Interpolation in Operator Theory

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0 Introduction

The first theorem on interpolation of linear operators was proved in 1911 by I. Schur [17], who showed that if T maps l^1 to l^1 and l^∞ to l^∞, then T maps l^2 to l^2, and $\|T\|_{l^2} \leq \|T\|_{l^1}^{1/2} \|T\|_{l^\infty}^{1/2}$. This was extended by M. Riesz in 1926 [16], when he proved that if T is bounded from $L^{p_0}(\mu)$ to $L^{p_0}(\mu)$ and from $L^{p_1}(\mu)$ to $L^{p_1}(\mu)$, then it is bounded from $L^p(\mu)$ to $L^p(\mu)$, for all $1 \leq p_0 \leq p \leq p_1 \leq \infty$. In the late 1930's, two quite different proofs of (generalizations of) Riesz's theorem were found: G. Thorin found a proof using complex analysis [22], and J. Marcinkiewicz a real variable proof (the theorem was announced in [14], but he was killed before he could publish a proof; A. Zygmund gave a proof in [23]).

Their ideas were extended to general schemes to produce a scale of Banach spaces X_s connecting a given pair of Banach spaces (X_0, X_1), where the pair is compatible in the sense of both spaces being embedded in some Hausdorff topological vector space (such as the space of measurable functions). The spaces X_s have the property that if T is a linear operator that maps X_0 to Y_0 and and X_1 to Y_1, then it maps X_s to Y_s, and moreover

$$\|T\|_{X_s \to Y_s} \leq C \|T\|_{X_0 \to Y_0}^{1-s} \|T\|_{X_1 \to Y_1}^{s}$$

(C will be used throughout the paper to denote a positive constant). The schemes are called, respectively, the complex method, due to A. Calderón [1], and the real method, due to J. Lions and J. Peetre [11]. Although they always produce interpolating spaces, being able to concretely identify the interpolating spaces is important in applications.

In seeking to apply interpolation theory to operator theory, the operators are often *a priori* defined only on a dense set. For example, for m in H^∞, the bounded analytic

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functions on the unit disk $\mathbb{D}$, let $T_{\bar{m}}$ denote the Toeplitz operator defined on polynomials by $T_{\bar{m}}p = \Pi\overline{m}p$, where Π is the truncation operator that takes a Fourier series and annihilates the negative powers of $e^{i\theta}$. Let σ denote Lebesgue measure on the circle, and for w a positive summable function, let $H^2(w)$ denote the closure in $L^2(w\sigma)$ of the polynomials. In [6] the question was raised whether the boundedness of some $T_{\bar{m}}$ on $H^2(|z-1|^{1-\epsilon})$ and on $H^2(|z-1|^{1+\epsilon})$ implied boundedness on $H^2(|z-1|)$. The question was answered in the affirmative in [5], where it was proved that if w_0 and w_1 are both log-integrable, then a necessary and sufficient condition that every operator T that is bounded on both $H^2(w_0)$ and $H^2(w_1)$ is also bounded on $H^2(w_0^{1-s}w_1^s)$ with norm at most

$$C\|T\|_{H^2(w_0)}^{1-s}\|T\|_{H^2(w_1)}^{s}$$

is that $\log(w_1/w_0)$ is in BMO, the space of functions of bounded mean oscillation.

A look at the case where neither weight is log-integrable is instructive. In this case, by Szegö's theorem [21], $H^2(w) = L^2(w\sigma)$ for $w = w_0, w_1, w_0^{1-s}w_1^s$. Now, E. Stein proved in [20] that if T is a linear operator defined on simple functions (*i.e.* linear combinations of step functions), and if T is bounded on $L^2(w_0\sigma)$ and $L^2(w_1\sigma)$, then it is bounded on $L^2(w_0^{1-s}w_1^s\sigma)$ (with the usual bound of the geometric mean of the norms on the end-spaces). One might think, then, that if the simple functions are replaced by the polynomials, which are dense in $L^2(w\sigma)$ for $w = w_0, w_1, w_0^{1-s}w_1^s$ by hypothesis, that boundedness of some $T_{\bar{m}}$ on $L^2(w_0\sigma)$ and $L^2(w_1\sigma)$ would imply boundedness on $L^2(w_0^{1-s}w_1^s\sigma)$. This is not true, as the following example, due to Ben Lotto, shows:

Example 1 [B. Lotto]

