NEVANLINNA-PICK KERNELS AND LOCALIZATION

JIM AGLER AND JOHN E. M$^{\text{C}}$CARTHY

ABSTRACT. We describe those reproducing kernel Hilbert spaces of holomorphic functions on domains in $\mathbb{C}^d$ for which an analogue of the Nevanlinna-Pick theorem holds, in other words when the existence of a (possibly matrix-valued) function in the unit ball of the multiplier algebra with specified values on a finite set of points is equivalent to the positvity of a related matrix. Our description is in terms of a certain localization property of the kernel.

§0. Introduction

This paper concerns a generalization of the following result due to Pick [**P**] and Nevanlinna [**N**].

Theorem 0.1 *Let n be a positive integer, let $\lambda_1, \dots, \lambda_n$ be distinct points in $\mathbb{D}$, the open unit disc in the complex plane centered at 0, and let $z_1, \dots, z_n \in \mathbb{C}$. There exists a holomorphic function φ on $\mathbb{D}$ with $\varphi(\lambda_i) = z_i$ for each i and $\sup\limits_{|z|<1} |\varphi(z)| \leq 1$ if and only if the $n \times n$ matrix $\left[(1 - z_j \overline{z}_i)(1 - \lambda_j \overline{\lambda}_i)^{-1}\right]$ is positive semidefinite.*

Theorem 0.1 will be generalized in three ways. First, the domain $\mathbb{D} \subseteq \mathbb{C}$ will be replaced by a bounded domain U in $\mathbb{C}^d$. Secondly, observe that if H^2 is the classical Hardy space of analytic functions on $\mathbb{D}$ with square integrable boundary values and φ is a holomorphic function on $\mathbb{D}$, then φ is a multiplier of H^2 (i.e. $\varphi H^2 \subseteq H^2$) if and only if $\sup\limits_{|z|<1} |\varphi(z)|$ is finite. Indeed the condition, $\sup\limits_{|z|<1} |\varphi(z)| \leq 1$, in Theorem 0.1 is equivalent to the condition that the norm of φ as a multiplier of H^2 (i.e. $\sup\limits_{\substack{f \in H^2 \\ \|f\|=1}} \|\varphi f\|$) is less than or equal to one. Furthermore, since the reproducing kernel for H^2 has the form $k_\lambda(\mu) = (1 - \overline{\lambda}\mu)^{-1}$ it is clear that the $n \times n$ matrix that appears

Partially supported by the National Science Foundation.
Partially supported by National Science Foundation grant DMS 9531967.
Math Subject Classification: Primary 47A20, Secondary 46E20.

in Theorem 0.1 can be written in the form $[(1-z_j\overline{z}_i)k_{\lambda_i}(\lambda_j)]$. The second way in which we shall generalize the statement of Theorem 0.1 will be to replace the bound condition $\sup_{|z|<1} |\varphi(z)| \leq 1$ on the holomorphic function φ with a bound condition of the form

$$\sup_{\substack{f\in\mathcal{H} \\ \|f\|=1}} \|\varphi f\| \leq 1 \tag{0.2}$$

where $\mathcal{H}$ is a Hilbert space of analytic functions on U and to replace the condition that the $n\times n$ matrix $[(1-z_j\overline{z}_i)(1-\overline{\lambda}_i\lambda_j)^{-1}]$ be positive semidefinite with the condition that $[(1-z_j\overline{z}_i)k_{\lambda_i}(\lambda_j)]$ be positive semidefinite where $k_\lambda(\mu)$ is the reproducing kernel for $\mathcal{H}$. Thirdly, we shall replace the vorgegebene Funktionswerte $z_1,\ldots,z_n$ of Theorem 0.1 with points $z_1,\ldots,z_n \in \mathcal{M}_m(\mathbb{C})$, the C^*-algebra of $m\times m$ matrices. Since the function φ in that event will be a holomorphic $\mathcal{M}_m(\mathbb{C})$-valued map on U we modify (0.2) to the form

$$\sup_{\substack{f\in\mathbb{C}^m\otimes\mathcal{H} \\ \|f\|=1}} \|\varphi f\| \leq 1, \tag{0.3}$$

and replace the condition that the $n\times n$ matrix $[(1-z_j\overline{z}_i)k_{\lambda_i}(\lambda_j)]$ be positive semidefinite with the condition that the $mn\times mn$ block matrix $[(1-z_jz_i^*)k_{\lambda_i}(\lambda_j)]$ be positive semidefinite. We thus have been led to ask the following question.

Question 0.4. Let U be a bounded domain in $\mathbb{C}^d$ and let $\mathcal{H}$ be a Hilbert space of analytic functions on $\mathcal{H}$ with reproducing kernel k. Fix a positive integer m. When are the following two conditions equivalent for all choices of positive integer n, all choices of distinct points $\lambda_1,\ldots,\lambda_n\in U$ and all choices of points $z_1,\ldots,z_n\in\mathcal{M}_m(\mathbb{C})$?

(i) There exists an $\mathcal{M}_m(\mathbb{C})$-valued holomorphic function φ on U with $\varphi(\lambda_i)=z_i$ for each i and with the norm of φ as a multiplier of $\mathbb{C}^m\otimes\mathcal{H}$ less than or equal to one.
(ii) The $mn\times mn$ matrix $[(1-z_jz_i^*)k_{\lambda_i}(\lambda_j)]$ is positive semidefinite.

In this paper we shall give an answer to Question 0.4 in the case when the multiplications by the coordinate functions of $\mathbb{C}^d$ form a bounded d-tuple of operators

M acting on $\mathcal{H}$ with the properties that $\sigma(M) = cl(U)$ and M^* is of sharp type ([**Agr-S**]). Specifically, let superscripts denote the standard coordinates in $\mathbb{C}^d$ and assume for each integer r in the range $1 \leq r \leq d$ that if M^r is defined on $\mathcal{H}$ by the formula

$$(M^r f)(\lambda) = \lambda^r f(\lambda), f \in \mathcal{H}, \lambda \in U,$$

then

$$M^r \mathcal{H} \subseteq \mathcal{H}. \tag{0.5}$$

Evidently, (0.5) will guarantee that, for each r, $M^r \in \mathcal{L}(\mathcal{H})$, the C^*-algebra of bounded linear transformations of $\mathcal{H}$. We shall assume that the d-tuple of commuting operators $M = (M^r)$ satisfies the following conditions.

(0.6) $\sigma(M)$, the Taylor spectrum of M, (see [**T1**]and [**T2**]) is a subset of $cl(U)$ (equivalently, $\sigma(M) = cl(U)$)

(0.7) $\sigma_e(M)$, the essential Taylor spectrum of M (see [**C**]), is a subset of ∂U.

$$\dim \bigcap_{r=1}^{d} \ker(\lambda^r - M^r)^* = 1 \quad \text{for all} \quad \lambda \in U. \tag{0.8}$$

Let us agree to say that a Hilbert space $\mathcal{H}$ of holomorphic functions on a bounded domain U is *regular* if (0.5) - (0.8) are satisfied. Note that condition (0.8) guarantees that an operator that commutes with every M^r is multiplication by a multiplier of $\mathcal{H}$, and conversely all multipliers give rise to operators that commute with M (see *e.g.* [**Agr-S**]).

We now describe our answer to Question 0.4. Let us agree to say that a Hilbert space of analytic functions on a bounded domain is an m-interpolation space if the two conditions in Question 0.4 are equivalent. For $\lambda = (\lambda_1, \ldots, \lambda_n)$ an n-tuple of distinct points in U, let $\mathcal{H}_\lambda = \{f \in \mathcal{H} : f(\lambda_i) = 0 \text{ for each } i\}$, and let $\mathcal{S}_\lambda$ denote the collection of commuting d-tuples of operators T such that $\sigma(T) \subseteq \{\lambda_1, \ldots, \lambda_n\}$ and $h(T) = 0$ whenever h is holomorphic on U and $h(\lambda_i) = 0$ for each i. For T a commuting d-tuple of operators let $\mathcal{A}_T$ denote the sigma-weak operator topology closed algebra generated by the components of T. Finally, let H_k^∞ denote the algebra of multipliers of $\mathcal{H}$. Observe that naturally $H_k^\infty \subseteq \mathcal{L}(\mathcal{H})$, so that $\mathcal{M}_m(\mathbb{C}) \otimes H_k^\infty$

carries a distinguished norm (that inherited from the tensor product of $\mathcal{M}_m(\mathbb{C})$ and $\mathcal{L}(\mathcal{H})$ as C^*-algebras). An elementary exercise is to ascertain that if $\mathcal{M}_m(\mathbb{C}) \otimes H_k^\infty$ is identified with the space of matrix multipliers of $\mathbb{C}^m \otimes \mathcal{H}$, then this distinguished norm is the same as the norm defined in (0.3).

Now, in the ground-breaking papers [**Arv1**] and [**Arv2**] Arveson introduced the notion of m-contractivity. A linear map ρ defined on a subspace S of a C^*-algebra A and taking values in a C^*-algebra B is said to be m-contractive if the map

$$id_m \otimes \rho : \mathcal{M}_m(\mathbb{C}) \otimes S \longrightarrow \mathcal{M}_m(\mathbb{C}) \otimes B$$

is contractive. Here, id_m denotes the identity mapping on $\mathcal{M}_m(\mathbb{C})$. A map ρ is said to be completely contractive if P is m-contractive for every m. Now, let $\mathcal{C}_m$ denote the category with objects the subalgebras of C^*-algebras and morphisms the m-contractive algebra homorphisms. We assume that all algebras contain a unit and that morphisms map units to units.

