operators.

Introduction

The basic idea of interpolation theory is as follows: one is given a linear operator T with some property P on two different topological vector spaces, X_0 and X_1, and one constructs a family of spaces X_s "between" X_0 and X_1, such that T also has property P on each $X_s, 0 < s < 1$. Normally, X_0 and X_1 are Banach spaces, and P is the property of being bounded.

One of the first such theorems was proved by M. Riesz [Ri], who showed that if T is bounded from $L^{p_0}(\mu)$ to $L^{p_0}(\mu)$ and from $L^{p_1}(\mu)$ to $L^{p_1}(\mu)$, then it is bounded from $L^p(\mu)$ to $L^p(\mu)$, for all $1 \leq p_0 \leq p \leq p_1 \leq \infty$. Since then, several methods have been developed for producing interpolating families X_s, the principal ones being the complex method due to Calderón [Ca] and the real method due to Lions and Peetre [LP]. Although these methods always produce interpolation spaces whenever (X_0, X_1) is a compatible couple, they are useful only in so far as the spaces X_s can be concretely realised. Thus Riesz' theorem above can be proved by showing that for either of these methods, the interpolation spaces are isomorphic to $L^p(\mu)$; but in general, it is far from clear what the interpolation spaces look like, or whether they are naturally occurring spaces at all.

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If one is working with Hilbert spaces (which we always assume to be separable), one would expect the situation to be more transparent, and indeed W. Donoghue gave a classification of all Hilbert spaces that can interpolate exactly between two given Hilbert spaces [Do]. Our first result is that if one makes an additional log-convexity assumption, the interpolating spaces are unique, and there is a canonical way of finding them.

Specifically, let $(\mathcal{H}_0, \mathcal{H}_1)$ be a compatible couple of Hilbert spaces, *i.e.* they both embed continuously in some Hausdorff topological vector space V, and $\mathcal{D} := \mathcal{H}_0 \cap \mathcal{H}_1$ is dense in both spaces (a typical example of a compatible couple is two spaces of analytic functions on a domain, *e.g.* the Hardy space and the Bergman space). Let $\mathcal{H}$ be another Hilbert space contained in V in which $\mathcal{D}$ is dense, so any linear operator defined on $\mathcal{H}_0$ and $\mathcal{H}_1$ is at least densely defined on $\mathcal{H}$. Then, for $0 < s < 1$, we shall say $\mathcal{H}$ is a *geometric interpolating space of exponent* s between $\mathcal{H}_0$ and $\mathcal{H}_1$ if it satisfies the following three properties for linear operators T:

i) If T maps $\mathcal{D}$ to $\mathcal{D}$ and satisfies $\|T\xi\|_{\mathcal{H}_0} \leq \lambda_0 \|\xi\|_{\mathcal{H}_0}$ and $\|T\xi\|_{\mathcal{H}_1} \leq \lambda_1 \|\xi\|_{\mathcal{H}_1}$, then $\|T\xi\|_{\mathcal{H}} \leq {\lambda_0}^{1-s} \lambda_1^s \|\xi\|_{\mathcal{H}}$.

ii) If T maps $\mathcal{D}$ into some Hilbert space $\mathcal{K}$, and $\|T\xi\|_{\mathcal{K}} \leq \lambda_0 \|\xi\|_{\mathcal{H}_0}$ and $\|T\xi\|_{\mathcal{K}} \leq \lambda_1 \|\xi\|_{\mathcal{H}_1}$, then $\|T\xi\|_{\mathcal{K}} \leq {\lambda_0}^{1-s} \lambda_1^s \|\xi\|_{\mathcal{H}}$.

iii) If T maps some Hilbert space $\mathcal{K}$ into $\mathcal{D}$ and $\|T\xi\|_{\mathcal{H}_0} \leq \lambda_0 \|\xi\|_{\mathcal{K}}$ and $\|T\xi\|_{\mathcal{H}_1} \leq \lambda_1 \|\xi\|_{\mathcal{K}}$, then $\|T\xi\|_{\mathcal{H}} \leq {\lambda_0}^{1-s} \lambda_1^s \|\xi\|_{\mathcal{K}}$.

