INTERPOLATION BETWEEN WEIGHTED HARDY SPACES

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Abstract. We prove that $H^p(w_0^{1-s}w_1^s)$ is an interpolation space of exponent s between $H^p(w_0)$ and $H^p(w_1)$ if and only if $log(w_1/w_0)$ is in BMO. If $log(w_1/w_0)$ fails to be in BMO, $H^p(w_0^{1-s}w_1^s)$ can still be an interpolation space, provided the range of w has sufficiently large gaps.

Introduction

The first theorem on the interpolation of linear operators is due to M. Riesz [Ri], who showed that if T is a linear operator that is continuous from $L^{p_0}(\mu)$ to $L^{p_0}(\mu)$ and from $L^{p_1}(\mu)$ to $L^{p_1}(\mu)$, then it is continuous from $L^p(\mu)$ to $L^p(\mu)$, for all $1 \leq p_0 \leq p \leq p_1 \leq \infty$. Many interpolation theorems have been proved since then; we mention only a result of E. Stein [St] that allows the measure to change: if T is continuous from $L^p(w_0\mu)$ to $L^p(w_0\mu)$, and from $L^p(w_1\mu)$ to $L^p(w_1\mu)$, then it is continuous from $L^p(w_0^{1-s}w_1^s\mu)$ to itself, for every $0 \leq s \leq 1$.

In this paper we consider the question of when linear operators defined and continuous on two weighted Hardy spaces must also be continuous on some other interpolating weighted Hardy space. By a *weight* we shall always mean a non-negative measurable function w on the unit circle, that satisfies $\int_{\mathbb{T}} |log(w)| d\sigma < \infty$, where σ is normalised Lebesgue measure. If w is also integrable, the weighted Hardy space $H^p(w)$ is the closure in $L^p(w\sigma)$ of the (analytic) polynomials, for $1 \leq p < \infty$. This is isometrically isomorphic to the space ψH^p, where H^p is the classical Hardy space (*i.e.* $w \equiv 1$), ψ is the outer function with modulus $w^{-1/p}$, and the norm of ψf in ψH^p is defined to be the norm of f in H^p (see [Ga] for the definition of an outer function). For an arbitrary weight, one can always

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find such an outer function (because of the log-integrability assumption), and so we define $H^p(w)$ to be ψH^p. For simplicity, we shall always assume that $p \geq 1$.

Given any two Banach spaces X_0 and X_1 that are 'compatible', in the sense of both being continuously embedded in some topological vector space, there are two principal ways of defining a family of intermediate spaces X_s such that any linear operator that is continuous from X_0 to X_0 and from X_1 to X_1 is also continuous from X_s to X_s, $0 < s < 1$: these are the so-called "real" and "complex" methods (see [BL] for details). Most interpolation theorems are proved by showing that the desired space is isomorphic to one of these X_s's. Both these methods have the property that the norm of an operator T on X_s is less than or equal to $\|T\|_{X_0}^{1-s}\|T\|_{X_1}^{s}$. By analogy with Stein's theorem, one might expect that the space $H^p(w_0^{1-s}w_1^s)$ would be isomorphic to an interpolation space between $H^p(w_0)$ and $H^p(w_1)$ obtained by one of these methods. In Section 1 we prove that this is true if and only if $log(w_1/w_0)$ is in the space BMO of functions of bounded mean oscillation (see [Ga] for a definition of BMO). If $log(w_1/w_0)$ is not in BMO, we show that $H^p(w_0^{1-s}w_1^s)$ is not an interpolation space of exponent s (*i.e.* it does not satisfy $\|T\|_{H^p(w_0^{1-s}w_1^s)} \leq C\|T\|_{H^p(w_0)}^{1-s}\|T\|_{H^p(w_1)}^{s}$ for all linear operators bounded on both $H^p(w_0)$ and $H^p(w_1)$).