If $\lambda_1, \dots, \lambda_n$ are n distinct points in U and $T \in \mathcal{S}_\lambda$ observe that Φ_T defined via the functional calculus [**T2**] by

$$\Phi_T(\varphi) = \varphi(T), \quad \varphi \in H_k^\infty$$

is an algebra homorphism from H_k^∞ onto $\mathcal{A}_T$. In particular, if T_λ is the commuting d-tuple of operators on $\mathcal{H}_\lambda^\perp$ defined by setting $T_\lambda^r = PM^r|\mathcal{H}_\lambda^\perp$ where P denotes the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_\lambda^\perp$, Φ_{T_λ} is a well-defined algebra homomorphism from H_k^∞ onto $\mathcal{A}_{T_\lambda}$. Also, it is clear that the map Φ_{T_λ} is completely contractive. We shall in future call the map Φ_{T_λ} the localization operator and denote it simply ρ. Observe that the localization operator depends only on $\mathcal{H}$ and the choice of points $\lambda_1, \dots, \lambda_n$. Let us agree to say that $\mathcal{H}$ possesses the m-contractive localization property if the following diagram can be completed in the category $\mathcal{C}_m$ whenever n is a positive integer, $\lambda_1, \dots, \lambda_n$ are n distinct points in U, $T \in \mathcal{S}_\lambda$, and Φ_T is an m-contraction.

$$\begin{tikzcd}
& \mathcal{A}_{T_\lambda} \arrow[dr, dotted] & \\
H_k^\infty \arrow[ur, "\rho"] \arrow[rr, "\Phi_T"] & & \mathcal{A}_T
\end{tikzcd}$$

Evidently, since $T \in \mathcal{S}_\lambda, \varphi(T) = 0$ whenever $\varphi \in \{\varphi \in H_k^\infty : \varphi(\lambda_i) = 0 \text{ for each } i\} = \ker \rho$, so that the map $\Psi_T : \mathcal{A}_{T_\lambda} \to \mathcal{A}_T$ defined by $\Psi_T(\rho(\varphi)) = \varphi(T)$ always completes the diagram. Thus, $\mathcal{H}$ has the m-contractive localization property if and only if whenever n is a positive integer, $\lambda_1, \ldots, \lambda_n$ are n-distinct points in $U, T \in \mathcal{S}_\lambda$, and Φ_T is m-contractive, it is the case that Ψ_T is also m-contractive.

We now can state our answer to Question 0.4.

Theorem 3.6. *Let $\mathcal{H}$ be a regular space on a bounded domain $U \subseteq \mathbb{C}^d$. $\mathcal{H}$ is an m-interpolation space if and only if $\mathcal{H}$ has the m-contractive localization property.*

We prove this theorem in Section 3. To understand the basic ideas behind our proof, we recall some ideas from [**A1**]. In [**A1**] the collection of operators that can be modelled using the space $\mathcal{H}$ was studied in the case where $d = 1$ and $U = \mathbb{D}$. Specifically, one lets k be a positive definite holomorphic kernel over $\mathbb{D}$ with the property that the associated Hilbert space $\mathcal{H}$ (i.e. the unique Hilbert space of analytic functions with reproducing kernel k) is regular and introduces the collection of operators $\mathcal{F}(k)$, consisting of all operators T for which there exist a Hilbert space $\mathcal{K}$, a unital representation $\pi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K})$, and a subspace $\mathcal{N} \subseteq \mathcal{K}$ with the properties that $\mathcal{N}$ is invariant for $\pi(M^*)$ and T is unitarily equivalent to $\pi(M^*)|\mathcal{N}$. In the case when k is the Szegö kernel, $k(\lambda, \mu) = (1 - \overline{\lambda}\mu)^{-1}, \mathcal{F}(k)$ is the set of operators that can be written as a coisometry restricted to an invariant subspace, the object of the Sz.-Nagy-Foias and de Branges-Rovnyak model theories.

There are no surprises in the generalization to several variables of the setup described in the previous paragraph. If U is a bounded domain in $\mathbb{C}^d$ and $\mathcal{H}$ is a regular Hilbert space of analytic functions on U with reproducing kernel k, then $\mathcal{F}(k)$ is defined as the collection of all commuting d-tuples of operators $T = (T^r)$ for which there exists a Hilbert space $\mathcal{K}$, a unital representation $\pi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K})$, and a subspace $\mathcal{N} \subseteq \mathcal{K}$ with the properties that $\mathcal{N}$ is invariant for $h(\pi(M^*))$ whenever h is holomorphic on a neighborhood of $\overline{cl(U)}$ and T is unitarily equivalent to $\pi(M^*)|\mathcal{N}$ (i.e. there exists a unitary operator X such that $T^r = X^*(\pi(M^r)^*|\mathcal{N})X$ for each r). Throughout the paper, we shall use $cl(E)$ to denote the closure of E, and $\overline{E}$ to denote the complex conjugate of E. Note in the definition of $\mathcal{F}(k)$ that since π is a unital representation, $\sigma(\pi(M^*)) \subseteq \overline{cl(U)}$ and consequently, $h(\pi(M^*))$ is well-defined by Taylor's functional calculus [**T2**].

From now on, we shall set $K = cl(U)$, and let $H(K)$ denote the space of germs of holomorphic functions on a neighborhood of K. If $T = (T^r)$ is a commuting d-tuple of operators acting on a space $\mathcal{K}$ and $\sigma(T) \subseteq K$, we say a subspace $\mathcal{N} \subseteq \mathcal{K}$ is $H(K)$-invariant for T if $h(T)\mathcal{N} \subseteq \mathcal{N}$ whenever $h \in H(K)$. Thus, if T is a commuting d-tuple with $\sigma(T) \subseteq \overline{K}, T \in \mathcal{F}(k)$ if and only if there exist a unital representation π of $\mathcal{L}(\mathcal{H})$ and an $H(\overline{K})$-invariant subspace $\mathcal{N}$ for $\pi(M^*)$ such that T is unitarily equivalent to $\pi(M^*)|_{\mathcal{N}}$. We shall abbreviate this latter condition in the sequel and simply say that T has an $H(\overline{K})$-extension to a tuple of the form $\pi(M^*)$.

Now a fundamental fact about the family $\mathcal{F}(k)$ is that $H(\overline{K})$-extensions localize. Precisely what this means is that if $\lambda_1, \ldots, \lambda_n$ are n distinct points in $U, T \in \mathcal{S}_{\overline{\lambda}}$, and T has an $H(\overline{K})$-extension to a tuple of the form $\pi(M^*)$, then T has an extension to a tuple of the form $\pi(M_z^*|\mathcal{H}_\lambda^\perp)$. This result, which is a key element in the proof of Theorem 3.6, is proved in Section 1 of this paper (Theorem 1.2).

Since extensions localize it is natural to ask whether dilations localize. Let us agree to say that a commuting d-tuple $T = (T^r)$ has an $H(\overline{K})$-dilation to a tuple of the form $\pi(M_z^*)$ if there exist a unital representation π of $\mathcal{L}(\mathcal{H})$ and a pair of $H(\overline{K})$-invariant subspaces $\mathcal{N}_1, \mathcal{N}_2$ for $\pi(M^*)$ with the properties that $\mathcal{N}_1 \subseteq \mathcal{N}_2$ and T is unitarily equivalent to $P_{\mathcal{N}_2 \ominus \mathcal{N}_1}\pi(M^*)|\mathcal{N}_2 \ominus \mathcal{N}_1$. Let us agree to say $\mathcal{H}$ has the *dilation* *localization property* if whenever n is a positive integer, $\lambda_1, \ldots, \lambda_n$ are n distinct points in $U, T \in \mathcal{S}_\lambda$, and T^* has an $H(\overline{K})$-dilation to an operator of the form $\pi(M^*)$, then T^* has an $H(\overline{K})$-dilation to an operator of the form $\pi(M^*|\mathcal{H}_\lambda^\perp)$. Equivalently, if T has an $H(K)$-dilation to an operator of the form $\pi(M)$, then T has an $H(K)$-dilation to an operator of the form $\pi(PMP)$, where P is the projection from $\mathcal{H}$ onto $\mathcal{H}_\lambda^\perp$).

Now a straightforward consequence of Arveson's theory of completely contractive algebra homomorphisms is that $\mathcal{H}$ has the dilation localization property if and only if $\mathcal{H}$ has the m-contractive localization property for every m. We thus obtain from Theorem 3.6 the following result:

Theorem 3.5. *Let $\mathcal{H}$ be a regular space on a bounded domain $U \subseteq \mathbb{C}^d$. $\mathcal{H}$ is a complete interpolation space if and only if $\mathcal{H}$ has the $H(\overline{K})$-dilation localization property.*

In Section 4 we give two concrete applications of Theorem 3.5; one in the multiplier norm of the Dirichlet space in one variable and the other in a norm on a space of holomorphic functions on the ball in $\mathbb{C}^d$ that agrees with the ordinary H^∞ norm when $d = 1$. These applications demonstrate that Theorem 3.6 and 3.5 already contain whatever concrete function theory one might believe is involved in the equivalence of 0.4 (i) and 0.4 (ii).

