MULTIPLIERS OF DE BRANGES SPACES

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0. Introduction

Hilbert spaces of functions on the unit circle $\mathbb{T}$ that can be extended to analytic functions on the disk $\mathbb{D}$ lie at the heart of harmonic analysis, from the solution of the prediction theory problems of Szegö [24], Kolmogorov [14], Helson-Szegö [13] and Helson-Sarason [12], to the success of the theory of de Branges and Rovnyak on spaces contractively contained in H^2 in [4]. In this paper we consider certain of these latter spaces.

Let b be a fixed function in the unit ball of H^∞, the bounded analytic functions on $\mathbb{D}$; let $D_b = (1 - T_b T_{\bar{b}})^{\frac{1}{2}}$, where $T_m : H^2 \to H^2$ denotes the Toeplitz operator of multiplication by the function m followed by projection onto H^2, the space of square-integrable functions on the circle whose negative Fourier coefficients vanish. Then the de Branges space $\mathcal{H}(b)$ is defined to be the range of D_b, with an inner product that makes D_b into a partial isometry. $\mathcal{H}(b)$ is a (not necessarily closed) subspace of H^2, on which the evaluation functionals at points in the disk are continuous.

The spaces $\mathcal{H}(b)$ were introduced by de Branges and Rovnyak in [5] and [6] and have been studied and utilised in various contexts, ranging from model theory [1] and kernels of Toeplitz operators [10], to exposed points in H^1 [23] and complex function theory [22]. The structure of the spaces, however, is still not well understood; a natural question to ask is, given a specific b, what are the multipliers of $\mathcal{H}(b)$, *i.e.* what functions (necessarily in H^∞) multiply $\mathcal{H}(b)$ into itself? This question has been addressed in [16],[17],[20] and [21]; we shed some further light on it below, but the general solution is still unknown.

For any (positive) measure μ on the circle, $H^2(\mu)$ will denote the closure of the (analytic) polynomials in $L^2(\mu)$. In Section 1 we introduce the measure μ_b associated with b; a function m is a multiplier of $\mathcal{H}(b)$ if and only if the Toeplitz operator $T_{\bar{m}}$, defined *a priori* on polynomials, is bounded on $H^2(\mu_b)$.

Let us call a measure μ *Szegö* if it is absolutely continuous with respect to Lebesgue measure σ on the circle, and if evaluation at any point of $\mathbb{D}$ is a continuous linear functional on $H^2(\mu)$; any Szegö measure can be written as $|h|^2\sigma$, where h is some H^2 function ([9,p.144]). If, in addition, the truncation operator Π that takes a formal Fourier series $\sum_{n=-\infty}^{\infty} a_n z^n$ to $\sum_{n=0}^{\infty} a_n z^n$ is bounded on $L^2(\mu)$, then μ is called *Helson-Szegö*. In Section 2 we show that if $\mu = |h|^2\sigma$ is a Szegö measure then a necessary condition for $T_{\bar{m}}$ to be bounded on $H^2(\mu)$ is that m be in the range of $T_{\bar{h}}$; in the special case that $\mu = |p|^2\rho$, where p is a polynomial and ρ is Helson-Szegö, we show that this condition is also sufficient (Theorem 2.3).

In section 3 we consider the effect of adding a singular measure to an absolutely continuous one. Let $T(\mu) = \{m \in H^\infty : T_{\bar{m}} \text{ is bounded on } H^2(\mu)\}$, and let $\mu = \mu_a + \mu_s$ be the decomposition of μ into absolutely continuous and singular parts (with respect to σ); then we can classify $T(\mu)$ in terms of $T(\mu_a)$ when μ_s consists of a finite number of atoms:

Theorem 3.8. *Let $\mu = |h|^2\sigma + \sum_{n=1}^{N} r_n \delta_{\lambda_n}$, where h is an outer function in H^2, and each r_n is positive. Then the following are equivalent:*
(i) m is in $T(\mu)$.
(ii) For each n, m is in $T(|h|^2\sigma + \delta_{\lambda_n})$.
(iii) (a) m is in $T(|h|^2\sigma)$
(b) For each n, the Fourier series of m at λ_n converges, to some number $m(\lambda_n)$ say.
(c) For each n, m is in $\Pi(\bar{\lambda}_n - \bar{z})\bar{h}H^2$.
Moreover, condition (c) is equivalent to:
(c′) For each n,

$$\int_{\mathbb{T}} |\Pi \frac{1}{\bar{h}(z)} \frac{m(z) - m(\lambda_n)}{z - \lambda_n}|^2 d\sigma(z) < \infty.$$

The absolutely continuous part of μ_b being Szegö is equivalent to the function b not being an extreme point of the unit ball of H^∞; whether or not this is the case introduces a dichotomy into the study of multipliers of $\mathcal{H}(b)$. For example, in [20] Sarason showed that a (non-constant) polynomial is a multiplier of $\mathcal{H}(b)$ if and only if b is not extreme. In section 4 we show that the polynomials are not the only functions that are always multipliers in the non-extreme case; in particular, we show:

Theorem 4.2. *The H^∞ function m is a multiplier of every $\mathcal{H}(b)$ space, when b is not an extreme point of the ball of H^∞, if and only if there is some constant $c > 0$ such that $\hat{m}(n) = O(exp(-c\sqrt{n}))$.*

In Section 5 we consider the extreme point case. We prove

Theorem 5.1. *If b is an extreme point of the ball of H^∞, and m is a non-constant multiplier of $\mathcal{H}(b)$, then m is a cyclic vector of S^*.*

In Section 6, we ask some questions, and make some conjectures, on what the multipliers of $\mathcal{H}(b)$ are in general.

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1. The measure μ_b

Let b be a fixed non-constant function in the unit ball of H^∞. The function $\frac{1+b}{1-b}$ then has positive real part, so can be represented as a Herglotz integral

$$\frac{1+b(z)}{1-b(z)} = \int_{\mathbb{T}} \frac{e^{i\theta}+z}{e^{i\theta}-z} d\mu_b(e^{i\theta}) + ic,$$

where c is a real constant and μ_b is a unique positive Borel measure.