Let $w_0(e^{i\theta}) = \chi_{[0,3\pi/2]}(\theta)$ and $w_1(e^{i\theta}) = \chi_{[-\pi,\pi/2]}(\theta)$, so $w_s(e^{i\theta}) = \chi_{[0,\pi/2]}(\theta) + \chi_{[\pi,3\pi/2]}(\theta)$, for all s in $(0,1)$. It follows from [12, Thm. 14.1] that $T_{\bar{m}}$ is bounded on $L^2(w\sigma)$, for one of these weights w, if and only if m (in H^∞) can be written as a constant plus the Cauchy transform of a bounded function times $w\sigma$. Let f be a smooth function on the circle, that is 1 between 0 and $\pi/2$, and is zero between π and $3\pi/2$. Let m be the Cauchy transform of $f\sigma$. Then $T_{\bar{m}}$ is bounded on $L^2(w_0)$ and on $L^2(w_1)$, but not on $L^2(w_s)$, because if g were a function supported on $[0,\pi/2]\cup[\pi,3\pi/2]$, with the Cauchy transforms of $f\sigma$ and $g\sigma$ differing by a constant, then $f-g$ would be co-analytic on the circle, with the constant value 1 between $\pi/2$ and π, and the value 0 between $3\pi/2$ and 2π, which is impossible. $\Box$

The problem is that, although the polynomials are dense in $L^2(w_0^{1-s}w_1^s)$, they are not dense in $L^2(w_0+w_1)$; thus one cannot approximate an arbitrary function f in $L^2(w_0^{1-s}w_1^s)$

by a sequence of polynomials converging in both $L^2(w_0)$ and $L^2(w_1)$, so $T_{\bar{m}}f$ is not well-defined. The moral is that, before applying an interpolation theorem to an operator defined on $\mathcal{H}_0$ and $\mathcal{H}_1$, one must check that the operator is well-defined on $\mathcal{H}_0 \cap \mathcal{H}_1$.

In the category of compatible couples of Hilbert spaces (*i.e.* pairs $(\mathcal{H}_0, \mathcal{H}_1)$ such that both $\mathcal{H}_0$ and $\mathcal{H}_1$ are embedded in some ambient Hausdorff topological vector space, and $\mathcal{H}_0 \cap \mathcal{H}_1$ is dense in both $\mathcal{H}_0$ and $\mathcal{H}_1$), there is a canonical way of interpolating with a geometric mean bound on the norm in the interpolation space:

Theorem 2 [15] *There is a unique functor $\mathcal{F}_s$ mapping the category of compatible couples of Hilbert spaces to the category of Hilbert spaces, with the property that, for any two compatible couples $(\mathcal{H}_0, \mathcal{H}_1)$ and $(\mathcal{K}_0, \mathcal{K}_1)$, if T is a linear operator that maps $\mathcal{H}_0$ to $\mathcal{K}_0$ with norm M_0, and maps $\mathcal{H}_1$ to $\mathcal{K}_1$ with norm M_1, then T maps $\mathcal{F}_s(\mathcal{H}_0, \mathcal{H}_1)$ to $\mathcal{F}_s(\mathcal{K}_0, \mathcal{K}_1)$ with norm less than or equal to $M_0^{1-s} M_1^s$.*

Moreover, there is an explicit way to construct the spaces $\mathcal{F}_s(\mathcal{H}_0, \mathcal{H}_1)$ (hereinafter written as $\mathcal{H}_s$). The inner product for $\mathcal{H}_1$ is a Hermitian form on a dense subspace of $\mathcal{H}_0$, so there is a positive (not necessarily bounded) operator A on $\mathcal{H}_0$ such that, for any ξ, η in $\mathcal{H}_0 \cap \mathcal{H}_1$, $(\xi, \eta)_1 = (\xi, A\eta)_0$. For $0 < s < 1$, define a new inner product on $\mathcal{H}_0 \cap \mathcal{H}_1$ by $(\xi, \eta)_s = (\xi, A^s \eta)_0$. The closure of $\mathcal{H}_0 \cap \mathcal{H}_1$ with respect to the norm given by this inner product is the space $\mathcal{H}_s$.

The construction of these spaces in [15] is not new: the spaces $\mathcal{H}_s$ turn out to be isometric to those produced by the complex method (this can be verified easily if A is diagonal, and the general case follows either from a limiting argument or from the uniqueness of $\mathcal{F}_s$ and the fact that the complex method is also an exact interpolating functor of degree s). Even the use of powers of A to define interpolating spaces was first done by Lions in 1958 [10]. The importance of the theorem is the uniqueness assertion, so geometric interpolation, as we call this way of interpolating between Hilbert spaces, is a *canonical* method of interpolation. It is thus a natural object of study in operator theory; we believe it will also prove to be a good source of examples and counter-examples.