In this paper we shall always adhere to the following notations. T will denote a commuting d-tuple of bounded operators acting on a Hilbert space. The components of T (as well as the components of elements of $\mathbb{C}^d$) will be denoted with superscripts. All operators on d-tuples will be assumed to act componentwise. Thus, for example, if T is a commuting d-tuple of operators acting on a space $\mathcal{H}$, then $T^* = (T^{r^*})$ and if $\pi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K})$ is a representation then $\pi(T) = (\pi(T^r))$. If $\{T_\alpha : \alpha \in \mathfrak{A}\}$ is a collection of d-tuples with T_α acting on $\mathcal{H}_\alpha$, then $\bigoplus_{\alpha \in \mathcal{H}_\alpha} T_\alpha$ denotes the d-tuple $\left(\bigoplus_{\alpha \in \mathfrak{A}} T_\alpha^r\right)$ which acts on $\bigoplus_{\alpha \in \mathfrak{A}} \mathcal{H}_\alpha$. If $\mathcal{H}$ is a Hilbert space and $\mathcal{S} \subseteq \mathcal{H}$, then $[s : s \in S]$ denotes the closed linear span of S in $\mathcal{H}$ and if $x, y \in \mathcal{H}$, then $x \otimes y \in \mathcal{L}(\mathcal{H})$ is defined by $x \otimes y(h) = \langle h, y \rangle x$.

Finally, we remark that because of the fundamental work of Aronszajn [**Aro**], it is well known that $\mathcal{H}$ and k determine each other uniquely. Accordingly, if a kernel is assumed to be regular, this means that the corresponding Hilbert space $\mathcal{H}$ is regular. Similarly, we make no particular distinction between kernels and the spaces determined by them with respect to the notions of being m-contractively localizable, dilation localizable, having the m-interpolation property, or having the complete interpolation property.

§1. Coanalytic Models

In this section U will be a bounded domain in $\mathbb{C}^d$, we shall set $K = cl(U)$ and $\mathcal{H}$ will be a regular space on U with reproducing kernel k. We emphasize that we are assuming $\sigma(M) \subseteq K$ and $\sigma_e(M) \subseteq \partial U$. Let $\mathcal{F}$ denote the model generated by M, i.e. the collection of all commuting d-tuples of operators on Hilbert space that are of the form $\pi(M^*)|\mathcal{N}$ where π is a unital representation of $\mathcal{L}(\mathcal{H})$ and $\mathcal{N}$ is an $H(\overline{K})$-invariant subspace for $\pi(M^*)$. Our first result is an obvious extension of Theorem 2.8 in [**A1**]. Accordingly, we merely provide a brief sketch of its proof.

Theorem 1.1. *Let $\mathcal{K}$ be a Hilbert space and let $J \in \mathcal{L}(\mathcal{K})^{(d)}$. Let $\mathcal{A}$ be a C^*-algebra containing M and all compact operators on $\mathcal{H}$. There exists a unital representation $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{K})$ with $\pi(M) = J$ if and only if J is unitarily equivalent to a tuple of the form $M^{(\nu)} \oplus \pi_0(M)$ where ν is a cardinal and π_0 is a unital representation of $\mathcal{A}$ that annihilates the compact operators on $\mathcal{H}$.*

Proof. First assume that $J \in \mathcal{L}(\mathcal{K})^{(d)}, \mathcal{K}_0$ is a Hilbert space, $\pi_0 : \mathcal{A} \to \mathcal{L}(\mathcal{K}_0)$ is a unital representation killing the compacts, and $U : \mathcal{K} \to H^{(\nu)} \oplus \mathcal{K}_0$ is a Hilbert space isomorphism with $J = U^*(M^{(\nu)} \oplus \pi_0(M))U$. Define $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{K})$ by

$$\pi(X) = U^*(X^{(\nu)} \oplus \pi_0(X))U.$$

Obviously, π is a unital representation and $\pi(M) = J$.

Conversely, assume that $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{K})$ is a unital representation and $J = \pi(M)$. Setting

$$\mathcal{K}_1 = [\pi(f \otimes g)x : f, g \in \mathcal{H},\ x \in \mathcal{K}]$$

and

$$\mathcal{K}_0 = \mathcal{K} \ominus \mathcal{K}_1,$$

it is easy to verify that $\pi = \pi_1 \oplus \pi_0$ where $\pi_1 : \mathcal{A} \to \mathcal{L}(\mathcal{K}_1)$ and $\pi_0 : \mathcal{A} \to \mathcal{L}(\mathcal{K}_0)$ are unital representations. An analysis of the definition of $\mathcal{K}_1$ reveals that π_0 kills the compact operators on $\mathcal{H}$. Finally, observing that

$$\mathcal{K}_1 = [\pi(k_\lambda \otimes g)x : \lambda \in U, g \in \mathcal{H}, x \in \mathcal{K}]$$

allows one to deduce as in [**A1**] that $\pi_1(M)$ is unitarily equivalent to $M^{(\nu)}$ where

$$\nu = \dim([\pi(k_\lambda \otimes g)x : g \in \mathcal{H}, x \in \mathcal{K}])$$

is independent of the choice of λ. This establishes Theorem 1.1. □

In the project of studying the contractions model theoretically the von Neumann-Wold decomposition theorem plays a crucial role. Once one has realized that it is the coisometries that will form the collection of modelling operators (i.e. the operators to which the general contraction extends) then the fact that the general coisometry has a particular concrete form is an important step in this project. In the present context where the contractions have been replaced by the more general class $\mathcal{F}$ the von Neumann-Wold decomposition is replaced by Theorem 1.1.

Fix n distinct points $\lambda_1, \ldots, \lambda_n \in U$ and consider the ideal $I_{\overline{\lambda}} \subseteq \mathbb{C}[x_1, \ldots, x_d]$, the ring of polynomials in d variables, defined by

$$I_{\overline{\lambda}} = \{p : p(\overline{\lambda}_i) = 0 \quad \text{whenever} \quad 1 \leq i \leq n\}.$$

Associated with $I_{\overline{\lambda}}$ is the localized model $\mathcal{F}_{\overline{\lambda}}$ which is defined as the set of all $T \in \mathcal{F}$ such that $p(T) = 0$ whenever $p \in I_{\overline{\lambda}}$. The following localization result is the key to studying the norm in the n-dimensional Banach algebra formed from H_k^∞ by factoring out $(H_k^\infty)_\lambda$, the ideal of functions in H_k^∞ that vanish at the points $\lambda_1, \ldots, \lambda_n$.

Theorem 1.2. *Let $\mathcal{G}$ be a Hilbert space and let $T \in \mathcal{L}(\mathcal{G})$. $T \in \mathcal{F}_{\overline{\lambda}}$ if and only if there exists a cardinal ν and an invariant subspace $\mathcal{N}$ for $(M^*|\,[k_{\lambda_1}, \ldots, k_{\lambda_n}])^{(\nu)}$ such that T is unitarily equivalent to $(M^*|\,[k_{\lambda_1}, \ldots k_{\lambda_n}])^{(\nu)}|\mathcal{N}$.*

Proof. First observe that $M^*|[k_{\lambda_1}, \ldots k_{\lambda_n}] \in \mathcal{F}_{\overline{\lambda}}$ and that $\mathcal{F}_{\overline{\lambda}}$ is closed with respect to the operations of forming direct sums and restricting to invariant subspaces.

Conversely assume that $T \in \mathcal{F}_{\overline{\lambda}}$. Theorem 1.1 and the definition of $\mathcal{F}$ imply that there exists a cardinal ν and a unital representation $\pi_0 : \mathcal{L}(H^2_i) \to \mathcal{L}(\mathcal{K}_0)$ which annihilates the compact operators on $\mathcal{H}$ and an isometry $V : \mathcal{G} \to (\mathcal{H})^{(\nu)} \oplus \mathcal{K}_0$ such that

$$(1.3) \qquad \begin{array}{l} h(T) = V^*[h(M^*)^{(\nu)} \oplus h(\pi_0(M^*))]V \\ \text{and} \\ VV^*[h(M^*)^{(\nu)} \oplus h(\pi_0(M^*))]VV^* = [h(M^*)^{(\nu)} \oplus h(\pi_0(M^*))]VV^* \end{array}$$

whenever $h \in H(\overline{K})$.

We claim first that in (1.3) it may be assumed that the $h(\pi_0(M^*))$ summand is absent. Equivalently, if V is decomposed,

$$V = \begin{bmatrix} V_1 \\ V_0 \end{bmatrix},$$

with respect to $(\mathcal{H})^{(\nu)} \oplus \{0\}$ and $\{0\} \oplus \mathcal{K}_0$, then $V_0 = 0$. To see this recall that $\sigma_e(M) \subseteq \partial U$ and π_0 kills the compacts. Hence $\sigma(\pi_0(M)) \subseteq \partial U$. On the other hand (1.3) implies that

$$V_0^* p(\pi_0(M^*))^* p(\pi_0(M^*))V_0 = 0$$

whenever $p \in I_{\overline{\lambda}}$. But these facts would imply a contradiction if $V_0 \neq 0$. For suppose $x \in \mathcal{G}$ and $V_0 x \neq 0$. Let i denote the first positive integer such that

$$p(\pi_0(M^*))V_0 x = 0$$

whenever $p \in I_{\{\overline{\lambda}_1,\ldots,\overline{\lambda}_i\}}$ and choose $p_0 \in I_{\{\overline{\lambda}_1,\ldots,\overline{\lambda}_{i-1}\}}$ such that

$$p_0(\pi_0(M^*))V_0 x \neq 0.$$

By construction,

$$p_0(\pi_0(M^*))V_0 x \in \bigcap_{r=1}^{d} \ker\left(\overline{\lambda_i^r} - \pi_0(M^r)^*\right),$$

that is

$$\overline{\lambda}_i \in \sigma_p(\pi_0(M^*)).$$

This contradicts $\sigma(\pi_0(M)) \subseteq \partial U$ and establishes our claim that the $\pi_0(M^*)$ summand is absent.