In Section 2 we give an example of a weight w whose logarithm is not in BMO, but such that $H^2(\sqrt{w})$ is an interpolation space between H^2 and $H^2(w)$. The example is a step function - such an example is necessary, as this behaviour cannot occur if the essential range of $log(w)$ does not have large gaps (for example if the sizes of the components of the complement are bounded).

In Section 3 we consider the question of when, for any fixed Banach space Y, the continuity of a linear map T from $H^p(w_0)$ to Y and from $H^p(w_1)$ to Y implies the continuity of T from $H^p(w_0^{1-s}w_1^s)$ to Y. The answer is that this holds if and only if $H^p(w_0^{1-s}w_1^s)$ is contained in $H^p(w_0) + H^p(w_1)$. We show that this fails if w_1/w_0 has one-sided limits of zero and infinity at the same point.

In Section 4 we apply the positive result of Section 1 to a problem raised in [DM$^{\mathrm{c}}$C], namely determining which Toeplitz operators are bounded on $H^2(|e^{i\theta} - 1|)$. This was the problem that originally inspired this paper.

We remark that there is another, non-equivalent, way of defining weighted Hardy spaces; for details, and for interpolation theorems proved for these spaces, see [ST].

Section 1

Let us fix weights w_0 and w_1, a real number $p \geq 1$, and a number s strictly between 0 and 1. Any interpolation theorem between $H^p(w_0)$ and $H^p(w_1)$ will depend only on the quotient $w = w_1/w_0$, so we shall assume that $w_0 = 1$, $w_1 = w$. For every positive real number t, define a pair of outer functions u_t and v_t by requiring that their moduli on the circle be given almost everywhere by

$$\begin{aligned} |u_t(e^{i\theta})| &= min(1, tw^{\frac{1}{p}}(e^{i\theta})) \\ |v_t(e^{i\theta})| &= min(1, \frac{1}{tw^{\frac{1}{p}}(e^{i\theta})}). \end{aligned} \tag{1.1}$$

The solution of the interpolation problem depends on whether one can find a positive uniform lower bound on $|u_t(z)| + |v_t(z)|$ as z ranges over the unit disk $\mathbb{D}$; the following lemma asserts that the bound exists only if $log(w)$ is in BMO.

Lemma 1.2. *Let u_t and v_t be defined as above. There exists $c > 0$ such that $|u_t(z)| + |v_t(z)| > c$ for all z in $\mathbb{D}$ and all $t > 0$ if and only if $log(w)$ is in BMO.*

Proof. By [Ga,p.225], $log(w^{\frac{1}{p}})$ fails to be in BMO if and only if, for any $M > 0$, there exists z in $\mathbb{D}$ such that

$$\int_{\mathbb{T}} |log(w^{\frac{1}{p}}(e^{i\theta})) - \int_{\mathbb{T}} log(w^{\frac{1}{p}}(e^{i\phi}))P_z(\phi)d\sigma(\phi)|P_z(\theta)d\sigma(\theta) > M, \tag{1.3}$$

where $P_z(.)$ is the Poisson kernel for z. The integral in (1.3) remains unchanged if w is replaced by tw; choose that t for which $\int log(tw^{\frac{1}{p}}(e^{i\theta}))P_z(\theta)d\sigma(\theta) = 0$. Then

$$log|u_t(z)| = \int_{\{\theta: tw^{\frac{1}{p}}(e^{i\theta})<1\}} log(tw^{\frac{1}{p}}(e^{i\theta}))P_z(\theta)d\sigma(\theta) < -\frac{M}{2}$$

and

$$log|v_t(z)| = -\int_{\{\theta: tw^{\frac{1}{p}}(e^{i\theta})>1\}} log(tw^{\frac{1}{p}}(e^{i\theta}))P_z(\theta)d\sigma(\theta) < -\frac{M}{2}.$$

So (1.3) implies the existence of $t > 0$ such that $|u_t(z)| + |v_t(z)| < 2e^{-M/2}$. The converse follows from a similar argument. $\square$

How small u and v can be made simultaneously controls the norm of the inclusion map from $H^p(w^s)$ into $H^p + H^p(w)$ (if the former space is not contained in the latter, we shall say the norm of the inclusion is infinite).