For each function f in $H^2(\mu_b)$, define a function $V_b f$ on $\mathbb{D}$ by

$$(V_b f)(z) = (\int \frac{f(e^{i\theta})}{1-ze^{-i\theta}} d\mu_b(e^{i\theta}))(1-b(z)).$$

Saracon proved in [23] that V_b is an isometry from $H^2(\mu_b)$ onto $\mathcal{H}(b)$.

For any measure μ on the circle, $H^2(\mu)$ decomposes into the direct sum of $L^2(\mu_s)$ and $H^2(\mu_a)$; if $log(\frac{d\mu_a}{d\sigma})$ is integrable (with respect to σ), μ_a is Szegö; if $log(\frac{d\mu_a}{d\sigma})$ is not integrable, then μ_a is not Szegö, and $H^2(\mu_a) = L^2(\mu_a)$. For a discussion, see *e.g.* [2] or [9].

Considering the measure μ_b, notice that the absolutely continuous part of μ_b is

$$\frac{1 - |b|^2}{|1 - b|^2}\sigma,$$

(look at radial limits of $\Re\frac{1+b}{1-b}$). By the above remarks, the absolutely continuous part of μ_b is therefore Szegö if and only if $log(1-|b|^2)$ is integrable; this integrability in turn is equivalent to b not being an extreme point of the unit ball of H^∞ [7]. So if b is an extreme point of the ball, $H^2(\mu_b) = L^2(\mu_b)$, and $H^2(\mu_b)$ has no analytic structure; if b is not an extreme point, then evaluation at a point of the disk is a continuous functional on $H^2(\mu_b)$, so this space can be thought of as having analytic extensions to the disk. This disparity is reflected by the very different behaviour of the multipliers of $\mathcal{H}(b)$ in the two cases.

For future reference, let us also note that μ_b is purely singular if and only if $|b| = 1$ σ-almost everywhere, *i.e.* if and only if b is inner.

By looking at what V_b^{-1} does to multiplication by m on $\mathcal{H}(b)$, Lotto showed in [15] that a function m in H^∞ is a multiplier of $\mathcal{H}(b)$ if and only if the operator $T_{\bar{m}}$, defined on polynomials by $T_{\bar{m}}(p) = \Pi\bar{m}p$, has a continuous extension to all of $H^2(\mu_b)$. Thus the problem of determining multipliers of $\mathcal{H}(b)$ is reduced to the more tractable one of finding which co-analytic Toeplitz operators are bounded on $H^2(\mu_b)$, *i.e.* finding $T(\mu_b)$.

Let $<,>$ denote the formal inner product of two Fourier series, *i.e.* if $f(z) \sim \sum_{n=-\infty}^{\infty} a_n z^n$, and $g(z) \sim \sum_{n=-\infty}^{\infty} b_n z^n$, then $< f, g >= \sum_{n=-\infty}^{\infty} a_n \bar{b}_n$. We shall often write $< f, g >$ as $\int f\bar{g}d\sigma$, where, if necessary to avoid convergence problems, we think of this integral as $\lim_{r\uparrow 1} \int f(re^{i\theta})\overline{g(re^{i\theta})}d\sigma(\theta)$.

Let S be the unilateral shift, which we think of as multiplication by the independent variable. Then the action of $T_{\bar{m}}$ on a polynomial p of degree d is given by

$$(T_{\bar{m}}p)(z) =< p, m > + < p, Sm > z + ...+ < p, S^d m > z^d. \tag{1.1}$$

2. Szegö measures

Suppose μ is a Szegö measure. Then it can be written as $|h|^2\sigma$ for some outer function h in H^2 (see *e.g.* [9]). Moreover, μ being Szegö means that the functional that assigns to a polynomial its zeroth coefficient is continuous on $H^2(\mu)$. So if $T_{\bar{m}}$ is bounded on $H^2(\mu)$, the functional $p \mapsto (T_{\bar{m}}p)(0)$ is bounded, which from (1.1) means there is a constant C such that

$$| < p, m > | \leq C\sqrt{\int_{\mathbb{T}} |p|^2 d\mu} \tag{2.1}$$

for all polynomials p. The set of analytic functions m that satisfy (2.1) can be identified with the dual of $H^2(\mu)$, and we shall denote this set by $H^2(\mu)^*$.

In [11] it is observed that the space $H^2(|h|^2\sigma)$ is the same as the space of quotients H^2/h, and consequently that m is in $H^2(\mu)^*$ if and only if it is in the range of $T_{\bar{h}}$, *i.e.* if and only if there exists some f in H^2 such that $m = \Pi\bar{h}f$. Thus we get:

Proposition 2.2. *A necessary condition for a function m in H^∞ to be in $T(|h|^2\sigma)$ is that m be in the range of $T_{\bar{h}}$.*

It follows from [21] that this condition is not, in general, sufficient; it does suffice, however, if the distant future and past are at a positive angle in $L^2(\mu)$. Let $\mathcal{F}_d$ be the set of analytic polynomials, whose first d coefficients are zero; and let $\mathcal{P}_d$ be the set of co-analytic polynomials whose first d coefficients vanish. In this terminology, a Helson-Szegö measure is one for which $\mathcal{P}_{-1}$ and $\mathcal{F}_0$ are at a positive angle. In [12] Helson and Sarason showed that $\mathcal{P}_{-1}$ and $\mathcal{F}_d$ are at a positive angle in $L^2(\mu)$ if and only if $\mu = |p|^2\rho$, where p is a polynomial of degree less than or equal to d, and ρ is a Helson-Szegö measure. For a measure of this form, we can prove the converse to Proposition 2.2:

Theorem 2.3. *Let $\mu = |h|^2\sigma = |p|^2\rho$, where h is outer, p is a polynomial and ρ is a Helson-Szegö measure. Then a function m is in $T(\mu)$ if and only if m is in the range of $T_{\bar{h}}$ intersected with L^∞.*

Proof. Necessity follows from Proposition 2.2.