We have shown that (1.3) can be reformulated as:

$$(1.4) \qquad h(T) = V^* h(M^*)^{(\nu)} V$$

whenever $h \in H(\overline{K})$. Since $p(T)^*p(T) = 0$ whenever $p \in I_{\overline{\lambda}}$, it follows from (0.8) that $\operatorname{ran} V \subseteq [k_{\lambda_1}, \dots, k_{\lambda_n}]^{(\nu)}$ which concludes the proof of Theorem 1.2. $\square$

A basic fact in Taylor's functional calculus [**T2**] is that if T is a commuting d-tuple of operators and $\sigma(T) \subseteq K$ then the map

$$h \longmapsto h(T)$$

defined on $H(K)$ is continuous on $H(K)$. If $f \in H(K)$ define $\check{f} \in H(\overline{K})$ by setting $\check{f}(\lambda) = \overline{f(\overline{\lambda})}$. It is then the case that $\check{f}(T^*) = f(T)^*$ whenever $f \in H(K)$ and $\sigma(T) \subseteq K$. Also the map

$$\check{h} \longmapsto \check{h}(T^*)$$

is continuous on $H(\overline{K})$. It follows by the nuclearity of $H(K)$ that the bilinear map defined on $H(\overline{K}) \times H(K)$ by

$$(\check{g}, f) \longmapsto \check{g}(T^*)f(T)$$

can be extended uniquely to a continuous map

$$h \longmapsto h(T)$$

defined on $H(\overline{K} \times K)$ hereafter referred to as the hereditary functional calculus for T. The following result is then clear (Theorem 1.5 in [**A1**]).

Theorem 1.3. *Let U be a bounded domain and let $\mathcal{H}$ be a regular space on U with associated kernel k. $T \in \mathcal{F}(k)$ if and only if $\sigma(T) \subseteq \overline{K}$ and the map*

$$h(M^*) \longmapsto h(T) \ , \quad h \in H(K \times \overline{K})$$

is completely positive.

As in [**A1**] Theorem 1.3 will allow us to derive a concrete condition for a tuple T with $\sigma(T) \subseteq \overline{U}$ to be an element of $\mathcal{F}(k)$. The key is the calculation that proves the following lemma whose proof is identical to the proof of Proposition 2.5 in [**A1**].

Lemma 1.4. *Let U be a bounded domain and let $\mathcal{H}$ be a regular space over U with kernel k. If $h \in \mathcal{M}_m(\mathbb{C}) \otimes H(K \times \overline{K})$, then $h(M^*) \geq 0$ if and only if the $\mathcal{M}_m(\mathbb{C})$-valued holomorphic kernel function,*

$$h(\mu, \overline{\lambda})k_\lambda(\mu)$$

is positive semi-definite on U.

We conclude this section with the promised concrete condition for $T \in \mathcal{F}(k)$ when $\sigma(T) \subseteq U$. For its proof follow the proof of Theorem 2.3 in [**A1**]. Remember that $\overline{U}$ is the complex conjugate of U.

Theorem 1.5. *Let* U *be a bounded domain in* $\mathbb{C}^d$ *and let* $\mathcal{H}$ *be a regular space over* U *with nonvanishing kernel* k. *If* T *is a commuting* d*-tuple of operators with* $\sigma(T) \subseteq \overline{U}$, *then* $T \in \mathcal{F}(k)$ *if and only if* $\frac{1}{k}(T) \geq 0$.

§2. Discrete Matrix Interpolation

In this section we shall review some of the ideas in Section I of [**A2**] and then derive a generalization of Proposition 1.18 of that paper. Our exposition will be somewhat terse; the reader is invited to consult [**A2**] for a chatty discussion.

Fix a positive integer n and let $N = \{i \in \mathbb{Z} : 1 \leq i \leq n\}$. If $g : N \times N \to \mathbb{C}$ we say $g \geq 0$ if the $n \times n$ matrix $(g(i,j))_{i,j \in N}$ is positive semidefinite. We define the support of g, $spt(g)$, by

$$spt(g) = \{i \in N : g(i,i) \neq 0\}.$$

If $g : N \times N \to \mathbb{C}$ and $g \geq 0$ we say g is a kernel on N if the matrix

$$(g(i,j))_{i,j \in spt(g)}$$

is positive definite. If g is a kernel on N, we define for each $i \in N$, $g_i : N \to \mathbb{C}$, by the formula $g_i(j) = g(i,j)$. We then form a Hilbert space H_g by defining an inner product on linear combinations of the form $\sum\limits_{i \in spt(g)} c_i g_i$ by setting

$$\langle \sum_i c_i g_i, \sum_j d_j g_j \rangle = \sum_{i,j} g(i,j) c_i \overline{d}_j.$$

Observe that H_g is a Hilbert space of functions on N and that g_i has the reproducing property $\langle f, g_i \rangle = f(i)$ whenever $f \in H_g$.

The alert reader will have noticed that we have just duplicated the construction that this paper began with, with the set U replaced by the set N but without the hypothesis that g be strictly positive definite. The analog of the operators $h(M)^*$ for $h \in H(K)$ would now be the class of operators with the property that the g_i are eigenfunctions for the operators. Accordingly for $z : N \to \mathbb{C}$ define $T_{g,z} \in \mathcal{L}(H_g)$ by requiring that

$$T_{g,z} g_i = \overline{z(i)} g_i \ , \quad i \in \ spt(g).$$

The adjoint is taken in this formula so that the notation will be consistent with (0.5). Also observe that $T_{g,z}$ depends only on the values of z on the set $spt(g)$.

Now if $I \subseteq N$ define $H_g(I) \subseteq H_g$ by $H_g(I) = [g_i : i \in I \cap spt(g)]$. If $I \subseteq N$ and $z : I \to \mathbb{C}$, define $T_{g,z} \in \mathcal{L}(H_g(I))$ by requiring that

$$T_{g,z} g_i = \overline{z(i)} g_i \quad i \in I \cap spt(g).$$

If one knows z then one knows the domain of z so this notation is unambiguous. Also observe that if $I_0 \subseteq I_1 \subseteq N,\ z_1 : I_1 \to \mathbb{C}$ and $z_0 = z_1|I_0$, then $H_g(I_0)$ is invariant for T_{g,z_1} and

$$T_{g,z_0} = T_{g,z_1}|H_g(I_0).$$

Now, if $I_0 \subseteq I_1 \subseteq N$ and $z : I_0 \to \mathbb{C}$, define $H_g(I_0, I_1) \subseteq H_g$ by

$$H_g(I_0, I_1) = H_g(I_1) \ominus H_g(I_1 \backslash I_0)$$

and define $T_{g,z}(I_1) \in \mathcal{L}(H_g(I_0, I_1))$ by

$$T_{g,z}(I_1) = P T_{g,\tilde{z}}|H_g(I_0, I_1),$$

where P is the orthogonal projection of $H_g(I_1)$ onto $H_g(I_0, I_1)$ and $\tilde{z}$ is any extension of z to I_1 (i.e. $\tilde{z} : I_1 \to \mathbb{C}$ and $\tilde{z}|I_0 = z$). It can be shown that $T_{g,z}(I_1)$ depends only on z and I_1. It does not depend on the choice of extension $\tilde{z}$.

We now extend these ideas in an obvious way to the vector valued case. For m a positive integer let $H_{g,m} = \mathbb{C}^m \otimes H_g$. Denoting $a \otimes g_i \in H_{g,m}$ by simply $a g_i$ when $a \in \mathbb{C}^m$ and $i \in spt(g)$ it is clear that the general element $f \in H_{g,m}$ can be represented uniquely in the form $\sum\limits_{i \in spt(g)} a_i g_i$. If $I \subseteq N$ set $H_{g,m}(I) = [a g_i : a \in \mathbb{C}^m, i \in spt(g) \cap I]$. If $I \subseteq N$ and $z : I \to \mathcal{M}_m(\mathbb{C})$ define $T_{g,z} \in \mathcal{L}(H_{g,m}(I))$ by requiring that

$$T_{g,z}(a g_i) = (z(i)^* a) g_i,\ \ a \in \mathbb{C}^m,\ \ i \in spt(g) \cap I.$$

As before observe that if $I_0 \subseteq I_1 \subseteq N, z_1 : I_1 \to \mathcal{M}_m(\mathbb{C})$, and $z_0 = z_1|I_0$, then $H_{g,m}(I_0)$ is invariant for T_{g,z_1} and $T_{g,z_0} = T_{g,z_1}|H_{g,m}(I_0)$. If $I_0 \subseteq I_1 \subseteq N$, define

$H_{g,m}(I_0, I_1) = H_{g,m}(I_1) \ominus H_{g,m}(I_1 \backslash I_0)$. If $I_0 \subseteq I_1 \subseteq N$ and $z : I_0 \to \mathcal{M}_m(\mathbb{C})$ define $T_{g,z}(I_1) \in \mathcal{L}(H_{g,m}(I_0, I_1))$ by

$$T_{g,z}(I_1) = PT_{g,\tilde{z}}|H_{g,m}(I_0, I_1),$$

where P is the orthogonal projection of $H_{g,m}(I_1)$ onto $H_{g,m}(I_0, I_1)$ and $\tilde{z}$ is any extension of z to I_1. As before the definition of $T_{g,z}(I_1)$ does not depend on the choice of $\tilde{z}$.