In Section 1 we show that for any compatible couple $(\mathcal{H}_0, \mathcal{H}_1)$, a geometric interpolating space of exponent s exists and is unique. In the language of category theory, this says that for each $0 < s < 1$ there is a unique functor $\mathcal{F}_s$ mapping the category of compatible couples of Hilbert spaces to the category of Hilbert spaces, with the property that, for any two compatible couples $(\mathcal{H}_0, \mathcal{H}_1)$ and $(\mathcal{K}_0, \mathcal{K}_1)$, if T is a linear operator that maps $\mathcal{H}_0$ to $\mathcal{K}_0$ with norm λ_0, and maps $\mathcal{H}_1$ to $\mathcal{K}_1$ with norm λ_1, then T maps $\mathcal{F}_s(\mathcal{H}_0, \mathcal{H}_1)$ to $\mathcal{F}_s(\mathcal{K}_0, \mathcal{K}_1)$ with norm less than or equal to $\lambda_0^{1-s} \lambda_1^s$.

In Section 2 we consider what properties other than boundedness are also preserved by geometric interpolation. We believe that this should be a valuable tool in building examples of operators that have certain properties, but not others. Specifically, consider the question of the existence of an operator that is polynomially hyponormal but not subnormal. This was open for a long time; recently R. Curto and M. Putinar [CP] have proved that such operators exist, but their proof is non-constructive, and no concrete examples are known. Now suppose interpolation preserved polynomial hyponormality (we cannot, alas, prove this); it does not preserve subnormality (see below), so one would get an easy example of an operator that was polynomially hyponormal but not subnormal.

Although we cannot answer the question of whether hyponormality is preserved by interpolation, we can answer the question, subject to one space being contained in the other, for normality (yes), subnormality (no) and Krein subnormality (no). On the way to the last result, we establish a representation theorem for cyclic Krein subnormal operators.

Section 1 : Geometric Interpolation

Let $(\mathcal{H}_0, \mathcal{H}_1)$ be a compatible couple of Hilbert spaces. Because the inner product for $\mathcal{H}_1$, $(.,.)_1$ is a Hermitian form on a dense subspace of $\mathcal{H}_0$, there is a positive (not necessarily bounded) operator A on $\mathcal{H}_0$ such that, for any ξ, η in $\mathcal{H}_0 \cap \mathcal{H}_1$, $(\xi, \eta)_1 = (\xi, A\eta)_0$. For $0 < s < 1$, define a new inner product on $\mathcal{H}_0 \cap \mathcal{H}_1$ by $(\xi, \eta)_s = (\xi, A^s \eta)_0$. The closure of $\mathcal{H}_0 \cap \mathcal{H}_1$ with respect to the norm given by the inner product we will call $\mathcal{H}_s$.

We remark that using powers of a positive operator to define interpolation spaces seems to have first appeared in a paper by J.L. Lions [Li], where, under the assumption that one of the Hilbert spaces is contained in the other, he proves essentially that the norm of an operator on an intermediate space will be bounded by some constant times the weighted geometric mean of the norms on the end-spaces. That the constants can be chosen to be one (again assuming that one Hilbert space contains the other) was proved by Krein and Petunin [KP] as part of their general theory of scales of Banach spaces (letting s range over $\mathbb{R}$ gives a natural scale of Hilbert spaces); see also the book by Krein, Petunin and Semenov [KPS].

Thus the fact that $\mathcal{H}_s$ is a geometric interpolating space is really already known; the significance of our proof is that it is simple and short, and that the operator theoretic approach naturally leads to the (new) result that the space is unique. This unicity, and the canonical construction of the spaces, make the geometric interpolating spaces natural objects of study for operator theory.

We note that the proof of existence is based on an idea of P. Halmos [Ha].