For sufficiency, let d be the degree of p. Because m is in $H^2(\mu)^*$, so are $Sm, S^2m, ..,$ and S^dm. Therefore there is a constant C_1 such that, for any polynomial q,

$$| < q, m > | + ... + | < q, S^d m > | \leq C_1\sqrt{\int_{\mathbb{T}} |q|^2 d\mu}.$$

Now, starting with a polynomial q, multiply it by $\bar{m}$: this increases the norm (in $L^2(\mu)$) by at most a factor of $\|m\|_\infty$. Subtract off the terms in $1, z, ...$ up to z^d : the norm is now increased by at most a factor of $\|m\|_\infty + C_1$. Now apply Π. Π will annihilate the negative terms in the Fourier series of $\bar{m}q$; whilst this in general will be an infinite series, and so not in $\mathcal{P}_{-1}$, it is the (complex) conjugate of a function in H^∞_0, and so lies in the $L^2(\mu)$ closure of $\mathcal{P}_{-1}$ (because H^∞ is contained in $H^2(\mu)$). Therefore, applying Π can increase the norm by at most a factor of twice the cosecant of the angle between $\mathcal{P}_{-1}$ and $\mathcal{F}_d$, a finite number, C_2 say. Now add back the terms in $1, z, ..., z^d$. The result is $T_{\bar{m}}q$, and the norm of q has been increased by at most a factor of $(C_2(\|m\|_\infty + C_1) + C_1)$. Therefore $T_{\bar{m}}$ is bounded on polynomials, as required. $\square$

It follows from the proof that if μ is actually Helson-Szegö, then all of H^∞ is in $T(\mu)$. The converse follows from the closed graph theorem:

Corollary 2.4. *$T(\mu) = H^\infty$ if and only if μ is Helson-Szegö.*

Proof. If μ is Helson-Szegö, then $T_{\bar{m}}$ is just multiplication by the bounded function $\bar{m}$ followed by the bounded projection Π, so the composition is bounded.

Conversely, if all of H^∞ is in $T(\mu)$, it is easy to check, using the closed graph theorem, that the inclusion map from H^∞ into the bounded operators on $H^2(\mu)$, that sends m to $T_{\bar{m}}$, is continuous, so there is some constant C_3 such that $\|T_{\bar{m}}\| \leq C_3\|m\|_\infty$. In particular, $\|T_{\bar{z}^n}|$ is less than or equal to C_3 for all n.

But if μ is not Helson-Szegö, then there exists some polynomial p of norm less than $1/2C_3$ for which $\|T_{\bar{z}^n}p\| = 1$. Therefore μ must be Helson-Szegö. $\square$

A measure $|h|^2\sigma$ satisfies the Helson-Szegö condition if and only if $|h|^2$ satisfies Muckenhoupt's A_2 condition, *viz.* the infimum over all intervals I of

$$(\frac{1}{|I|}\int_I |h|^2 d\sigma)(\frac{1}{|I|}\int_I \frac{1}{|h|^2} d\sigma)$$

must be positive (see [9]). One can construct bounded functions h that satisfy this condition for which the essential infimum of $|h|$ is zero; in other words the Toeplitz operator $T_{\bar{h}}$

need not be invertible; however, its range must always contain all of H^∞, by combining Corollary 2.4 and Proposition 2.2.

In [21] Sarason found necessary and sufficient conditions for H^∞ to be the multipliers of $\mathcal{H}(b)$ in terms of the functions b and a, where a is the outer function satisfying $|a|^2+|b|^2=1$ (so $h=a/(1-b)$). Combining his last condition with Corollary 2.4, we get:

Corollary 2.5. *Suppose a is outer, b is in H^∞, and $|a|^2+|b|^2=1$. Then*
(i) a and b form a Corona pair, and $|a|^2$ satisfies A_2
is equivalent to
(ii) $|\frac{a}{1-b}|^2$ satisfies A_2.

3. Adding a singular measure

In general, there are no inclusion relations between $T(\mu_1+\mu_2)$ and $T(\mu_1)$, $T(\mu_2)$, except for the obvious one $T(\mu_1)\cap T(\mu_2)\subseteq T(\mu_1+\mu_2)$. However, if μ_1 is absolutely continuous and μ_2 is singular, then $T(\mu_1+\mu_2)\subseteq T(\mu_1)$ (unless μ_1 is zero, in which case $T(\mu_2)$ is just the constants - see 5.3).

Proposition 3.1. *If μ is a measure on $\mathbb{T}$ with μ_a non-zero, and m is any function in $T(\mu)$, then m is in $T(\mu_a)$.*

Proof. Let χ_a be a measurable function that is 1 σ-almost everywhere, and 0 μ_s-almost everywhere. By Forelli's lemma ([8,p43]), there exist polynomials q_n that tend to χ_a $(\sigma+\mu)$-almost everywhere, and whose infinity-norms remain bounded. By hypothesis, there is some constant C such that

$$\int_{\mathbb{T}} |T_{\bar{m}}p|^2 d\mu \leq C\int_{\mathbb{T}} |p|^2 d\mu$$

for all polynomials p. Therefore

$$\int_{\mathbb{T}} |T_{\bar{m}}(q_n p)|^2 d\mu_a \leq \int_{\mathbb{T}} |T_{\bar{m}}(q_n p)|^2 d\mu \leq C\int_{\mathbb{T}} |q_n p|^2 d(\mu_a+\mu_s). \tag{3.2}$$

Since m is in $T(\mu)$, $T_{\bar{m}}$ has a continuous extension, R say, to $H^2(\mu)$.
Claim: $\chi_a R(\chi_a p) = T_{\bar{m}}(p)$ μ_a-almost everywhere.

Given the claim, (3.2) then yields, as n tends to infinity,

$$\int_{\mathbb{T}} |T_{\bar{m}}p|^2 d\mu_a \leq C \int_{\mathbb{T}} |p|^2 d\mu_a,$$

which is the desired conclusion.