We now are ready to state and prove the promised generalization of Proposition 1.18 from [**A2**]. Let us agree to say g is a discrete m-interpolation kernel on N if for all $I \subseteq N$ and all $z : I \to \mathbb{C}$

$$\|T_{g,z}\| = \inf_{\substack{\tilde{z}:N \to \mathcal{M}_m(\mathbb{C}) \\ \tilde{z}|_I = z}} \|T_{g,\tilde{z}}\|.$$

Proposition 2.1. *Let g be a kernel on N. The following three conditions are equivalent.*

(i) *g is a discrete m-interpolation kernel.*

(ii) *If $I_0 \subseteq I_1 \subseteq N$ and $z : I_0 \to \mathcal{M}_m(\mathbb{C})$, then*

$$\|T_{g,z}(I_1)\| \leq \|T_{g,z}\|.$$

(iii) *If $I_0 \subseteq I_1 \subseteq N$, $I_1 \backslash I_0$ consists of a single point and $z : I_0 \to \mathcal{M}_m(\mathbb{C})$, then*

$$\|T_{g,z}(I_1)\| \leq \|T_{g,z}\|.$$

Proof. That (i) $\Rightarrow$ (ii) follows in exactly the same way as in the proof of Proposition 1.18 in [**A2**]. Indeed, by hypothesis (i), any z defined on I_0 can be extended to $\tilde{z}$ defined on all of N so that $\|T_{g,\tilde{z}}\|$ is arbitrarily close to $\|T_{g,z}\|$; and compressing $T_{g,\tilde{z}}$ to $H_g(I_0, I_1)$ can not increase the norm.

Obviously (ii) $\Rightarrow$ (iii). There remains to show that (iii) $\Rightarrow$ (i). Accordingly fix a kernel g, assume that (iii) holds, let $I \subseteq N, z : I \to \mathcal{M}_m(\mathbb{C})$, let $i' \in N \backslash I$ and set $I' = I \cup \{i'\}$. We shall show that

$$\inf_{\substack{w:I' \to \mathcal{M}_m(\mathbb{C}) \\ w|_I = z}} \|T_{g,w}\| = \|T_{g,z}\|. \tag{2.2}$$

Since I, i', and z are arbitrary, condition (i) will then follow by iteration.

Now by an argument similar to that which occurs in[**A2**], the infimum in (2.2) is actually attained. Choose $\tilde{z} : I' \to \mathbb{C}$ such that $\tilde{z}|I = z$ and such that

$$\rho_1 = \|T_{g,\tilde{z}}\|^2 = \inf_{\substack{w:I' \to \mathcal{M}_m(\mathbb{C}) \\ w|I=z}} \|T_{g,w}\|.$$

Set $\rho_0 = \|T_{g,z}\|^2$. Thus, Proposition 2.1 will be established if we can show that $\rho_1 = \rho_0$. We shall argue by contradiction. Accordingly assume that

$$\rho_0 < \rho_1. \tag{2.3}$$

Now choose $\omega \in H_g(I') \ominus H_g(I)$ with the property that $\langle g_{i'}, \omega \rangle = 1$. The existence of ω is guaranteed by (2.3). Fix an orthonormal basis $\{e_r\} \subseteq \mathbb{C}^m$ and observe that if $\delta \in \mathcal{M}_m(\mathbb{C})$ then

$$\varphi_\delta(t) = \|T_{g,\tilde{z}} + t \sum_r (\delta e_r) g_{i'} \otimes e_r \omega\|^2$$

is a candidate for the infimum in (2.2). Now φ_δ is not in general differentiable at $t = 0$. However it is differentiable from the right at $t = 0$ and its derivative can be calculated in the following manner. Set $\mathcal{M} = \ker(\rho_1 - T^*_{g,\tilde{z}} T_{g,\tilde{z}})$. Then

$$\lim_{t \to 0+} \frac{1}{t} \left(\varphi_\delta(t) - \varphi_\delta(0)\right) = \sup_{\substack{\gamma \in \mathcal{M} \\ \|\gamma\|=1}} 2Re\langle T^*_{g,\tilde{z}} \left(\sum_r (\delta e_r) g_{i'} \otimes e_r \omega \right) \gamma, \gamma \rangle. \tag{2.4}$$

Since 0 is a local minimum for φ_δ whenever $\delta \in \mathcal{M}_m(\mathbb{C})$ we deduce from (2.4) that

$$0 \leq \sup_{\substack{\gamma \in \mathcal{M} \\ \|\gamma\|=1}} 2Re\langle T^*_{g,\tilde{z}} \left(\sum_r (\delta e_r) g_{i'} \otimes e_r \omega \right) \gamma, \gamma \rangle \tag{2.5}$$

whenever $\delta \in \mathcal{M}_m(\mathbb{C})$. Now let $\mathcal{P}$ denote the convex set of positive operators $A \in \mathcal{L}(H_{g,m}(I'))$ with the properties ran$A \subseteq \mathcal{M}$ and $trA = 1$. Since the operators of the form $\gamma \otimes \gamma$ with $\gamma \in \mathcal{M}$ and $\|\gamma\| = 1$ are in $\mathcal{P}$ we deduce from (2.5) that

$$0 \leq \inf_{\delta \in \mathcal{M}_m(\mathbb{C})} \sup_{A \in \mathcal{P}} 2Re\ tr \left(T^*_{g,\tilde{z}} \left(\sum_r (\delta e_r) g_{i'} \otimes e_r \omega \right) A \right). \tag{2.6}$$

Now observe in (2.6) that the objective in the min-max problem is a real bilinear function in δ and A. Consequently, the von Neumann minimax theorem (or indeed the Hahn-Banach theorem) implies that there exists $A_0 \in \mathcal{P}$ such that

$$0 \leq 2Re\ tr\left(T^*_{g,\tilde{z}}\left(\sum_r (\delta e_r)g_{i'} \otimes e_r\omega\right)A_0\right)$$

for all $\delta \in \mathcal{M}_m(\mathbb{C})$. Replacing δ by $e^{i\theta}\delta$ with $e^{i\theta}$ appropriately chosen thus yields that in fact,

$$0 = tr\left(T^*_{g,\tilde{z}}\left(\sum_r (\delta e_r)g_{i'} \otimes e_r\omega\right)A_0\right)$$

for all $\delta \in \mathcal{M}_m(\mathbb{C})$. In this last equality letting $\delta = x \otimes e_s$ where $x \in \mathbb{C}^m$ and $1 \leq s \leq m$ reveals that

$$0 = tr\ T^*_{g,\tilde{z}}(xg_{i'} \otimes e_s\omega)A_0$$

whenever $x \in \mathbb{C}^m$ and $1 \leq s \leq m$. Hence

$$T^*_{g,\tilde{z}}(\mathbb{C}^m g_{i'}) \perp A_0(\mathbb{C}^m\omega). \tag{2.7}$$

Now, we claim that $A_0(\mathbb{C}^m\omega) \neq \{0\}$. For otherwise, $\mathbb{C}^m\omega \perp \text{ran}A_0 \subseteq \mathcal{M}$. Since $tr\ A_0 = 1$ this would imply the existence of a nonzero vector in $\mathcal{M}$ (the space on which $T_{g,\tilde{z}}$ attains its norm) that is orthogonal to $\mathbb{C}^m\omega$ contradicting (2.3). Choose $y \in A_0(\mathbb{C}^m\omega)$ with $\|y\| = 1$. Observe that since $\text{ran}A_0 \subseteq \mathcal{M}, T^*_{g,\tilde{z}}T_{g,\tilde{z}}y = \rho_1 y$. Also (2.7) implies that

$$T_{g,\tilde{z}}y \perp \mathbb{C}^m g_{i'}. \tag{2.8}$$

Now, if $a \in \mathbb{C}^m$, then

$$\begin{aligned}\langle y, ag_{i'}\rangle &= \frac{1}{\rho_1}\langle T^*_{g,\tilde{z}}T_{g,\tilde{z}}y, ag_{i'}\rangle\\ &= \frac{1}{\rho_1}\langle T_{g,\tilde{z}}y, (\tilde{z}(i')^*a)g_{i'}\rangle\\ &= 0.\end{aligned}$$

Hence we also have that

$$y \perp \mathbb{C}^m g_{i'}. \tag{2.9}$$

We now use (iii), (2.8), and (2.9) to derive a contradiction to (2.3). Let P denote the orthogonal projection of $\mathcal{H}_{g,m}(I')$ onto $\mathcal{H}_{g,m}(I, I')$. Observe that (2.8) and (2.9) imply that y and $T_{g,\tilde{z}}y$ are in $\mathcal{H}_{g,m}(I,I')$. Hence

$$\begin{aligned}
\rho_1 &= \|T_{g,\tilde{z}}y\|^2 \\
&= \|(PT_{g,\tilde{z}}|\mathcal{H}_{g,m}(I,I'))y\|^2 \\
&= \|T_{g,\tilde{z}}(I')y\|^2 \\
&\leq \|T_{g,\tilde{z}}(I')y\|^2 \\
&\leq \|T_{g,z}\| \\
&= \rho_0,
\end{aligned}$$

a contradiction which concludes the proof of Proposition 2.1. □

§3. Holomorphic Interpolation Kernels

In this section we shall give a concrete model theoretic condition on the family $\mathcal{F}(k)$ which is both necessary and sufficient for the Nevanlinna-Pick interpolation result to be true for the space H_k^∞. Specifically, we introduce the following definition.