Lemma 1.4. *Let u_1 and v_1 be as in (1.1). Suppose there is a point z in $\mathbb{D}$ such that $|u_1(z)|^s + |v_1(z)|^{1-s} \leq \epsilon$. Then the norm of the inclusion from $H^p(w^s)$ into $H^p + H^p(w)$ is at least $(1-\epsilon)/\epsilon$.*

Proof. Suppose the norm of the embedding is C, so any f in $H^p(w^s)$ can be written as $g + h$, with

$$\|g\|_{H^p} + \|h\|_{H^p(w)} \leq C\|f\|_{H^p(w^s)}. \tag{1.5}$$

Writing

$$\begin{aligned} F &= f(\frac{u_1}{v_1})^s, \\ G &= g, \\ H &= h(\frac{u_1}{v_1}), \end{aligned}$$

(1.5) is equivalent to saying that, for any F in H^p of norm one, there exist G, H in H^p, with $\|G\| + \|H\| \leq C$, and satisfying

$$(\frac{v_1}{u_1})^s F = G + (\frac{v_1}{u_1})H.$$

Now,

$$\begin{aligned} |G(z)| &\leq exp[\int_{\mathbb{T}} log|G(e^{i\theta})|P_z(\theta)d\sigma(\theta)] \\ &\leq exp[\int_{\{\theta:|v_1(e^{i\theta})|<1\}} log(|v_1^s(e^{i\theta})F(e^{i\theta})| + |v_1(e^{i\theta})H(e^{i\theta})|)P_z(\theta)d\sigma(\theta) + \\ &\qquad \int_{\{\theta:|v_1(e^{i\theta})|=1\}} log|G(e^{i\theta})|P_z(\theta)d\sigma(\theta)] \\ &\leq |v_1(z)|^s exp[\int_{\mathbb{T}} log(|F(e^{i\theta})| + |H(e^{i\theta})| + |G(e^{i\theta})|)P_z(\theta)d\sigma(\theta)]. \end{aligned} \tag{1.6}$$

So if $M(z,p)$ is the maximum value an H^p function of norm one can attain at z, (1.6) asserts that $|G(z)| \leq \| \quad |F|+|G|+|H| \quad \| M(z,p)|v_1(z)|^s \leq (C+1)M(z,p)|v_1(z)|^s$. Similarly, $|H(z)| \leq (C+1)M(z,p)|u_1(z)|^{1-s}$, so

$$|F(z)| \leq (C+1)M(z,p)(|u_1(z)|^s + |v_1(z)|^{1-s}). \tag{1.7}$$

Letting F be that function which attains the value $M(z,p)$ at z, (1.7) and the hypothesis of the lemma imply that C is at least $(1-\epsilon)/\epsilon$, as desired. □

We can now prove that log-convex interpolation is possible if and only if $log(w)$ is in BMO.

Theorem 1.8. *Let weights w_0 and w_1 be given, and let $0 < s < 1$. Then all linear operators T, continuous from $H^p(w_0)$ to $H^p(w_0)$ and from $H^p(w_1)$ to $H^p(w_1)$, extend to be continuous from $H^p(w_0^{1-s}w_1^s)$ to $H^p(w_0^{1-s}w_1^s)$, and satisfy $\|T\|_{H^p(w_0^{1-s}w_1^s)} \leq C\|T\|_{H^p(w_0)}^{1-s}\|T\|_{H^p(w_1)}^s$, for some constant C independent of T, if and only if $log(w_1/w_0)$ is in BMO.*

Proof. Without loss of generality, assume $w_0 = 1$ and $w_1 = w$.