To prove the claim, we shall show that $\chi_a R(\chi_a p)\sqrt{\frac{d\mu}{d\sigma}}$ and $T_{\bar{m}}(p)\sqrt{\frac{d\mu}{d\sigma}}$, both functions in $L^2(\sigma)$, have the same Fourier coefficients, and therefore are equal σ-almost everywhere. To this end, note that

$$\lim_{n\to\infty} \int_{\mathbb{T}} |[R(q_n p) - R(\chi_a p)]\sqrt{\frac{d\mu}{d\sigma}}|^2 d\sigma = 0.$$

Therefore

$$\begin{aligned}
[\chi_a R(\chi_a p)\sqrt{\frac{d\mu}{d\sigma}}]\widehat{\ }(l) &= \int_{\mathbb{T}} \chi_a R(\chi_a p)\sqrt{\frac{d\mu}{d\sigma}}\bar{z}^l d\sigma \\
&= \lim_{n\to\infty} \int_{\mathbb{T}} \chi_a R(q_n p)\sqrt{\frac{d\mu}{d\sigma}}\bar{z}^l d\sigma \\
&= \lim_{n\to\infty} \int_{\mathbb{T}} T_{\bar{m}}(q_n p)\sqrt{\frac{d\mu}{d\sigma}}\bar{z}^l d\sigma \\
&= \lim_{n\to\infty} \int_{\mathbb{T}} (\Pi \bar{m} q_n p)\overline{(\sqrt{\frac{d\mu}{d\sigma}}z^l)} d\sigma \\
&= \lim_{n\to\infty} \int_{\mathbb{T}} (\bar{m} q_n p)\overline{(\Pi\sqrt{\frac{d\mu}{d\sigma}}z^l)} d\sigma \\
&= \int_{\mathbb{T}} (\bar{m} p)\overline{(\Pi\sqrt{\frac{d\mu}{d\sigma}}z^l)} d\sigma \\
&= \int_{\mathbb{T}} T_{\bar{m}}(p)\sqrt{\frac{d\mu}{d\sigma}}\bar{z}^l d\sigma \\
&= [T_{\bar{m}}p\sqrt{\frac{d\mu}{d\sigma}}]\widehat{\ }(l)
\end{aligned}$$

as required. □

Adding a point mass to a measure μ also cannot increase $T(\mu)$, even if μ is not absolutely continuous:

Proposition 3.3. *Let μ be any measure on $\mathbb{T}$, δ_λ be the point mass at λ, for some λ on $\mathbb{T}$. If m is in $T(\mu+\delta_\lambda)$, then m is in $T(\mu)$.*

Proof. Without loss of generality, we can assume $\mu(\lambda) = 0$. By hypothesis, there is a constant C such that

$$\int_{\mathbb{T}} |T_{\bar{m}} p|^2 d(\mu+\delta_\lambda) \leq C \int_{\mathbb{T}} |p|^2 d(\mu+\delta_\lambda). \tag{3.4}$$

Let $q_n(z) = [\frac{1}{2}(1+\bar{\lambda}z)]^n$; q_n converges to 1 at λ, and zero everywhere else.
Claim: $\lim_{n\to\infty}[T_{\bar{m}}((1-q_n)p)](\zeta) = [T_{\bar{m}}(p)](\zeta)$, for all $\zeta \neq \lambda$ on the circle.
Given the claim, (3.4), with $(1-q_n)p$ instead of p, yields, in the limit as n tends to infinity,

$$\int_{\mathbb{T}} |T_{\bar{m}} p|^2 d\mu \leq C \int_{\mathbb{T}} |p|^2 d\mu,$$

as desired.

The claim is equivalent, by writing p as a sum of monomials and applying the backward shift to m, to showing that, for all $\zeta \neq \lambda$, $\lim_{n\to\infty}[T_{\bar{m}} q_n](\zeta) = 0$. Now

$$\begin{aligned}
[T_{\bar{m}} q_n](\zeta) &= \frac{1}{2^n} \sum_{l=0}^{n} \bar{\lambda}^l \tbinom{n}{l} \sum_{k=0}^{l} \overline{\hat{m}(k)} \zeta^{l-k} \\
&= \frac{\bar{\lambda}^n}{2^n} \sum_{k=0}^{n} \overline{\hat{m}(k)} \zeta^{n-k} \sum_{j=0}^{n-k} \tbinom{n}{j} (\lambda\bar{\zeta})^j.
\end{aligned} \tag{3.5}$$

First, observe that if m is a monomial, $m(z) = z^k$, then the absolute value of (3.5) is less than or equal to $\frac{1}{2^n}\{|1+\lambda\bar{\zeta}|^n + \sum_{i=0}^{k} \binom{n}{i}\}$, which tends to zero as n tends to infinity. So we can add any polynomial to m and not change the behaviour of (3.5) in the limit; let us therefore fix $\epsilon > 0$, and subtract a finite number of terms from m so that $\int_{\mathbb{T}} |m|^2 d\sigma < (\frac{\epsilon}{C(\lambda\bar{\zeta})})^2$, where $C(\lambda\bar{\zeta})$ is the constant from Lemma 3.6.

Applying Lemma 3.6, (3.5) becomes

$$\begin{aligned}
|(T_{\bar{m}} q_n)(\zeta)| &\leq \frac{C(\lambda\bar{\zeta})}{\sqrt{n}} \sum_{k=0}^{n} |\hat{m}(k)| \\
&\leq C(\lambda\bar{\zeta}) \|m\|_2 \\
&< \epsilon.
\end{aligned}$$

So for large enough n, (3.5) can be made arbitrarily small, so $[T_{\bar{m}} q_n](\zeta)$ tends to zero, as required. $\square$

In the proof, we used the following computational lemma:

Lemma 3.6. *Suppose ξ is on the unit circle, but not equal to 1. Then there is a constant $C(\xi)$ such that*

$$|\frac{1}{2^n}\sum_{j=0}^{l}\binom{n}{j}\xi^j| \leq C(\xi)\frac{1}{\sqrt{n}},$$

for all $l \leq n$.