Definition 3.1. Let $U \subseteq \mathbb{C}^d$ be a bounded domain and let k be a kernel on U of the type described in the introduction. Then k is a holomorphic m-interpolation kernel on U if for each positive integer n, each choice of distinct points $\lambda_1, \ldots, \lambda_n \in U$ and each choice of matrices $z_1, \ldots, z_n \in \mathcal{M}_m(\mathbb{C})$, there exists a function $\varphi \in H_{k,m}^\infty$ with $\|\varphi\| \leq 1$ and $\varphi(\lambda_i) = z_i$ for each i if and only if the $mn \times mn$ matrix $[(1 - z_i^* z_j)k_{\lambda_i}(\lambda_j)]$ is positive semidefinite. We say k is a holomorphic complete interpolation kernel if k is a holomorphic m-interpolation kernel for every m.

Our first result is a holomorphic analog of Proposition 2.1.

Proposition 3.2. *The kernel k is a holomorphic m-interpolation kernel on U if and only if for each positive integer n and each choice of distinct points $\lambda_1, \ldots, \lambda_n \in U$ the kernel g on N defined by $g(i,j) = k_{\lambda_i}(\lambda_j)$ for $1 \leq i, j \leq n$ is a discrete m-interpolation kernel on N.*

To prove the proposition mimic the argument from [**A2**] that deduced Theorem 1.27 from Lemma 1.26: Given $z_1, \ldots, z_n$ on $\lambda_1, \ldots, \lambda_n$, choose a countable set of uniqueness in U, and extend z point by point to this set. In the limit, one gets a bounded operator whose adjoint commutes with M_z^* on a dense set, and so comes from a function ϕ.

Now, if g is a discrete kernel on N as in §2 of this paper, then there is no particularly distinguished operator of the form $T_{g,z}$ with $z : N \to \mathbb{C}$. However, if g arises as in Proposition 3.1 by localizing the holomorphic kernel k to n distinct points $\lambda_1, \ldots, \lambda_n$, then Theorem 1.2 provides ample evidence that the d-tuple of operators, $M_z^*|[k_{\lambda_i}, \ldots, k_{\lambda_n}]$, which in the $T_{g,z}$ notation has the form $T_\lambda = (T_{g,\lambda^1}, \ldots, T_{g,\lambda^d})$ where for each $r, \lambda^r : N \to \mathbb{C}$ is defined by $\lambda^r(i) = \lambda_i^r$, is highly distinguished. To exploit this tuple we introduce the following notion.

Observe that Theorem 1.2 is equivalent to the following assertion.

(3.3) If T has an $H(\overline{K})-$extension to an operator of the form $\pi(M_z^*)$ and $T \in S_{\overline{\lambda}}$, then T has an $H(\overline{K})-$extension to an operator of the form $\pi(M_z^*|[k_{\lambda_1}, \ldots, k_{\lambda_n}])$.

The assertion of (3.3) is a property of the kernel k. We introduce the following definition which results from replacing the 2 occurrences of the word "extension" by the word "dilation" in property (3.3).

Definition 3.4. Let $U \subseteq \mathbb{C}^d$ be a bounded domain and let k be a kernel on U of the type described in the introduction. Then k is *dilation localizable* if for all choices of n distinct points $\lambda_1, \ldots, \lambda_n \in U$ (3.3) holds with the word extension replaced by the word dilation.

We now can state the principal result of this paper.

Theorem 3.5. *Let $U \subseteq \mathbb{C}^d$ be a bounded domain and let k be a kernel on U of the type described in the introduction. Then k is a holomorphic complete interpolation kernel if and only if k is dilation localizable.*

We shall deduce Theorem 3.5 as a corollary of the Arveson dilation machinery and Theorem 3.6 below. If $T \in \mathcal{S}_\lambda$ it is clear that $\sigma(T) = \{\lambda_1, \ldots, \lambda_n\} \subseteq U$ and that the functional calculus map

$$\Phi_T(h) = h(T) \ , \quad h \in H(K)$$

extends by continuity to a continuous unital algebra homomorphism onto $\mathcal{A}_T$ of H_k^∞. Also, it should be clear on the level of algebra that if P denotes the orthogonal projection of $\mathcal{H}$ onto $[k_{\lambda_1}, \ldots, k_{\lambda_n}]$, then

$$\Psi_T(h(PM|[k_{\lambda_1}, \ldots, k_{\lambda_n}])) = h(T)$$

defines a unital algebra homomorphism of $\mathcal{A}_{PM|\,[k_{\lambda_1},\ldots,k_{\lambda_n}]}$ onto $\mathcal{A}_T$.

Recall that k has the *m-contractive localization property* if for all positive integers n, all choices of distinct points $\lambda_1, \ldots, \lambda_n \in U$, and all $T \in \mathcal{S}_\lambda$ if Φ_T is m-contractive then Ψ_T is m-contractive. Define $\rho : H_k^\infty \to \mathcal{A}_{PM|\,[k_{\lambda_1},\ldots,k_{\lambda_n}]}$ by $\rho(h) = h(PM|\,[k_{\lambda_1}, \ldots, k_{\lambda_n}])$. To say that k has the m-contractive localization property means that the diagram below can always be completed in the category of operator algebras with morphisms the unital m-contractive algebra homomorphisms.

$$\begin{tikzcd}
 & \mathcal{A}_{PM_z|\,[k_{\lambda_1},\ldots,k_{\lambda_n}]} \arrow[dr, dotted, "\Psi_T"] & \\
H_k^\infty \arrow[ur, "\rho"] \arrow[rr, "\Phi_T"] & & \mathcal{A}_T
\end{tikzcd}$$

Theorem 3.6. *Let $U \subseteq \mathbb{C}^d$ be a bounded domain and let k be a kernel on U of the type described in the introduction. Then k is a holomorphic m-interpolation kernel if and only if k has the m-contractive localization property.*

Proof. Suppose k is an m-interpolation kernel and assume that $T \in S_\lambda$ and $\|id_m \otimes \Phi_T\| \leq 1$. We wish to show that if $h \in \mathcal{M}_m(\mathbb{C}) \otimes H(K)$ and $\|h(PM_z|\,[k_{\lambda_1}, \ldots, k_{\lambda_n}])\| \leq 1$, then $\|h(T)\| \leq 1$. Since k is an m-interpolation kernel, if $\|h(PM_z|\,[k_{\lambda_1}, \ldots, k_{\lambda_n}])\| \leq 1$, then there exists $\tilde{h} \in \mathcal{M}_m(\mathbb{C}) \otimes H_k^\infty$ with $\|\tilde{h}\|_\infty \leq 1$ and $\tilde{h}(\lambda_i) = h(\lambda_i)$ for each i.

Let us belabor the proof of this last assertion, as it is key.

$$\begin{aligned}\|h((PM_z|\,[k_{\lambda_1},\dots,k_{\lambda_n}])\| \;=\; \|h(PMP)\| &= \|Ph(M)P\| \\ &= \|Ph(M)^*P\| \\ &= \|\check{h}(M^*)P\|\end{aligned}$$

Now, the fact that k is an m-interpolation kernel means that $\check{h}$ can be replaced by h_1 where $h_1(\overline{\lambda}_i) = \check{h}(\overline{\lambda}_i)$ for each i and

$$\|\check{h}_1\|_{H^\infty_k} = \|h_1(M^*)P\| = \|\check{h}(M^*)P\| \leq 1.$$

Put $\tilde{h} = \check{h}_1$ and it satisfies the assertion.

As T is in $\mathcal{S}_\lambda$, $\tilde{h}(T) = h(T)$. Consequently,

$$\|h(T)\| = \|\tilde{h}(T)\| = \|id_m \otimes \Phi_T(\tilde{h})\| \leq 1.$$

Now assume that k has the m-contractive localization property. By Proposition 3.2, k will be an m-interpolation kernel if $g_i(j) = k_{\lambda_i}(\lambda_j)$ is a discrete m-interpolation kernel. We prove this by verifying condition (iii) of Proposition 2.1. Accordingly fix $\lambda_1,\dots,\lambda_{n+1} \in U$ and let $N_1 = \{1,2,\dots,n+1\}$. We wish to show that if $z : N \to \mathcal{M}_m(\mathbb{C})$, then

$$\|PT_{k,\tilde{z}}|_{\mathcal{H}_g \ominus \mathbb{C}g_{n+1}}\| \leq \|T_{g,z}\| \tag{3.7}$$

where P denotes the orthogonal projection of $\mathcal{H}_g$ onto $\mathcal{H}_g \ominus \mathbb{C}g_{n+1}$ and $\tilde{z}$ is any extension of N to N_1. Now exploiting the definition of g it is clear that if the map Ω is defined by

$$\Omega(\check{h}(M^*|_{[k_{\lambda_1},\dots,k_{\lambda_n}]})) = \check{h}(QM^*|_{[k_{\lambda_1},\dots,k_{\lambda_{n+1}}]\ominus\mathbb{C}k_{\lambda_{n+1}}}),\ h \in H(K) \tag{3.8}$$

where Q denotes the orthogonal projection onto $[k_{\lambda_1},\dots,k_{\lambda_{n+1}}] \ominus \mathbb{C}k_{\lambda_{n+1}}$, then (3.7) is equivalent to the m-contractivity of Ω. If $\check{\Omega}$ is defined by $\check{\Omega}(S) = \Omega(S^*)^*$ we obtain that (3.7) will follow from the m-contractivity of $\check{\Omega}$.