To prove sufficiency, we can show that $H^p(w^s)$ is isomorphic, under the hypothesis $log(w) \in$BMO, to either the real interpolation space $K_{s,p}(H^p, H^p(w))$ or the complex interpolation space $[H^p, H^p(w)]_s$. Both these assertions are true, but as the complex space is a little simpler, we shall deal with it (see [Ca] or [BL] for a definition).

Because the complex method gives $L^p(w^s\sigma)$ when applied to L^p and $L^p(w\sigma)$, the complex interpolation space between H^p and $H^p(w\sigma)$ must be contained in $L^p(w\sigma)$, and hence, since all the elements in the interpolation space are also included in $H^p + H^p(w^s)$, they must be analytic (in particular in the Smirnov class) and so the interpolation space is always included in $H^p(w^s)$.

To get the reverse inclusion, letting ψ be the outer function with modulus $w^{-1/p}$, one must find, for any function $\psi^s f$ in $H^p(w^s)$, a function F, continuous on $\{\zeta \in \mathbb{C} : 0 \leq \Re(\zeta) \leq 1\}$ and analytic on the interior of the strip, that takes values in $H^p + H^p(w^s)$, on the line $\{\zeta \in \mathbb{C} : \Re(\zeta) = 0\}$ takes values in H^p, on the line $\{\zeta \in \mathbb{C} : \Re(\zeta) = 1\}$ takes values in $H^p(w)$, vanishes at infinity, and satisfies $F(s) = \psi^s f$ (the values of such functions F at s are, by definition, the elements of the space $[H^p, H^p(w)]_s$). The appropriate function is $F(\zeta) = e^{\zeta^2 - s^2} \psi^\zeta f$; this will work if $\Im(log(\psi))$ is bounded, *i.e.* if $log(w)$ has a bounded conjugate. Multiplying w by a weight that's bounded above and below will only affect constants, so this method will work if $log(w)$ is the sum of a bounded function and one with bounded conjugate - which is equivalent to $log(w)$ being in BMO [Ga,p.248].

(Necessity) Assume $log(w)$ is not in BMO, and let u_t and v_t be as in (1.1). Then Lemmata 1.2 and 1.4 imply that, for any constant K, there exists a t so that the norm of the inclusion from $H^p((t^p w)^s)$ into $H^p + H^p(t^p w)$ is larger than K. By duality, there is a linear functional Λ, of norm at most one on both H^p and $H^p(t^p w)$, and norm at least K on $H^p((t^p w)^s)$. Now, let f be any non-zero function in $H^p \cap H^p(w^s) \cap H^p(w)$ (this intersection is dense in all three spaces). Then the rank-one operator $\xi \mapsto \Lambda(\xi) f$ has norm at most $\|f\|_{H^p}$ on H^p, at most $t\|f\|_{H^p(w)}$ on $H^p(w)$, and at least $Kt^s \|f\|_{H^p(w^s)}$ on $H^p(w^s)$. Choosing K larger than $C\|f\|_{H^p}^{1-s} \|f\|_{H^p(w)}^{s} \|f\|_{H^p(w^s)}^{-1}$ yields a contradiction. $\square$

Remark: The theorem is also true for $0 < p < 1$. The necessity argument remains unchanged; for sufficiency, one can use the real method of interpolation (see [BL]). To get an upper bound on $K(t, f)$ one solves the corona problem for u_t and v_t to get functions U_t and V_t satisfying $u_t U_t + v_t V_t = 1$, and with $\|U_t\|$ and $\|V_t\|$ bounded independently of t (this can be done because of Lemma 1.2). Writing $f = u_t U_t f + v_t V_t f$ yields that the quasi-norms of f in $K_{s,p}(H^p, H^p(w))$ and $H^p(w^s)$ are equivalent.