Proof. The quantity on the left-hand side is a partial sum of a Fourier series, and can be written using the Dirichlet kernel as

$$\frac{1}{2^n}\sum_{j=0}^{l}\binom{n}{j}\xi^j = \frac{1}{2\pi}\int_0^{2\pi}(\frac{1+\xi e^{-it}}{2})^n \frac{Sin(l+\frac{1}{2})t}{Sin\frac{1}{2}t}dt. \tag{3.7}$$

Let θ be the argument of ξ, and let $\alpha = min(\frac{\theta}{2}, \frac{2\pi-\theta}{2}, \frac{\pi}{4})$. Choose n so large that $n(n+1)(\sqrt{\frac{1+Cos\alpha}{2}})^n < 1$. The integral in (3.7) can be written as the sum of the integral from $\theta-\alpha$ to $\theta+\alpha$, and the integral over the rest of the circle. For the latter integral, $|\frac{1+\xi e^{-it}}{2}| \leq \sqrt{\frac{1+Cos\alpha}{2}}$, so the modulus of the integral is less than $2(l+\frac{1}{2})(\sqrt{\frac{1+Cos\alpha}{2}})^n$, which is less than $\frac{2}{n}$. In the former integral, we have control over the second term:

$$\begin{aligned} |\int_{\theta-\alpha}^{\theta+\alpha}(\frac{1+\xi e^{-it}}{2})^n \frac{Sin(l+\frac{1}{2})t}{Sin\frac{1}{2}t}dt| &\leq \frac{1}{Sin\frac{\alpha}{2}}\frac{1}{2^n}\int_0^{2\pi}|1+e^{it}|^n dt \\ &= \frac{1}{Sin\frac{\alpha}{2}}\frac{1}{2^n}\int_0^{2\pi}(2+2Cos\quad t)^{\frac{n}{2}}dt \\ &= \frac{2}{Sin\frac{\alpha}{2}}\int_0^{\pi}|Cos^n u|du. \end{aligned}$$

This last integral can be worked out explicitly; applying Stirling's formula to the result, one gets that it is bounded by a constant times $\frac{1}{\sqrt{n}}$, as desired. $\square$

Before stating our next theorem, we must say what we mean by $\Pi f/\bar{h}$, when h and f are in H^2, and h is outer. In general, $f/\bar{h}$ will not be integrable on the circle; however, $1/\bar{h}(z) = \sum_{n=0}^{\infty}\beta_n\bar{z}^n$ is a harmonic function on the unit disk, with a power series that converges almost uniformly. Suppose $f(z) = \sum_{n=0}^{\infty}a_n z^n$. Then $f(z)/\bar{h}(z) =$

$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n\beta_m z^n\bar{z}^m$, where the series converges almost uniformly on the disk. Truncate this by annihilating those terms in $z^n\bar{z}^m$ where m is greater than n, to get the function $\tilde{g}(z) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} a_n\beta_m z^n\bar{z}^m$; let $g(z) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} a_n\beta_m z^{n-m}$ (one could think of this as the radial limit in the sense of Abel of $\tilde{g}$). We define $\Pi f/\bar{h}$ to be this function g; this will agree with the usual definition when $f/\bar{h}$ is summable, but will be defined (at least as an analytic function on the disk) for all h and f.

Observe that defining truncations via passage to the disk means that $\Pi\bar{h}\Pi f = \Pi\bar{h}f$, whenever h is analytic; so, in particular, $\Pi\frac{1}{\bar{h}}\Pi\bar{h}f = \Pi f$. Moreover, an analytic function f is in the range of $T_{\bar{h}}$ (*i.e.* can be written $\Pi\bar{h}k$, for some H^2 function k) if and only if $\Pi f/\bar{h}$ is in H^2.

We can now say what additional restrictions the addition of a finite number of atoms to a Szegö measure imposes on the Toeplitz operators. The special case of the measure $\sigma + 2\delta_1$ (which corresponds to the function $b(z) = \frac{1+z}{2}$) has been studied in [21] and [16].

Theorem 3.8. *Let $\mu = |h|^2\sigma + \sum_{n=1}^{N} r_n\delta_{\lambda_n}$, where h is an outer function in H^2, and each r_n is positive. Then the following are equivalent:*
(i) m is in $T(\mu)$.
(ii) For each n, m is in $T(|h|^2\sigma + \delta_{\lambda_n})$.
(iii) (a) m is in $T(|h|^2\sigma)$
(b) For each n, the Fourier series of m at λ_n converges, to some number $m(\lambda_n)$ say.
(c) For each n,

$$\int_{\mathbb{T}} |\Pi\frac{1}{\bar{h}(z)}\frac{m(z)-m(\lambda_n)}{z-\lambda_n}|^2 d\sigma(z) < \infty. \tag{3.9}$$

(iv) (a) m is in $T(|h|^2\sigma)$
(b) For each n, the Fourier series of m at λ_n converges, to some number $m(\lambda_n)$ say.
(c$'$) For each n, m lies in $\Pi(\bar{\lambda}_n - \bar{z})\bar{h}H^2$.

Proof. We will show $(i) \iff (ii)$, $(ii) \iff (iii)$, and $(c) \iff (c')$.

$(i) \implies (ii)$: This follows from Proposition 3.3.

$(ii) \implies (i)$: Since there are only a finite number of atoms, there is some constant C

such that, for all polynomials p and for each n,

$$\int_{\mathbb{T}} |T_{\bar{m}}p|^2(|h|^2 d\sigma + d\delta_{\lambda_n}) \leq C \int_{\mathbb{T}} |p|^2(|h|^2 d\sigma + d\delta_{\lambda_n}).$$

Therefore

$$\sum_{n=1}^{N} r_n \int_{\mathbb{T}} |T_{\bar{m}}p|^2(|h|^2 d\sigma + d\delta_{\lambda_n}) \leq C \sum_{n=1}^{N} \int_{\mathbb{T}} r_n |p|^2(|h|^2 d\sigma + d\delta_{\lambda_n}),$$

and so

$$\int_{\mathbb{T}} |T_{\bar{m}}p|^2(|h|^2 d\sigma + d\sum_{n=1}^{N} r_n \delta_{\lambda_n}) \leq \frac{max(1, \sum_{n=1}^{N} r_n)}{min(1, \sum_{n=1}^{N} r_n)} C \int_{\mathbb{T}} |p|^2(|h|^2 d\sigma + d\sum_{n=1}^{N} r_n \delta_{\lambda_n}).$$

$(ii) \implies (iii)$: (a) follows from Proposition 3.1.