Now set $T = QM|[k_{\lambda_1},\dots,k_{\lambda_{n+1}}] \ominus \mathbb{C}k_{\lambda_{n+1}}$. Evidently, $\check{\Omega} = \Psi_T$ and Φ_T is completely contractive. Since in particular Φ_T is m-contractive we deduce from the m-contractive localization property that $\check{\Omega}$ is m-contractive. This concludes the proof of Theorem 3.6. □

Proof of Theorem 3.5. By [**Arv1**], the statement that T^* has an $H(\overline{K})$-dilation to an operator of the form $\pi(M_z^*)$ is equivalent to the assertion that Φ_{T^*} is a complete contraction; and that T^* has an $H(\overline{K})$-dilation to an operator of the form $\pi(M_z^*)|[k_{\lambda_1}, \dots, k_{\lambda_n}]$ is equivalent to the assertion that Ψ_{T^*} is a complete contraction. So, by interchanging T and T^*, the statement of the theorem is equivalent to saying that k is a holomorphic complete interpolation kernel if and only if whenever Φ_T is completely contractive, then Ψ_T is completely contractive.

It follows from Theorem 3.6 that if k is a holomorphic complete interpolation kernel, then if Φ_T is completely contractive, then Ψ_T is m-contractive for all m, and hence completely contractive. Conversely, if the complete contractivity of Φ_T implies that of Ψ_T, then the map Ω of (3.8) will be completely contractive, so the same argument used in the proof of Theorem 3.6 shows that k is a holomorphic complete interpolation kernel.

§4. Some Examples and Remarks

In this section we shall give two concrete applications of Theorem 3.6. Our intent is more to demonstrate that Theorem 3.6 contains interesting function theoretic content rather than to work out the most general possible concrete interpolation theorem that would follow from the ideas in this section. Our first application is a matrix valued generalization of Theorem 0.1 in [**A2**].

Theorem 4.1. *If $\mathcal{H}$ is the Dirichlet space $\Big($i.e. the Hilbert space of analytic functions on $\mathbb{D}$ with reproducing kernel defined by $k_\lambda(\mu) = \dfrac{1}{\overline{\lambda}\mu} \log \dfrac{1}{1-\overline{\lambda}\mu}\Big)$, then $\mathcal{H}$ is a complete interpolation space.*

Our second application constitutes a generalization of the classical Nevanlinna-Pick interpolation theorem to the ball.

Theorem 4.2. *Let d be a positive integer and let B denote the open unit ball in $\mathbb{C}^d$. If $\mathcal{H}$ is the Hilbert space of analytic functions on B with reproducing kernel defined by $k_\lambda(\mu) = \dfrac{1}{1-\langle \mu, \lambda\rangle}$, then $\mathcal{H}$ is a complete interpolation space.*

Theorem 4.2 should be contrasted with Theorem 3.16 in [**A2**] which derives an analog of the Nevanlinna-Pick interpolation theorem on the polydisc. In that solution the first departure of the canonical interpolation norm from the H^∞ norm occurs in dimension 3. In Theorem 4.2 already when $d = 2, H_k^\infty \neq H^\infty(B)$ (this has also been observed in [**Arv3**]). Indeed if $n = (n_1, n_2)$ is a multi-index, then

$$\|z^n\|_{\mathcal{H}}^2 = \begin{pmatrix} n_1 + n_2 \\ n_2 \end{pmatrix}^{-1}$$

and

$$\sup_{z \in B} |z^n|^2 = \left(\frac{n_1}{n_1 + n_2}\right)^{n_1} \left(\frac{n_2}{n_1 + n_2}\right)^{n_2}.$$

Consequently,

$$\begin{aligned} \|z^n\|_{H_k^\infty}^2 &\geq \frac{\|M_{z^n} z^n\|_{\mathcal{H}}^2}{\|z^n\|_{\mathcal{H}}^2} \\ &= \frac{\|z^{2n}\|_{\mathcal{H}}^2}{\|z^n\|_{\mathcal{H}}^2} \\ &= \begin{pmatrix} n_1 + n_2 \\ n_1 \end{pmatrix} \begin{pmatrix} 2(n_1 + n_2) \\ 2n_1 \end{pmatrix}^{-1} \left(\frac{n_1}{n_1 + n_2}\right)^{-n_1} \left(\frac{n_2}{n_1 + n_2}\right)^{-n_2} (\sup_{z \in B} |z^n|)^2. \end{aligned}$$

Setting $n_1 = n_2 = j$ and letting $j \to \infty$ we deduce that the inclusion map of H_k^∞ into $H^\infty(B)$ is not bounded below. Hence by the open mapping theorem, $H_k^\infty \neq H^\infty(B)$.

In connection with the problem of ordinary H^∞-interpolation on B observe that the facts that $H_k^\infty \subseteq H^\infty(B)$ and $\|\varphi\|_{H_k^\infty} \leq \|\varphi\|_{H^\infty(B)}$ whenever $\varphi \in H_k^\infty$ imply via Theorem 4.2 that

$$\left(\frac{1 - z_j \overline{z}_i}{1 - \langle \lambda_j, \lambda_i \rangle}\right) \geq 0$$

is a sufficient condition for there to exist a holomorphic function φ on B such that $\varphi(\lambda_i) = z_i$ for each i and $\sup_{\lambda \in B} |\varphi(\lambda)| \leq 1$. On the other hand since $\frac{1}{(1 - \langle \mu, \lambda \rangle)^d}$ is the Szegö kernel for B it is clear from the usual proof of the Nevanlinna-Pick on $\mathbb{D}$ that

$$\left(\frac{1 - z_j \overline{z}_i}{(1 - \langle \lambda_j, \lambda_i \rangle)^d}\right) \geq 0$$

is a necessary condition for there to exist a holomorphic function φ on B such that $\varphi(\lambda_i) = z_i$ for each i and $\sup_{\lambda \in B} |\varphi(\lambda)| \leq 1$. These considerations prompt one to ask whether there is a kernel g on the ball (somehow intermediate between $(1 - \langle \mu, \lambda \rangle)^{-1}$ and $(1 - \langle \mu, \lambda \rangle)^{-d}$) with the property that $((1 - z_j \overline{z}_i) g_{\lambda_i}(\lambda_j)) \geq 0$ is both necessary and sufficient for ordinary H^∞ interpolation. We close this section with an argument which shows that such a kernel g does not exist.

Theorem 4.1 and Theorem 4.2 will both be deduced from the following fact.

Proposition 4.3. *Let $\mathcal{H}$ be a regular Hilbert space of analytic functions on a bounded domain $U \subseteq \mathbb{C}^d$ with kernel k. If $T \in \mathcal{F}(k)$ whenever T has a $H(K)$-dilation to an element of $\mathcal{F}(k)$, then $\mathcal{H}$ is a complete interpolation space.*

Proof. The proposition will follow from Theorem 3.6 if we can establish that $\mathcal{H}$ has the $H(\overline{K})$-dilation property. Accordingly, assume that $\lambda_1, \ldots, \lambda_n$ are n distinct points in $U, T \in \mathcal{S}_{\overline{\lambda}}$ and T has an $H(\overline{K})$-dilation to an operator of the form $\pi(M^*)$. We need to show that T has an $H(\overline{K})$-dilation to an operator of the form $\pi(M^*|\mathcal{H}_\lambda^\perp)$. This follows immediately from the hypotheses and Theorem 1.2. □

We now prove Theorem 4.1. Let $\mathcal{H}$ denote the Dirichlet space and set $k_\lambda(\mu) = \dfrac{1}{\overline{\lambda}\mu}\log\dfrac{1}{1-\overline{\lambda}\mu}$. By Lemma 2.7 in [**A2**] there exists a positive sequence $a_1, a_2, \ldots$ such that

$$\frac{1}{k_\lambda(\mu)} = 1 - \sum_{n=1}^{\infty} a_n(\overline{\lambda}\mu)^n, \tag{4.4}$$

where the series in (4.4) converges uniformly on compact subsets of $\mathbb{D}\times\mathbb{D}$. We claim that

$$T \in \mathcal{F}(k) \quad \text{if and only if} \quad rT \in \mathcal{F}(k) \quad \text{whenever} \quad 0 \le r < 1. \tag{4.5}$$

To prove (4.5) first assume that $T \in \mathcal{F}(k)$ and fix r with $0 \le r < 1$. Since $T \in \mathcal{F}(k), \sigma(T) \subseteq \mathbb{D}^-$. Hence $\sigma(rT) \subseteq r\mathbb{D}^-$, a compact subset of $\mathbb{D}$. Thus, by Theorem 1.5 that $rT \in \mathcal{F}(k)$ will follow if we can show that $\dfrac{1}{k}(rT) \ge 0$. But Theorem 1.3 and Lemma 1.4 imply that $\dfrac{1}{k}(rT) \ge 0$ follows from the positive definiteness of $\dfrac{k_\lambda(\mu)}{k_{r\lambda}(r\mu)}$ on $\mathbb{D}$. Since (4.4) implies that

$$\frac{k_\lambda(\mu)}{k_{r\lambda}(r\mu)} = 1 + \sum_{n=1}^{\infty} a_n(1-r^{2n})(\overline{\lambda}\mu)^n k_\lambda(\mu)$$

we deduce that $rT \in \mathcal{F}(k)$ as was to be shown. Now assume that $rT \in \mathcal{F}(k)$ whenever $0 \leq r < 1$. It follows from Theorem 1.3 that the hereditary functional calculus map

$$h(M^*) \longrightarrow h(rT)$$

is completely positive whenever $0 \leq r < 1$. Since for each $h \in H(\mathbb{D}\times\overline{\mathbb{D}})$, $\lim\limits_{r\to 1^-} h(rT) = h(T)$ we deduce that the map

$$h(M^*) \longrightarrow h(T)$$

is also completely positive. Hence Theorem 1.3 implies that $T \in \mathcal{F}(k)$. This establishes (4.5).