When $log(w)$ is not in BMO, the interpolation spaces $[H^p, H^p(w)]_s$ are still ones in which the polynomials are dense, evaluation at each point of the disk is continuous, and the shift is a cyclic contraction. Are the norms equivalent to ones realisable in some other way, *e.g.* as the $L^p(\mu)$-norm for some measure μ defined on the disk? In general, when is a complex interpolation space between $P^2(\nu_0)$ and $P^2(\nu_1)$ naturally isomorphic to some $P^2(\mu)$ (where $P^2(\mu)$ denotes the closure of the polynomials in $L^2(\mu)$)?

Notice that Theorem 1.8 does not prove that, if $log(w)$ fails to be in BMO, one cannot get $H^p(w^s)$ as an interpolation space between H^p and $H^p(w)$, only that one cannot get $H^p(w^s)$ as an interpolation space of exponent s. The method of proof of Theorem 1.8 will yield this stronger conclusion if one can pick the function f to have norm approximately 1 in H^p, $1/t$ in $H^p(w)$ and $1/t^s$ in $H^p(w^s)$. This in turn can be done if there is a set of positive measure where w is approximately t^{-p}, by choosing f to be a function with modulus one on this set, and modulus very small on the rest of the circle. For simplicity, we state this result for one special case:

Porism 1.9. *Suppose $log(w)$ is not in BMO, and the supremum of the diameters of the bounded components of the complement of the essential range of $log(w)$ is finite. Then there is a linear operator T, continuous from H^p to H^p and from $H^p(w)$ to $H^p(w)$, that fails to be continuous from $H^p(w^s)$ to $H^p(w^s)$.*

The hypotheses are satisfied, for example, by the weight $w(e^{i\theta}) = \theta, 0 \leq \theta < 2\pi$.

We note that an indication that it may be possible to interpolate but not with log-convex bounds is given by considering the simple example of w_0 being 1 on the upper semi-circle, ϵ on the lower semi-circle; and w_1 being ϵ on the upper semi-circle and 1 on the lower semi-circle. It is easy to check that for any linear operator T, $\|T\|_{H^2(\sqrt{w_0 w_1})} \leq \sqrt{2} max(\|T\|_{H^2(w_0)}, \|T\|_{H^2(w_1)})$. It is also easy to find operators with norms approximately one on both $H^2(w_0)$ and $H^2(\sqrt{w_0 w_1})$, and of order ϵ on $H^2(w_1)$. This behaviour makes the example in the next section easier to understand.

Section 2

We now give an example of a weight w whose logarithm is not in BMO, but for which $H^2(\sqrt{w})$ is an interpolation space between H^2 and $H^2(w)$.

Example 2.1. Parametrize the unit circle by θ going from $-\pi$ to π. For any point z in the disk, let ω_z denote harmonic measure on the circle with respect to z.

Define sets E_j and numbers α_j inductively as follows: Let $E_0 = \{e^{i\theta} : \pi \geq |\theta| \geq \pi/2\}$ and $\alpha_0 = \pi/2$. For each $j \geq 1$, choose $\alpha_j > 0$ so that the set $E_j := \{e^{i\theta} : \alpha_{j-1} > |\theta| \geq \alpha_j\}$ has the property that for each z in $\mathbb{D}$, either $\omega_z(\cup_{i=0}^{j-1} E_i) \leq 2^{-2^j}$, or $\omega_z(\mathbb{T}\backslash\cup_{\daleth=\nvDash}^{\daleth}\mathbb{E}_{\daleth}) \leq \nvDash^{-\nvDash^{\daleth}}$.

Let w be the weight that is 2^{2^j} on each E_j; clearly $log(w)$ is not in BMO. Moreover, by choosing α_j even smaller, if necessary, one can ensure that w is integrable (or in any desired L^p class).