(b): Let χ_n be the function that is one at λ_n, zero elsewhere. In $H^2(|h|^2\sigma + \delta_{\lambda_n})$, the functions $\bar{\lambda}_n^l z^l - \chi_n$ tend weakly to zero as l tends to infinity, so, since $T_{\bar{m}}$ is continuous and therefore weakly continuous,

$$(T_{\bar{m}}(\bar{\lambda}_n^l z^l), \chi_n)_{H^2(|h|^2\sigma+\delta_{\lambda_n})} \to (T_{\bar{m}}\chi_n, \chi_n)_{H^2(|h|^2\sigma+\delta_{\lambda_n})}.$$

But the left-hand side is just $\bar{\lambda}_n^l [T_{\bar{m}} z^l](\lambda_n)$, which is $\overline{\hat{m}(0)} + \overline{\hat{m}(1)}\bar{\lambda}_n + ... + \overline{\hat{m}(l)}\bar{\lambda}_n^l$. Therefore the Fourier series of m at λ_n converges to $(\chi_n, T_{\bar{m}}\chi_n)_{H^2(|h|^2\sigma+\delta_{\lambda_n})}$.

(c): There is some constant C such that, for all polynomials p,

$$|[T_{\bar{m}}p](\lambda_n)| \leq C \sqrt{\int_{\mathbb{T}} |p|^2(|h|^2 d\sigma + d\delta_{\lambda_n})}.$$

Therefore there is some function g in $H^2(|h|^2\sigma + \delta_{\lambda_n})$ such that

$$[T_{\bar{m}}p](\lambda_n) = (p, g)_{H^2(|h|^2\sigma+\delta_{\lambda_n})}.$$

g can be written as $g' + \alpha\chi_n$, for some g' in $H^2(|h|^2\sigma)$, and g' in turn can be written as k/h, for some k in H^2. Therefore we have

$$< p, \frac{m}{1 - \bar{\lambda}_n z} > = < p, \bar{h}k > + \quad \bar{\alpha} p(\lambda_n),$$

and so

$$< p, \frac{m-\alpha}{1-\bar{\lambda}_n z} > = < p, \bar{h}k > .$$

Setting $p(z) = z^l$, this says that the positive Fourier coefficients are the same, *i.e.*

$$\frac{m-\alpha}{1-\bar{\lambda}_n z} = \Pi \bar{h} k.$$

Letting l tend to infinity, one also gets that $\alpha = m(\lambda_n)$. Therefore

$$\begin{aligned} \Pi \frac{1}{\bar{h(z)}} \frac{m(z)-m(\lambda_n)}{z-\lambda_n} &= -\Pi \frac{\lambda_n}{\bar{h}} \Pi \bar{h} k \\ &= -\lambda_n k, \end{aligned}$$

and this is square-integrable, as required.

(iii) $\implies$ (ii): It is sufficient to prove that there exists some constant C such that

$$|[T_{\bar{m}-\bar{m}(\lambda_n)} p](\lambda_n)| \leq C \sqrt{\int_{\mathbb{T}} |p|^2 |h|^2 d\sigma} \tag{3.10}$$

for all polynomials p, for then the norm of $T_{\bar{m}}$ on $H^2(|h|^2\sigma + \delta_{\lambda_n})$ is less than or equal to $(C + |m(\lambda_n)| + \|T_{\bar{m}}\|_{H^2(|h|^2\sigma)})$.

The right-hand side of (3.10) is $C| < ph, ph > |$, and the left-hand side is

$$\begin{aligned} | < p, \frac{m-m(\lambda_n)}{1-\bar{\lambda}_n z} > | &= | < ph, \frac{1}{\bar{h}} \frac{m-m(\lambda_n)}{1-\bar{\lambda}_n z} > | \\ &= | < ph, \Pi \frac{1}{\bar{h}} \frac{m-m(\lambda_n)}{1-\bar{\lambda}_n z} > |. \end{aligned}$$

The supremum of this over all polynomials with $| < ph, ph > | = 1$ is just

$$\sqrt{\int_T |\Pi \frac{1}{\overline{h(z)}} \frac{m(z)-m(\lambda_n)}{1-\bar{\lambda}_n z}|^2 d\sigma},$$

which is finite as required.

$(c) \implies (c')$: As remarked earlier, condition (3.9) is equivalent to saying that

$$\frac{m(z)-m(\lambda_n)}{z-\lambda_n} = \Pi \bar{h}(z) k(z),$$

for some function k in H^2. Therefore

$$\begin{aligned} m(z) - m(\lambda_n) &= (z-\lambda_n)\Pi\bar{h}(z)k(z) \\ &= \Pi(\bar{z}-\bar{\lambda}_n)(-\lambda_n z)\Pi\bar{h}(z)k(z) \\ &= \Pi(\bar{z}-\bar{\lambda}_n)\bar{h}(z)\{(-\lambda_n)[zk(z) - \frac{1}{\bar{\hat{h}}(0)} < zk(z), h(z) >]\}. \end{aligned}$$

Therefore m is in $\Pi(\bar{\lambda}_n - \bar{z})\bar{h}H^2$.

$(c') \implies (c)$: Suppose $m = \Pi(\bar{z}-\bar{\lambda}_n)\bar{h}f$, for some f in H^2. Then

$$m(z) + \bar{\lambda}_n < f, h >= (z-\lambda_n)\Pi\bar{h}[-\lambda_n S^* f]. \tag{3.11}$$

Noting that the k^{th} partial sum of the Fourier series of m at λ_n is $-\lambda_n < \bar{h}f, 1-\bar{\lambda}_n z^{k+1} >$, (3.11) implies that

$$\frac{m(z)-m(\lambda_n)}{z-\lambda_n} = \Pi\bar{h}[-\lambda_n S^* f],$$

from which (3.9) follows. □

Remark: The problem in trying to generalise the theorem to an infinite number of atoms is that one loses control of the norms when trying to localise: a sufficient condition for $T_{\bar{m}}$ to be bounded on $H^2(|h|^2\sigma + \sum r_n \delta_{\lambda_n})$ is that $\|T_{\bar{m}}\|_{H^2(|h|^2\sigma+\delta_{\lambda_n})} \leq C$ for all n; a necessary condition is that $\|T_{\bar{m}}\|_{H^2(|h|^2\sigma+r_n\delta_{\lambda_n})} \leq C$ for all n. One can construct examples that show that the former condition is not necessary; we do not know any examples that prove that the latter condition is insufficient.