1 Two-sided Interpolation

By a two-sided interpolation property we mean one that, if possessed by T on both $\mathcal{H}_0$ and $\mathcal{H}_1$, is then also possessed by T on all the intermediate spaces. The foremost of these, of course, and the starting point of the theory, is boundedness. Assuming that T is actually bounded on $\mathcal{H}_0$ and $\mathcal{H}_1$, and therefore on all $\mathcal{H}_s$, what other two-sided properties are there? Notice that T on $\mathcal{H}_1$ is unitarily equivalent to $A^{1/2}TA^{-1/2}$ on $\mathcal{H}_0$, so the question becomes:

Question 1 *For what properties P does it follow that, if T and $A^{1/2}TA^{-1/2}$ both have property P, then $A^{s/2}TA^{-s/2}$ has property P for all $0 < s < 1$?*

Self-adjointness is such a property; in [15] it is proved that so is normality, subject to the restriction that $\mathcal{H}_0$ is contained in $\mathcal{H}_1$ (or *vice versa*), *i.e.* that A is bounded (or bounded below).

Question 2 *Is normality a two-sided interpolation property?*

How does the spectrum behave under interpolation? In [7], D. Herrero gave an example where, letting T_s denote T on $\mathcal{H}_s$, $\sigma(T_0) = \sigma(T_1) = \partial\mathbb{D} \cup \{0\}$, and, for $0 < s < 1$, $\sigma(T_s) = \overline{\mathbb{D}}$ and $T_s - \lambda$ is Fredholm of index -1 for all $\lambda \in \mathbb{D} \setminus \{0\}$. However, assuming A is bounded, Herrero proved in [8] that if T_0 and T_1 are both semi-Fredholm of the same index, then so is each T_s; moreover, if T_0 and T_1 are both invertible, then so is each T_s. Therefore, when one space is contained in the other, invertibility (and semi-Fredholmness of a fixed index) is a two-sided interpolation property; but, even if $\sigma(T_0) = \sigma(T_1)$, the spectrum of T_s can be strictly smaller. Without the assumption that one space is contained in the other, invertibility ceases to be a two-sided interpolation property.

Although the map $s \mapsto \sigma(T_s)$ can be discontinuous at 0 and 1, I. Sneiberg proved that it is upper semicontinuous on $(0, 1)$ (with respect to the Hausdorff metric on compacta in $\mathbb{C}$) [18]. Herrero and K. Saxe proved that it is continuous at points s_0 where $\sigma(T_{s_0})$ has no interior [9], and asked:

Question 3 *Is $s \mapsto \sigma(T_s)$ a continuous map on $(0, 1)$?*

J. Stapel proved [19] that if T_0 is compact and T_1 is bounded, then

$$\sigma(T_s) = \sigma(T_0) \ , \quad 0 \leq s < 1.$$

It was proved in [15] that neither subnormality nor Krein subnormality are two-sided interpolation properties. It is of interest to know whether hyponormality or polynomial hyponormality (*i.e.* every polynomial in the operator is hyponormal) are:

Question 4 *Is hyponormality or polynomial hyponormality a two-sided interpolation property?*

It was an open question for many years whether polynomially hyponormal operators had to be subnormal; the answer was recently shown by R. Curto and M. Putinar to be no, in a subtle and non-constructive proof [3]. If the answer to Question 4 is yes, it would follow that by interpolating between two subnormal operators one could produce a polynomially hyponormal non-subnormal operator. As a specific instance of Question 4 (that gives non-subnormal operators):

Question 5 *Is the weighted shift given by*

$$We_n = \left(\frac{4^{n+1}+1}{4^{n+1}+4}\right)^{\frac{1-s}{2}} e_{n+1}$$

polynomially hyponormal for $0 < s < 1$*?*

Studying interpolation of subnormal operators raises questions about moment problems. Let us call a sequence $\{\gamma_n\}$ a moment sequence if there is a positive finite Borel measure ν on $[0,1]$ such that $\gamma_n = \int_0^1 r^n d\nu(r)$. Let μ be the radial measure on $\mathbb{D}$ given by $\int f d\mu = \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 f(re^{i\theta})d\nu(r)d\theta$, and let $P^2(\mu)$ denote the closure of the polynomials in $L^2(\mu)$. Then the monomials z^n form an orthogonal basis in both $P^2(\mu)$ and H^2, so if one interpolates between $P^2(\mu)$ and H^2 one gets the space $\mathcal{H}_s$ of holomorphic functions on the disk with inner product

$$(\sum_0^\infty a_n z^n, \sum_0^\infty b_n z^n)_s = \sum_0^\infty a_n \overline{b_n} \gamma_{2n}^{1-s}. \qquad (*)$$

Therefore multiplication by z is subnormal on $\mathcal{H}_s$ if and only if γ_{2n}^{1-s} are the even elements of a moment sequence (if this is true for all s, the measure ν is called infinitely divisible). Question (5) arises from taking ν to be the sum of two equal point masses at $\frac{1}{2}$ and 1.

Multiplication by z on $\mathcal{H}_s$ is similar to a subnormal operator (as opposed to unitarily equivalent) if γ_{2n}^{1-s} is merely similar to a moment sequence, *i.e.* there is a moment sequence $\{\alpha_n\}$ and a positive constant C such that $\frac{1}{C}\alpha_{2n} \le \gamma_{2n}^{1-s} \le C\alpha_{2n}$ for all n.