Theorem 1.1. *Let* $(\mathcal{H}_0, \mathcal{H}_1)$ *and* $(\mathcal{K}_0, \mathcal{K}_1)$ *be compatible couples of Hilbert spaces. Suppose* T *is a linear operator that maps* $\mathcal{H}_0$ *to* $\mathcal{K}_0$ *with norm* λ_0*, and maps* $\mathcal{H}_1$ *to* $\mathcal{K}_1$ *with norm* λ_1*. Then* T *maps* $\mathcal{H}_s$ *to* $\mathcal{K}_s$ *with norm less than or equal to* $\lambda_0^{1-s}\lambda_1^s$*. Moreover,* $\mathcal{H}_s$ *is the unique geometric interpolation space of exponent* s *between* $\mathcal{H}_0$ *and* $\mathcal{H}_1$*.*

Proof.

(Existence) Let A be the positive operator on $\mathcal{H}_0$ that gives the $\mathcal{H}_1$ inner product, and B be the positive operator on $\mathcal{K}_0$ that gives the $\mathcal{K}_1$ inner product. First, suppose A^{-1} and B are bounded, and that $s = \frac{1}{2}$. Then we have

$$\begin{aligned} \|T\|_{\mathcal{H}_0 \to \mathcal{K}_0} &= \lambda_0 \\ \|B^{\frac{1}{2}} T A^{-\frac{1}{2}}\|_{\mathcal{H}_0 \to \mathcal{K}_0} &= \lambda_1. \end{aligned}$$

The norm of T from $\mathcal{H}_{\frac{1}{2}}$ to $\mathcal{K}_{\frac{1}{2}}$ equals the norm of $S := B^{\frac{1}{4}} T A^{-\frac{1}{4}}$ from $\mathcal{H}_0$ to $\mathcal{K}_0$. Letting ρ denote the spectral radius, we have

$$\begin{aligned} \|S\|^2 = \|S^*S\| &= \rho(S^*S) \\ &= \rho(A^{\frac{1}{4}} S^* S A^{-\frac{1}{4}}) \\ &\leq \|A^{\frac{1}{4}} S^* S A^{-\frac{1}{4}}\| \leq \|T^*\|_{\mathcal{K}_0 \to \mathcal{H}_0} \|B^{\frac{1}{2}} T A^{-\frac{1}{2}}\|_{\mathcal{H}_0 \to \mathcal{K}_0} \\ &\leq \lambda_0 \lambda_1. \end{aligned}$$

So the theorem is true for $s = \frac{1}{2}$. Now interpolating between $\mathcal{H}_0$ and $\mathcal{H}_{\frac{1}{2}}$ gives the theorem for $s = \frac{1}{4}$, and interpolating between $\mathcal{H}_{\frac{1}{2}}$ and $\mathcal{H}_1$ gives the theorem for $s = \frac{3}{4}$. Similarly, the theorem holds for any dyadic rational in $[0, 1]$, and, by continuity of $\|B^{s/2} T A^{-s/2}\|$, any real s in $[0, 1]$.

The assumption that A^{-1} and B be bounded can now be dropped by approximating them by the "truncated" operators $A_n^{-1} := \int_0^n t dE_{A^{-1}}(t)$ and $B_n := \int_0^n t dE_B(t)$, where

$E_{A^{-1}}(.)$ and $E_B(.)$ are the respective spectral projections, for it is easy to check that $\|B^{s/2}TA^{-s/2}\| = \lim_{n\to\infty} \|B_n^{s/2}TA_n^{-s/2}\|$.

(Uniqueness) Let $\mathcal{G}$ be another geometric interpolation space. Let $E(.)$ be the spectral measure of A. Fix $b > a > 0$; let $\mathcal{K}_{ab}$ be the space $E[a,b)\mathcal{H}_0$ with the $\mathcal{H}_0$ norm. Let T be the operator of orthogonal projection onto $\mathcal{K}_{ab}$, and R be the inclusion of $\mathcal{K}_{ab}$ into $\mathcal{H}_0 \cap \mathcal{H}_1$. Then the norm of T on $\mathcal{G}$ is at most $\frac{1}{a^{(1-s)/2}}$, and the norm of R into $\mathcal{G}$ is at most $b^{(1-s)/2}$. So it follows that

$$a^{(1-s)/2}\|\xi\|_{\mathcal{H}_0} \leq \|\xi\|_{\mathcal{G}} \leq b^{(1-s)/2}\|_{\mathcal{H}_0}. \tag{1.2}$$