Nevertheless, this (strong) sufficient condition extends to all singular measures, and the following result enables us, in the next section, to characterise universal multipliers:

Theorem 3.12. *Suppose that $\mu = \mu_a + \mu_s$ is a measure on $\mathbb{T}$, and that there is a set D, dense in the support of μ_s, and a constant C, such that $\|T_{\bar{m}}\|_{H^2(\mu_a+\delta_\lambda)} \leq C$ for all λ in D. Then m is in $T(\mu)$.*

Proof. Let ν_k be a sequence of atomic measures, with atoms lying in D, that converge weak-star to μ_s (such a sequence exists because any continuous function that vanishes on

D must be annihilated by μ_s). By the argument in the proof of $(ii) \implies (i)$ in Theorem 3.8,

$$\|T_{\bar{m}}\|_{H^2(\mu_a+\nu_k)} \leq \frac{max(1, \|\nu_k\|)}{min(1, \|\nu_k\|)} C \leq C',$$

for some C'.

For any polynomial p,

$$\begin{aligned}\int_{\mathbb{T}} |T_{\bar{m}} p|^2 d\mu &= \lim_{k\to\infty} \int_{\mathbb{T}} |T_{\bar{m}} p|^2 d(\mu_a + \nu_k) \\ &\leq C'^2 \lim_{k\to\infty} \int_{\mathbb{T}} |p|^2 d(\mu_a + \nu_k) \\ &= C'^2 \int_{\mathbb{T}} |p|^2 d\mu,\end{aligned}$$

so m is in $T(\mu)$, as required. □

4. Universal Multipliers

In Section 2, we defined the dual $H^2(\mu)^*$ to be the set of analytic functions whose formal (H^2) inner product with a polynomial is bounded by the $H^2(\mu)$-norm of the polynomial, *i.e.* the set of f for which

$$| < p, f > | \leq C \sqrt{\int_{\mathbb{T}} |p|^2 d\mu}.$$

If $\mu = |h|^2\sigma$ is Szegö, this means there is some g in $H^2(\mu) = H^2/h$ such that

$$\begin{aligned}< p, f > &= (p, g)_{H^2(\mu)} \\ &= \int_{\mathbb{T}} p\bar{h}\overline{(hg)} d\sigma,\end{aligned}$$

so f is in $\Pi\bar{h}H^2$ (which is the range of $T_{\bar{h}}$, where, if h is not in H^∞, $T_{\bar{h}}$ is thought of as mapping H^2 into the set of functions analytic on $\mathbb{D}$); conversely, any f in $\Pi\bar{h}H^2$ is in $H^2(\mu)^*$.

A multiplier of $H^2(\mu)^*$ is a function m such that mf lies in $H^2(\mu)^*$ whenever f does; from the preceding paragraph, it coincides in the $|h|^2\sigma$ case with multipliers of $ranT_{\bar{h}}$. The multipliers of $H^2(\mu)^*$ are the same as the co-analytic Toeplitz operators on $H^2(\mu)$ (though in general $H^2(\mu_b)^*$ is not identical to $\mathcal{H}(b)$).

Proposition 4.1. *Let μ be a measure on the circle. Then m is a multiplier of $H^2(\mu)^*$ if and only if m is in $T(\mu)$.*

Proof. If m is a multiplier of $H^2(\mu)^*$, then, by the closed graph theorem, multiplication by m, M_m, is a bounded linear operator on $H^2(\mu)^*$. The proof then consists of first observing that $H^2(\mu)^*$ really is the dual of the Hilbert space $H^2(\mu)$ (since any functional is determined by its action on polynomials), and secondly that the adjoint of M_m is $T_{\bar{m}}$, because if either $T_{\bar{m}}$ or M_m is bounded, then

$$\begin{aligned} < M_m^* p, f > &= \lim_{r \uparrow 1} \int p(re^{i\theta}) \overline{m(re^{i\theta}) f(re^{i\theta})} d\sigma(\theta) \\ &= < T_{\bar{m}} p, f > . \end{aligned}$$

So if m is a multiplier of $H^2(\mu)^*$, $T_{\bar{m}}$ is bounded; and if $T_{\bar{m}}$ is bounded, mf is in $H^2(\mu)^*$ whenever f is. □

In [18], it is proved that a function m is a multiplier of $H^2(\mu)^*$ for every Szegö measure μ if and only if m is in every $H^2(\mu)^*$, which in turn holds if and only if there is some constant c such that $\hat{m}(n) = O(exp(-c\sqrt{n}))$. These functions are multipliers of every $\mathcal{H}(b)$ when b is non-extreme: if μ_b is absolutely continuous, this follows from Proposition 4.1; if it isn't, we can use the fact that $\hat{m}(n)$ decays so rapidly to apply Theorem 3.12.

Theorem 4.2. *The H^∞ function m is a multiplier of every $\mathcal{H}(b)$ space, when b is not an extreme point of the ball of H^∞, if and only if there is some constant $c > 0$ such that $\hat{m}(n) = O(exp(-c\sqrt{n}))$.*

Proof. Necessity follows from the preceding paragraph.