Question 6 *If* γ_n *is a moment sequence, and* $0 < s < 1$, *is* γ_n^s *always similar to a moment sequence?*

2 One-sided Interpolation

By a one-sided interpolation property we mean one that if possessed by T on $\mathcal{H}_0$ is then possessed by T on all $\mathcal{H}_s$, $0 \le s < 1$, provided only that T is bounded on $\mathcal{H}_1$. One such property is compactness: it was shown by M. Cwikel [4] that compactness is a one-sided interpolation property for the real method of interpolation between any two Banach spaces (it is still unknown if this is true for the complex method). One can show that geometric interpolation between Hilbert spaces will yield spaces isomorphic (though not isometric) to the spaces obtained from the real method, so compactness is indeed a one-sided interpolation property for geometric interpolation.

One might argue that if A is bounded and invertible, and T is compact on $\mathcal{H}_0$, then $A^{s/2}TA^{-s/2}$ is compact for all s, so approximating an arbitrary A by, say, A restricted to the spectral subspaces corresponding to the intervals $[\frac{1}{n}, n]$, this result should stay true in the limit (where the assumption that $A^{1/2}TA^{-1/2}$ is bounded keeps things well-behaved). However, the same argument would also apply to any ideal, such as the trace-class operators, or any Schatten-von Neumann p-class, and simple examples show that being in a fixed Schatten-von Neumann p-class is not a one-sided interpolation property.

Stapel has given a direct proof that in geometric interpolation, compactness is a one-sided property [19]. He also studied membership in the Schatten-von Neumann classes, and showed that it behaved rather nicely. For any $0 < p < \infty$, let $\mathcal{S}_p$ denote the Schatten-von Neumann class , *i.e.* those operators T such that the eigenvalues of $(T^*T)^{1/2}$ are in l^p; and let $\mathcal{S}_\infty$ be the set of all bounded operators. Then Stapel proved [19]:

Theorem 3 J. Stapel *Suppose T_0 is in $\mathcal{S}_{p_0}$, and T_1 is in $\mathcal{S}_{p_1}$, for $0 < p_0 \le p_1 \le \infty$. Then T_s is in $\mathcal{S}_{p_s}$, and indeed*

$$\|T_s\|_{\mathcal{S}_{p_s}} \le \|T_0\|_{\mathcal{S}_{p_0}}^{1-s} \|T_1\|_{\mathcal{S}_{p_1}}^{s}$$

where

$$\frac{1}{p_s} = \frac{1-s}{p_0} + \frac{s}{p_1}.$$

Let A be normalized area measure on $\mathbb{D}$. This is a radial measure, with corresponding $\gamma_{2n} = \frac{1}{n+1}$. For any s between $-\infty$ and ∞ one can define an inner product as in $(*)$, to get a holomorphic Hilbert space on $\mathbb{D}$. If $-\infty < s < 1$, the space is actually $P^2(A_s)$, where

$$dA_s(z) = \frac{1}{\Gamma(1-s)} (\log(\frac{1}{|z|^2}))^{-s} dA(z).$$

These spaces form a scale $\{\mathcal{K}_s : -\infty < s < \infty\}$, such that interpolating between any two $\mathcal{K}_{s_0}$ and $\mathcal{K}_{s_1}$ yields the spaces $\mathcal{K}_s$ for $s_0 < s < s_1$. In [13] B. MacCluer and J. Shapiro proved that if a composition operator is compact on $\mathcal{K}_{s_0}$ for some $s_0 < 1$, and bounded on $\mathcal{K}_{s_1}$ for some $1 < s_1 < 3$, then it is compact on $\mathcal{K}_1$ (which is the Hardy space). As we see from the above discussion, this turns out not to be a special property of composition operators, but follows from compactness being a one-sided interpolation property.

Question 7 *What are other one-sided interpolation properties?*

There is a curious result of C. Cowen [2] that is in the spirit of one-sided interpolation, though it is not clear how it fits in to the theory of this paper. Let us continue to use $\mathcal{K}_s$ to denote the scale of holomorphic Hilbert spaces discussed above; an operator is called *co-subnormal* if its adjoint is subnormal.

Theorem 4 **[C. Cowen]** *If ϕ is a holomorphic map from $\mathbb{D}$ to $\mathbb{D}$ such that the composition operator C_ϕ is co-subnormal on $\mathcal{K}_1$, then C_ϕ is co-subnormal on $\mathcal{K}_n$ for all integers $n \leq 1$.*

Cowen's proof uses the Schur product theorem, and cannot be applied to non-integer indices.

Question 8 *Is Cowen's theorem true if n is replaced by an arbitrary real number less than 1?*

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