If we can show that whenever $[a_0,b_0) \cap [a_1,b_1)$ is empty, $E[a_0,b_0)\mathcal{H}_0$ and $E[a_1,b_1)\mathcal{H}_0$ are orthogonal in $\mathcal{G}$, then (1.2) yields, as a tends to b, that the $\mathcal{G}$-norm and the $\mathcal{H}_s$-norm coincide. But those two subspaces must be orthogonal, or else the projection $E[a_0,b_0)$, of norm one on both $\mathcal{H}_0$ and $\mathcal{H}_1$, would have norm greater than one on $\mathcal{G}$. $\square$

We remark that Calderón's complex method of interpolation also yields a geometric interpolation space, so by uniqueness it must be the same space as above.

Section 2 : Normality and Subnormality

Interpolation theory started as an effort to prove operators were bounded on a range of spaces, knowing how to prove this directly on the end-spaces. But it is of interest to ask what other properties of linear operators are preserved by geometric interpolation. Calderón showed that compactness is preserved [Ca]; we now prove that so is normality, provided one space is actually contained in the other (we don't believe this stipulation is actually necessary, but we cannot prove the result in general).

For the rest of the paper, $(\mathcal{H}_0, \mathcal{H}_1)$ will be a compatible couple of Hilbert spaces, and A will be the (perhaps densely defined) positive operator satisfying $(\xi,\eta)_{\mathcal{H}_1} = (A^{1/2}\xi, A^{1/2}\eta)_{\mathcal{H}_0}$ for all ξ,η in $\mathcal{H}_0 \cap \mathcal{H}_1$. The geometric interpolating space $\mathcal{H}_s$ will be as in the previous section, and s will always lie in the interval $[0,1]$. Note that A is bounded if and only if $\mathcal{H}_0$ is contained in $\mathcal{H}_1$.

Theorem 2.1. *Suppose $\mathcal{H}_0$ is contained in $\mathcal{H}_1$, and the operator $N : \mathcal{H}_0 \to \mathcal{H}_0$ is normal on both $\mathcal{H}_0$ and $\mathcal{H}_1$. Then N is normal on $\mathcal{H}_s$ for all s.*

Proof. The operator $N : \mathcal{H}_1 \to \mathcal{H}_1$ is unitarily equivalent to the operator $M := A^{1/2} N A^{-1/2}$ on $\mathcal{H}_0$. So M and N are normal operators, satisfying $MA^{1/2} = A^{1/2} N$. By the Fuglede-Putnam theorem - see Conway [Co, p.81], we also have $M^* A^{1/2} = A^{1/2} N^*$, and taking adjoints gives $A^{1/2} M = N A^{1/2}$. Therefore

$$AN = A^{1/2} M A^{1/2} = N A^{1/2} A^{1/2} = NA.$$

So N commutes with A, and hence all powers of A. Therefore on each $\mathcal{H}_s$, the operator N is unitarily equivalent to N on $\mathcal{H}_0$, and hence is normal. □

Note that if one assumes that N is actually self-adjoint on both $\mathcal{H}_0$ and $\mathcal{H}_1$, one does not have to appeal to the Fuglede-Putnam theorem, so there is no restriction that A be bounded above or below.

An operator is *subnormal* if it is the restriction of a normal operator to an invariant subspace. If μ is a compactly supported measure on $\mathbb{C}$, and $P^2(\mu)$ is the closure of the polynomials in $L^2(\mu)$, multiplication by the independent variable on $P^2(\mu)$ is a cyclic subnormal operator, denoted S_μ; moreover all cyclic subnormal operators arise this way [Co]. The *domain of analyticity* of $P^2(\mu)$ is the largest open set $U \subset \mathbb{C}$ such that if a sequence of polynomials converges in norm in $P^2(\mu)$, it converges uniformly on compact subsets of U. It then makes sense to assign values to elements of $P^2(\mu)$ at points of U. Moreover, if $P^2(\mu)$ has no L^2-summand (equivalently, S_μ has no reducing subspace on which it is normal), then an element of $P^2(\mu)$ is uniquely determined by its values on U, and $P^2(\mu)$ can be thought of as a space of analytic functions on U - Thomson [Th]. So given two pure spaces $P^2(\mu)$ and $P^2(\nu)$, with each component of the domain of analyticity of one intersecting the domain of analyticity of the other, they form a compatible couple (embedded in the space of analytic functions on the intersection of their respective domains). One can ask what the interpolation spaces look like, though they seem somewhat inaccessible in general. A special case was considered in [CMcCW].