Let us fix some m such that $\hat{m}(n) = O(exp(-c\sqrt{n}))$; we must show that for every measure μ of the form $|h|^2\sigma + \mu_s$, m lies in $T(\mu)$. By Theorem 3.12, it is sufficient to show that for every outer function h there is a constant C such that

$$\|T_{\bar{m}}\|_{H^2(|h|^2\sigma+\delta_\lambda)} \leq C;$$

this in turn is equivalent to showing

$$\sup_{\lambda \in \mathbb{T}} \|\frac{m(z) - m(\lambda)}{z - \lambda}\|_{H^2(|h|^2\sigma)^*} < \infty.$$

A straightforward computation shows that there is some constant M independent of λ such that the n^{th} Fourier coefficient of $\frac{m(z)-m(\lambda)}{z-\lambda}$ is smaller in modulus than $Me^{-c\sqrt{n}/2}$. Therefore the family of functionals $\{\frac{m(z)-m(\lambda)}{z-\lambda} : \lambda \in \mathbb{T}\}$ is pointwise bounded on $H^2(|h|^2\sigma)$, and, by the Banach-Steinhaus theorem, equicontinuous, as desired. $\square$

Remark: In [23], Sarason shows that for every non-extreme b, $\mu_{\lambda b}$ is absolutely continuous for almost every λ on the unit circle; since $\mathcal{H}(\lambda b) = \mathcal{H}(b)$, this result, together with Proposition 4.1, also proves 4.2.

5. The extreme point case

The next theorem states that the multipliers of $\mathcal{H}(b)$, when b is extreme, cannot be non-cyclic for S^* (the adjoint of the unilateral shift). A function f is non-cyclic for S^* if and only if there is an inner function I such that f lies in $H^2 \ominus IH^2$ (this follows immediately from Beurling's theorem). In particular, no function that is a polynomial of an inner function is cyclic. For an analytic condition on whether a function is cyclic, in terms of pseudo-continuations, see [19]. We note that this theorem has also been proved independently by Lotto and Sarason [17].

Theorem 5.1. *If b is an extreme point of the ball of H^∞, and m is a non-constant multiplier of $\mathcal{H}(b)$, then m is a cyclic vector of S^*.*

Proof. Our hypotheses are that $H^2(\mu_b) = L^2(\mu_b)$, and that $T_{\bar{m}}$ extends to a continuous operator on $H^2(\mu_b)$. Let Z be the operator of multiplication by the independent variable on $L^2(\mu_b)$. The commutator of $T_{\bar{m}}$ with Z is bounded, and on any polynomial p,

$$(T_{\bar{m}}Z - ZT_{\bar{m}})p = < p, S^*m > \mathbb{1}.$$

By hypothesis, the norm of this latter (constant) function is bounded by some constant C times the norm of p, so

$$| < p, S^*m > | \leq C\sqrt{\int_{\mathbb{T}} |p|^2 d\mu_b}.$$

Letting $n = S^*m$, this says that there is some function g in $L^2(\mu_b)$, such that

$$\int_{\mathbb{T}} p\bar{n} d\sigma = \int_{\mathbb{T}} p\bar{g} d\mu_b.$$

Since $(\bar{n}\sigma - \bar{g}\mu_b)$ annihilates the polynomials, the F. and M. Riesz theorem says there must be some function k in H^1_0 so that $(\bar{n}\sigma - \bar{g}\mu_b) = k\sigma$, *i.e.*

$$\bar{g}\mu_b = (\bar{n} - k)\sigma. \tag{5.2}$$

Because g is in $L^2(\mu_b)$, and $\frac{d\mu_b}{d\sigma}$ is not log-integrable (with respect to σ), neither is $\frac{d\bar{g}\mu_b}{d\sigma}$. But if n were not cyclic for S^*, there would be some inner function I such that $\Pi\bar{I}n = 0$; $I\bar{n}$ would then be in H^∞, so so would $I(\bar{n} - k)$, and the modulus of this, which is the same as the modulus of $\frac{d\bar{g}\mu_b}{d\sigma}$, would have to be log-integrable. Therefore n, and so m, is cyclic, as desired. □

As an immediate consequence, we get the result of Sarason mentioned earlier, that a polynomial is a multiplier of $\mathcal{H}(b)$ if and only if b is not extreme. From the proof we also get a new proof of a theorem of Lotto [16], that if b is inner, $\mathcal{H}(b)$ has no non-constant multipliers:

Porism 5.3. *If b is inner, $\mathcal{H}(b)$ has no non-constant multipliers.*

Proof. If b is inner, μ_b is purely singular, so (5.2) can only hold if both sides are zero, so n must be zero, so m must be a constant. □

6. Questions

Many questions remain open. For example:

If $\mu = |h|^2\sigma$ is absolutely continuous, when is $T(\mu) = ranT_{\bar{h}} \cap H^\infty$? In [3], it is proved that if $log|h_0/h_1|$ is in BMO, the space of functions of bounded mean oscillation on the circle, then $T(|h_0h_1|\sigma)$ contains $T(|h_0|^2\sigma) \cap T(|h_1|^2\sigma)$. It follows from this, and Theorem 2.3, that a sufficient condition that m be in $T(|z-1|\sigma)$ is that $m = T_{(\overline{z-1})^{(1/2)+\epsilon}} f$, for any $\epsilon > 0$ and some f in H^2. Can the epsilon be removed, to get agreement with the necessary condition from Proposition 2.2?

Suppose one adds an atomic measure with an infinite number of atoms, $\sum r_n\delta_{\lambda_n}$, to $|h|^2\sigma$. Is the condition $\|T_{\bar{m}}\|_{H^2(|h|^2\sigma + r_n\delta_{\lambda_n})} \leq C$ for all n a sufficient condition for m to be in $T(|h|^2\sigma + \Sigma r_n\delta_{\lambda_n})$? What if one adds a non-atomic singular measure? What is even a reasonable conjecture for a necessary and sufficient condition for $T_{\bar{m}}$ to be bounded on $H^2(\mu_a + \mu_s)$?

The space $\mathcal{M}(\bar{a})$, defined to be the range of $T_{\bar{a}}$, is intimately connected to $\mathcal{H}(b)$ - see [21]. Does it equal the intersection over all inner functions J of $H^2(\mu_{Jb})^*$?

When is $\mu_{\lambda b}$ absolutely continuous for all λ on the circle? (This question is related to characterising exposed points in H^1 - see [23]).

We note that Lotto and Sarason have studied the multipliers of de Branges spaces in [17], concentrating mainly on the extreme point case, and have obtained some very interesting results.

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