The proof of Theorem 4.1 is now easy to conclude. Assume that $T \in \mathcal{L}(\mathcal{K})$, $T \in \mathcal{F}(k), \mathcal{M} \subseteq \mathcal{K}$ is a semi-invariant subspace for T, and let P denote the orthogonal projection of $\mathcal{K}$ onto $\mathcal{M}$. Theorem 4.1 will follow from Proposition 4.3 if we can show that $PT|\mathcal{M} \in \mathcal{F}(k)$. Fix $f \in \mathcal{K}$ and let $r < 1$. Employing (4.4) we see that

$$\begin{aligned} \langle \frac{1}{k}(rPT|\mathcal{M})x, x\rangle &= \|x\|^2 - \sum_{n=1}^{\infty} a_n r^{2n} \|PT^n x\|^2 \\ &\geq \|x\|^2 - \sum_{n=1}^{\infty} a_n r^{2n} \|T^n x\|^2 = \langle \frac{1}{k}(rT)x, x\rangle. \end{aligned}$$

This inequality, (4.5) and Theorem 1.5 imply that $\frac{1}{k}(rPT|\mathcal{M}) \geq 0$. Since $r < 1$ is arbitrary we conclude via Theorem 1.5 and (4.5) that $PT|\mathcal{M} \in \mathcal{F}(k)$ establishing Theorem 4.1. ☐

The proof of Theorem 4.2 follows the general outline of the proof of Theorem 4.1. The equation (4.4) is replaced by the equation

$$\frac{1}{k_\lambda(\mu)} = 1 - \overline{\lambda}^1 z^1 - \overline{\lambda}^2 z^2,$$

and (4.5) is trivial.

We now prove the claim made at the beginning of this section that there does not exist a kernel k on the ball with respect to which the Nevanlinna-Pick theorem is true in H^∞ norm. In fact much more is true.

Proposition 4.6. *Let $\mathcal{H}$ be a regular Hilbert space of analytic functions on B, the open unit ball in $\mathbb{C}^d$. Let k denote the kernel for $\mathcal{H}$ and assume that 0.4 (ii) is* *equivalent to the solvability of the $H^\infty(B)$ Pick problem whenever $m = 1, n = 2$, and $\lambda_1, \lambda_2 \in B$.*

Then there exists a nonvanishing holomorphic function f on B such that $k_\lambda(\mu) = \overline{f(\lambda)}f(\mu)(1 - \langle \mu, \lambda \rangle)^{-1}$. In particular, M is unitarily equivalent to the d-tuple M of Theorem 4.2 and if $\lambda_1, \ldots, \lambda_n \in B$ and $z_1, \ldots, z_n \in \mathcal{M}_m(\mathbb{C})$, then

$$((1 - z_j z_i^*)k_{\lambda_i}(\lambda_j)) \geq 0$$

if and only if

$$\left(\frac{(1 - z_j z_i^*)}{1 - \langle \lambda_j, \lambda_i \rangle}\right) \geq 0.$$

Proof. First observe by considering the Carathéodory metric on B that if $\lambda \in B$ and $z \in \mathbb{D}$ then there exists a holomorphic mapping $h : B \to \mathbb{D}$ with $h(0) = 0$ and $h(\lambda) = z$ if and only if $|z| \leq \|\lambda\|$. On the other hand the assumed equivalence of 0.4 (i) and 0.4 (ii) implies that there exists a holomorphic mapping $h : B \to \mathbb{D}$ with $h(0) = 0$ and $h(\lambda) = z$ if and only if

$$\begin{bmatrix} k_0(0) & k_0(\lambda) \\ k_\lambda(0) & (1 - |z|^2)k_\lambda(\lambda) \end{bmatrix} \geq 0.$$

Hence we conclude that if $\lambda \in B$ and $z \in \mathbb{D}$, then $|z| \leq \|\lambda\|$ if and only if $|z|^2 k_0(0)k_\lambda(\lambda) \leq k_0(0)k_\lambda(\lambda) - |k_0(\lambda)|^2$. In particular,

$$k_\lambda(\lambda) = \frac{|k_0(\lambda)|^2}{k_0(0)} \frac{1}{1 - |\lambda|^2} \tag{4.7}$$

whenever $\lambda \in B$. If we define holomorphic functions g and h on $B \times B$ by the formulae

$$g(\mu, \lambda) = k_{\overline{\lambda}}(\mu)$$

and

$$h(\mu, \lambda) = \frac{\overline{k_0(\overline{\mu})}k_0(\lambda)}{k_0(0)} \frac{1}{1 - \langle \lambda, \overline{\mu} \rangle}$$

we deduce from (4.7) and the fact that $\{(\mu, \overline{\mu}) : \mu \in B\}$ is a set of uniqueness that $g = h$, and hence $k_\lambda(\mu) = \overline{f(\lambda)}f(\mu)(1 - \langle \mu, \lambda \rangle)^{-1}$ if f is defined by $f(\mu) = k_0(\mu)k_0(0)^{-\frac{1}{2}}$. The other conclusions of Proposition 4.6 follow immediately. $\square$

Proposition 4.6 implies that there is no kernel on the ball with respect to which the Nevanlinna-Pick theorem with H^∞ norm can be true. For by Proposition 4.6 such a kernel could be taken to be defined by $k_\lambda(\mu) = (1 - \langle \mu, \lambda \rangle)^{-1}$. Since $H^\infty_k \neq H^\infty(B)$, there exists an $\varphi \in H^\infty_k$ such that $\|\varphi\|_{H^\infty_k} = \rho > 1 = \|\varphi\|_{H^\infty(B)}$. To show that the Nevanlinna-Pick theorem doesn't hold, we must find an n-tuple $\lambda_1, \ldots, \lambda_n$ such that the matrix

$$\left(\frac{1 - \varphi(\lambda_j)\overline{\varphi(\lambda_i)}}{1 - \langle \lambda_j, \lambda_i \rangle} \right) \tag{4.8}$$

is not positive. But the positivity of (4.8) is equivalent to saying M_φ^* is a contraction on $[k_{\lambda_1}, \ldots, k_{\lambda_n}]$, and as finite linear combinations of the kernel functions are dense in $\mathcal{H}$, this would contradict the fact that $\|\varphi\|_{H^\infty_k} > 1$.

Note that the kernel in Theorem 4.2 has also been studied by Arveson [**Arv3**].

References

[A1] Agler, J., "The Arveson extension theorem and coanalytic models", *Int. Eq. and Op. Th.* **5**(1982), 608-631.

[A2] Agler, J., "Some interpolation theorems of Nevanlinna-Pick type", preprint.

[Agr-S] Agrawal, O. P. and Salinas, N., "Sharp kernels and canonical subspaces", *Am. Jour. of Math.* **110**(1987), 23-48.

[Aro] Aronszajn, N., "Theory of reproducing kernels", *T.A.M.S.* **68**(1950), 337-404.

[Arv1] Arveson, W. B., "Subalgebras of C^*-algebras", *Acta Math.* **123**(1969), 141-224.

[Arv2] Arveson, W. B., "Subalgebras of C^*-algebras II", *Acta Math.* **128**(1972), 271-308.

[Arv2] Arveson, W. B., "Subalgebras of C^*-algebras III: Multivariable Operator Theory", preprint.

[C] Curto, R., "Applications of several complex variables to multiparameter spectral theory", *Surveys of some recent results in Operator Theory, Vol. II,* ed. J.B. Conway and B.B. Morrel, Longman, Harlow, 1988, 25-90.

[N] Nevanlinna, R., "Über Beschränkte Funktionen die in gegebene Punkten vorgeschriebene Werte annehmen", *Ann. Acad. Sci. Fenn. Ser.* **A13**(1919), No. 1.

[P] Pick, G., "Über die Beschränkungen analytischer Funktionen, welche durch vorgegebene Funktionswerte bewirkt werden", *Math. Ann.* **77**(1916),7-23.

[T1] Taylor, J. L., "A joint spectrum for several commuting operators", *Jour. Func. Anal.* **6**(1970), 172-191.

[T2] Taylor, J. L., "The analytic functional calculus for several commuting operators", *Acta Math.* **125**(1970), 1-38.

University of California at San Diego, La Jolla California 92093

Washington University, St. Louis, Missouri 63130