For each j, let w_j be the weight $min(w, 2^{2^j})$. Notice that $\sqrt{w} \leq \sum_{i=0}^{\infty} \frac{1}{2^{2^{i-1}}} w_i \leq 2\sqrt{w}$. So to prove that every linear operator T continuous from H^2 to H^2 and from $H^2(w)$ to $H^2(w)$ is also continuous from $H^2(\sqrt{w})$ to itself, it is enough to prove T is continuous from each $H^2(w_j)$ to itself, with a bound on the norm of T independent of j. And this in turn will follow if we can show that each function f of norm one in $H^2(w_j)$ can be written as $g + h$, with $\|g\|_{H^2} \leq C/2^{2^{j-1}}$ and $\|h\|_{H^2(w)} \leq C$ for some universal constant C.

Let t be $2^{-2^{j-1}}$, and let $u := u_t$ and $v := v_t$ be given by (1.1). Claim: there is some $\delta > 0$, independent of j, such that $\inf_{z \in \mathbb{D}} |u(z)| + |v(z)| \geq \delta$. Assuming the claim for the moment, one can solve the corona problem for u and v to get functions U, V in H^∞, with norms less than some constant C depending only on δ, such that $uU + vV = 1$. Writing $f = uUf + vVf$ then yields the desired decomposition.

To prove the claim, pick any point z in $\mathbb{D}$. Either $\omega_z(\cup_{i=0}^{j-1} E_i) \leq 2^{-2^j}$ or, for all $k \geq j$, $\omega_z(\cup_{i=0}^{k-1} E_i) \geq 2^{-2^k}$, so $\omega_z(E_k) \leq 2^{-2^{k-1}}$ for all $k \geq j+1$. In the former case,

$$\begin{aligned} log|u(z)| &= \sum_{i=0}^{j-1} [\frac{1}{2} log(2)(2^i - 2^j)\omega_z(E_i)] \\ &> 2^{-2^j}(-j2^j), \end{aligned}$$

and this is bounded below independently of j. In the latter case,

$$\begin{aligned} log|v(z)| &= \sum_{i=j+1}^{\infty} [\frac{1}{2} log(2)(2^j - 2^i)\omega_z(E_i)] \\ &> -\sum_{i=j}^{\infty} 2^i 2^{-2^i}, \end{aligned}$$

and this too is bounded below. So the claim is proved. □

Despite this behaviour, the proof of Theorem 1.8 can be sharpened to yield the following:

If $log(w)$ is not in BMO, there exists some constant $C > 0$ such that, for any $\epsilon > 0$, there exists a rank-one operator with norm less than C on both H^2 and $H^2(w)$, greater than 1 on $H^2(\sqrt{w})$, and less than ϵ on either H^2 or $H^2(w)$.

Section 3

Suppose X_0 and X_1 are Banach spaces, and X is another Banach space in which $X_0 \cap X_1$ is dense. Then the following proposition, whose proof is elementary, holds:

Proposition 3.1. *Let Y be a Banach space. Any linear map T, that is continuous from X_0 to Y, and from X_1 to Y, is also continuous from X to Y, if and only if X is continuously embedded in $X_0 + X_1$.*

Corollary 3.2. *All linear functionals continuous on $H^p(w_0)$ and $H^p(w_1)$ are also continuous on $H^p(w_0^{1-s}w_1^s)$ if and only if, for $w = w_1/w_0$, the functions u_1 and v_1 defined in (1.1) form a corona pair, i.e. satisfy $\inf_{z\in\mathbb{D}} |u_1(z)| + |v_1(z)| > 0$.*

Proof. Necessity follows from Lemma 1.4 and Proposition 3.1. To prove sufficiency, observe that u_1^s and v_1^{1-s} also form a corona pair, so there are H^∞ functions U and V satisfying $u_1^s(z)U(z) + v_1^{1-s}(z)V(z) = 1$. As in 2.1, any f in $H^p(w_0^{1-s}w_1^s)$ can be written as $Uu_1^s f + Vv_1^{1-s} f$, where the first term is in $H^p(w_0)$ and the second is in $H^p(w_1)$. □

Clearly $L^p(\sqrt{w_0 w_1}) \subseteq L^p(w_0) + L^p(w_1)$; however, this need not be true if L^p is replaced by H^p, as the following example shows.