Let us look, however, at radial measures, $d\mu(re^{i\theta}) = \frac{1}{2\pi} d\tau(r) d\theta$. Then $\frac{1}{2\pi} \int z^n \overline{z^m} d\tau(r) d\theta = \delta_{nm} \alpha_{2n}$ where $\{\alpha_n\}$ is the moment sequence of τ (let us call τ the *radial part* of μ). The domain of analyticity is a disk centered at 0. Suppose ν is another radial measure, and let β_n be the moments of its radial part. Then the interpolating space of exponent s between $P^2(\mu)$ and $P^2(\nu)$ is again a space of analytic functions, in which the inner product of two polynomials $\sum a_n z^n$ and $\sum b_m z^m$ is $\sum a_n \overline{b}_n (\alpha_{2n})^{1-s} (\beta_{2n})^s$. The operator of multiplication by z on this interpolation space is subnormal if and only if $(\alpha_{2n})^{1-s}(\beta_{2n})^s$ are the even moments of some measure compactly supported in $\mathbb{R}^+$.

If we take all the β_n's to be 1, then $P^2(\nu)$ is the Hardy space H^2, and S_ν is the unilateral shift. If, for all $0 < s < 1$, $\{\alpha_n^s\}$ is a moment sequence, the measure τ is called *infinitely divisible* (notice that, for a measure supported on $[0, L]$, the even moments determine the odd moments). Thus we have:

Theorem 2.2. *Let μ be a radial measure, with radial part τ. Let $\mathcal{H}_0 = P^2(\mu)$, and $\mathcal{H}_1 = H^2$. Then the operator of multiplication by z is subnormal on every interpolating space $\mathcal{H}_s$ if and only if τ is infinitely divisible.*

Infinitely divisible measures have been studied by R.A. Horn [Ho].

Example 2.3. Let τ be Lebesgue measure on $[0, 1]$. This is infinitely divisible, as can be checked by Hausdorff's moment theorem. So is the measure "rdr", so interpolating between the Bergman shift and the Hardy shift always yields a subnormal operator.

Example 2.4. Now let τ have an atom at $\frac{1}{2}$ and an atom at 1, both of weight 1. Then $\alpha_n = 1 + \frac{1}{2^n}$. The sequence $\{\sqrt{\alpha_n}\}$ is not a moment sequence, as the 3-by-3 Hankel matrix

$$\begin{pmatrix} \sqrt{2} & \sqrt{\frac{3}{2}} & \sqrt{\frac{5}{4}} \\ \sqrt{\frac{3}{2}} & \sqrt{\frac{5}{4}} & \sqrt{\frac{9}{8}} \\ \sqrt{\frac{5}{4}} & \sqrt{\frac{9}{8}} & \sqrt{\frac{17}{16}} \end{pmatrix}$$

is not positive. Thus we have an example of interpolating between two subnormal operators to get an operator that is not subnormal (in the language of Curto and Putinar, it is not even 2-hyponormal). As we interpolate between $\mathcal{H}_0$ and $\mathcal{H}_1$, we get a family of weighted shifts given by

$$We_n = \left(\frac{4^{n+1} + 1}{4^{n+1} + 4} \right)^{\frac{1-s}{2}} e_{n+1}.$$

For s equal to 0 or 1, these shifts are subnormal. Are they polynomially hyponormal for all s? If so, they would provide the first example of an operator that is polynomially hyponormal but not subnormal (the existence of such operators has recently been proved by Curto and Putinar [CP]).