Example 3.3. *Let w_0 and w_1 be weights that satisfy*

$$\lim_{\theta\downarrow\alpha} \frac{w_0(e^{i\theta})}{w_1(e^{i\theta})} = 0$$

$$\lim_{\theta\uparrow\alpha} \frac{w_1(e^{i\theta})}{w_0(e^{i\theta})} = 0.$$

Then, for any $0 < s < 1$, $H^p(w_0^{1-s}w_1^s)$ is not contained in $H^p(w_0) + H^p(w_1)$.

Proof. By a theorem of Lindelöf [Ga,p.92], if an H^∞ function has a one-sided limit of zero on the circle, it also has zero as a radial limit at that point; so both u_1 and v_1 will have radial limit zero at $e^{i\alpha}$, and cannot form a corona pair. □

Section 4

Let $m(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic. The Toeplitz operator $T_{\bar{m}}$ is the linear operator whose action on monomials is given by $T_{\bar{m}}(z^k) = \overline{a_0} z^k + \overline{a_1} z^{k-1} + ... + \overline{a_k} z^0$. A problem that has been much studied recently is to determine, for a given weight w (or in general any measure on the circle), exactly which H^∞ functions m give rise to bounded operators $T_{\bar{m}}$ on $H^2(w)$ - see, *e.g.*, [Sa] and [LS].

A *Helson-Szegö* weight ρ is one that can be written as $e^{u+\tilde{v}}$, where u is in L^∞, and $\tilde{v}$ is the conjugate of a function in L^∞ of norm less than $\frac{\pi}{2}$. In [DMcC,Thm. 2.3], it was proved that if $w = |p|^2\rho$, where p is a polynomial and ρ is Helson-Szegö, then an H^∞ function m gives rise to a bounded operator $T_{\bar{m}}$ on $H^2(w)$ if and only if $m = T_{\bar{h}} f$, for some H^2 function f, where h is the outer function with modulus $w^{1/2}$ on the circle.

This theorem then solves the problem for all weights of the form $w(e^{i\theta}) = |e^{i\theta} - 1|^\alpha$, where $\alpha > -1$ is not an odd integer, because $|e^{i\theta} - 1|^\alpha$ is Helson-Szegö for $-1 < \alpha < 1$; using the affirmative part of Theorem 1.8, we can now almost solve the problem for α an odd integer:

Theorem 4.1. *Let $w(e^{i\theta}) = |e^{i\theta} - 1|^\alpha$, for any $\alpha > -1$. A necessary condition that the H^∞-function m give rise to a bounded Toeplitz operator $T_{\bar{m}}$ on $H^2(w)$ is that $m = T_{\overline{(z-1)}^{\alpha/2}} f$, for some f in H^2; a sufficient condition is that $m = T_{\overline{(z-1)}^{(\alpha/2)+\epsilon}} f$, for some $\epsilon > 0$ and some f in H^2.*

Proof. Necessity follows from [DMcC,Prop. 2.2]; sufficiency follows from the theorem quoted above, and interpolating between the spaces $H^2(|e^{i\theta}-1|^{\alpha+2\epsilon})$ and $H^2(|e^{i\theta}-1|^{\alpha-2\epsilon})$, which works because $log(|e^{i\theta} - 1|)$ is in BMO. $\square$

How can Theorem 4.1 be improved to one in which the necessary and sufficient conditions coincide?

Toeplitz operators are a special subset of linear operators. For example, whatever weight w one chooses, if $T_{\bar{m}}$ is bounded on $H^2(w)$ and $H^2(\frac{1}{w})$, it will automatically be bounded on H^2, with norm less than or equal to the minimum of the norms on the other two spaces. Can one say under what conditions the boundedness of a Toeplitz operator on $H^2(w_0)$ and on $H^2(w_1)$ implies its boundedness on $H^2(\sqrt{w_0 w_1})$?

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