Section 3 : Krein Subnormality

A *Krein space* is a Hilbert space $\mathcal{K}, (.,.)$ with an additional indefinite inner product $< .,. >$ defined, where this indefinite form is given in terms of a symmetry (a self-adjoint unitary) J by

$$< x, y >= (x, Jy).$$

Recently, operators on Krein spaces have attracted much attention - see *e.g.* the book by Azizov and Iokhvidov [AI] and the review of it by Rodman [Ro]. C. Cowen and S. Li initiated the study of Krein subnormal operators and their connection with moment problems [CL].

A linear operator T_1 on a Krein space $\mathcal{K}_1$ is a *Krein extension* of the operator T_0 on $\mathcal{K}_0$ if

(i) $\mathcal{K}_0$ is a closed subspace of $\mathcal{K}_1$;

(ii) $< x, y >_0 =< x, y >_1$ for all x, y in $\mathcal{K}_0$; and

(iii) $T_0 x = T_1 x$ for all x in $\mathcal{K}_0$.

There is a natural decomposition $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$ into the positive and negative eigenspaces of J. A linear operator T on $\mathcal{K}$ is called *fundamentally reducible* if it leaves these spaces invariant (and in this case the adjoint of T is the same with respect to either inner product). An operator on a Hilbert space $\mathcal{H}$ is called *Krein subnormal* if it has a continuous fundamentally reducible Krein extension that is normal (*i.e.* commutes with its adjoint). Note that $\mathcal{H}$ need not be contained in $\mathcal{K}^+$, it can be at an angle to it; but it can't intersect $\mathcal{K}^-$ because the inner product on $\mathcal{H}$ is positive. Let us also remark that Wu J. has studied a weaker form of subnormality, in which $\mathcal{K}_0$ is not required to be closed in $\mathcal{K}_1$, and the extension need not be fundamentally reducible; he proved that every bounded operator is subnormal in this weaker sense [Wu].

Let $\mu = \mu_+ - \mu_-$ be a real measure. Then $L^2(|\mu|)$ is a Krein space with inner product

$$< f, g >= \int f\overline{g}d\mu_+ - \int f\overline{g}d\mu_-.$$

Call this space $K^2(\mu)$. Just as all cyclic subnormal operators can be represented as multiplication by z on $P^2(\nu)$ for some positive measure ν, Cowen and Li proved that all cyclic Krein subnormal operators can be represented as multiplication by z on some $Q^2(\mu)$, the closure of the polynomials in $K^2(\mu)$ [CL, Theorem 9]. However, not all real measures μ can be obtained; here is a characterization of those that can.

Theorem 3.1. *Let μ have Jordan decomposition $\mu_+ - \mu_-$. Multiplication by the independent variable on $Q^2(\mu)$ is Krein subnormal if and only if there exists some constant $c < 1$ such that, for all polynomials p,*

$$\int |p|^2 d\mu_- \leq c \int |p|^2 d\mu_+. \tag{3.2}$$

Proof. ($\Rightarrow$) By [CL, Theorem 5] the 'forgetful map' from $K^2(\mu)$ to $L^2(|\mu|)$ that forgets about the Krein space structure implements a similarity between M_z on $Q^2(\mu)$ and M_z on $P^2(|\mu|)$. Therefore there is some constant c_1 such that

$$\int |p|^2 d|\mu| \leq c_1 \int |p|^2 d\mu,$$

and so

$$\int |p|^2 d\mu_- \leq \frac{c_1 - 1}{c_1 + 1} \int |p|^2 d\mu_+.$$

($\Leftarrow$) Multiplication by z on $K^2(\mu)$ is clearly a normal fundamentally reducible Krein extension. So all that remains is to prove that $Q^2(\mu)$ is a Hilbert space, *i.e.* that the sesquilinear form $< f, g >= \int f\overline{g}d\mu$ is positive definite. So suppose p_n are polynomials that converge to some non-zero element f of $Q^2(\mu)$. Then p_n converges to f in $L^2(|\mu|)$, so $|p_n|^2$ converges to $|f|^2$ in $L^1(|\mu|)$, and hence

$$\begin{aligned}
\int f\overline{f}d\mu &= \lim_{n\to\infty}\Big(\int |p_n|^2 d\mu_+ - \int |p_n|^2 d\mu_-\Big) \\
&\geq (1-c)\lim_{n\to\infty}\int |p_n|^2 d\mu_+ \\
&\geq (1-c)\int |f|^2 d\mu_+ \\
&\geq \frac{1-c}{1+c}\int |f|^2 d|\mu| \\
&> 0.
\end{aligned}$$

□

Note that if μ_- is non-zero, the operator cannot be subnormal, because if a measure ν satisfies $\int |p|^2 d\nu = \int |p|^2 d\mu$ for all polynomials p, then $\nu = \mu$.

If we take μ_+ to be normalized area measure on the unit disk, and μ_- to be an atom of weight α at zero, we get a Krein subnormal weighted shift for $\alpha < 1$. Direct calculation shows that this operator is hyponormal if $\alpha \leq 1/4$. This shows that the answer to Question 4 of [CL], whether a hyponormal Krein subnormal operator must be subnormal, is no (Cowen and Li say in a note added in proof that they have also found an example).

Obviously a theorem similar to 2.2 holds if one wants interpolation between H^2 and $Q^2(\mu)$ to preserve Krein subnormality for multiplication by z - the powers of the moments of the radial part of μ must themselves be moments of the radial part of a measure satisfying (3.2). That this does not always hold is shown by the following example.

Example 3.3. Let λ be linear measure on the multiplicative semi-group $[0,1]$, and let $*$ denote the convolution of measures with respect to the semi-group. Let $x = 0.58$ (which is slightly greater than $1/\sqrt{3}$). Let $\tau = \lambda + \delta_0 - \delta_x$, where δ_x is a point mass at x, and let $\sigma = \tau * \tau$.

Claim: The measure μ given by $d\mu(z) = \frac{1}{2\pi}d\sigma(r)d\theta$ satisfies (3.2) for $c = .99996$.

Proof: We must show that for any polynomial $p(z) = \sum_{n=0}^{N} a_n z^n$,

$$\int |p|^2 d(2\delta_x * \lambda + \delta_0)(r)d\theta \leq c\int |p|^2 d(\lambda * \lambda + 2\lambda * \delta_0 + \delta_{x^2})(r)d\theta.$$

As the moments of τ are $\frac{1}{n+1} - x^n$ for $n > 0$, and 1 for $n = 0$, and because the moments of σ are the squares of the moments of τ, this is equivalent to:

$$3|a_0|^2 + \sum_{n=1}^{N} |a_n|^2 (\frac{2x^{2n}}{2n+1}) \leq c[4|a_0|^2 + \sum_{n=1}^{N} |a_n|^2 (x^{4n} + \frac{1}{(2n+1)^2})].$$

The verification of this inequality is elementary.

Claim: The measure ν given by $d\nu(z) = \frac{1}{2\pi} d\tau(r) d\theta$ does not satisfy (3.2) for any $c \leq 1$.
Proof: Let $p(z) = z$. Then $\int |z|^2 d\nu_+ = \frac{1}{3}$, which is smaller than $\int |z|^2 d\nu_- = x^2$.

Claim: Let $\mathcal{H}_0 = Q^2(\mu)$ and $\mathcal{H}_1 = H^2$. Then the operator of multiplication by z is Krein subnormal on $\mathcal{H}_0$ and $\mathcal{H}_1$ but not on $\mathcal{H}_{\frac{1}{2}}$.
Deny. Then there would be a radial real measure κ with radial part ρ satisfying

$$\begin{aligned} \|z^n\|^2_{\mathcal{H}_{\frac{1}{2}}} &= \int_{[0,1]} r^{2n} d\rho(r) \\ &= \sqrt{\|z^n\|^2_{\mathcal{H}_0}} \\ &= \sqrt{\int_{[0,1]} r^{2n} d\sigma(r)} \\ &= |\int_{[0,1]} r^{2n} d\tau(r)|. \end{aligned}$$

But all of τ's moments, except for the first and second, are positive. So ρ must equal τ, and hence κ equals ν and does not satisfy (3.2). □

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