Non-commutative manifolds, the free square root and symmetric functions in two non-commuting variables

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Abstract

The richly developed theory of complex manifolds plays important roles in our understanding of holomorphic functions in several complex variables. It is natural to consider manifolds that will play similar roles in the theory of holomorphic functions in several non-commuting variables. In this paper we introduce the class of *nc-manifolds*, the mathematical objects that at each point possess a neighborhood that has the structure of an *nc-domain* in the *d-dimensional nc-universe* $\mathbb{M}^d$. We illustrate the use of such manifolds in free analysis through the construction of the non-commutative Riemann surface for the matricial square root function. A second illustration is the construction of a non-commutative analog of the elementary symmetric functions in two variables. For any symmetric domain in $\mathbb{M}^2$ we construct a 2-dimensional non-commutative manifold such that the symmetric holomorphic functions on the domain are in bijective correspondence with the holomorphic functions on the manifold. We also derive a version of the classical Newton-Girard formulae for power sums of two non-commuting variables.

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1   Introduction

Free analysis, the study of holomorphic functions in several non-commuting variables, dates back to 1973 and the seminal paper [22] by J. L. Taylor. The theory has picked up ever greater momentum in the past decade. The monograph [17], by D. S. Kaliuzhnyi-Verbovetskyi and V. Vinnikov contains a panoramic survey of the field up to the time of its writing. Since then, there have been further breakthroughs in both geometry (see *e.g.* [9, 12, 13, 14, 15] and function theory (see *e.g.* [7, 8, 16]).

Taylor's founding idea for non-commutative analysis was that analytic functions in several non-commuting variables should have the same basic algebraic properties as free polynomials have when viewed as functions on tuples of matrices (of indeterminate size). Here a *free polynomial* means a polynomial in non-commuting variables over the field $\mathbb{C}$ of complex numbers.

Traditionally, free analysis has dealt with functions defined on subsets of the *d-dimensional non-commutative universe* $\mathbb{M}^d$, which comprises the space of d-tuples of square matrices, for $d \geq 1$. Thus

$$\mathbb{M}^d = \bigcup_{n=1}^{\infty} \mathbb{M}_n^d \tag{1.1}$$

where $\mathbb{M}_n$ denotes the algebra of $n \times n$ matrices over $\mathbb{C}$ and

$$\mathbb{M}_n^d = \{(x^1, x^2, \ldots, x^d) \,|\, x^r \in \mathbb{M}_n \text{ for } r = 1, 2, \ldots, d\}.$$

In this paper we introduce the notion of an nc- or *non-commutative manifold*, which bears the same relation to $\mathbb{M}^d$ as complex manifolds bear to $\mathbb{C}^d$. This natural extension is needed even for such a basic notion as the "free Riemann surface" of the matricial square root function, an object that we construct in Section 6.

We were led to introduce topological nc-manifolds while seeking a non-commutative version of the anciently-known[1] fact that any symmetric holomorphic function of d variables can be expressed as a holomorphic function

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[1]See Section 11.

of the elementary symmetric functions in d variables on a suitable domain. In particular, if $f(z, w)$ is a holomorphic function on $\mathbb{C}^2$ that is symmetric, in the sense that $f(z, w) = f(w, z)$ for all $z, w \in \mathbb{C}$, then there exists a holomorphic function g on $\mathbb{C}^2$ such that $f(z, w) = g(z + w, zw)$ for all $z, w \in \mathbb{C}$. We asked whether there is a non-commutative version of this result.

Any non-commutative analog must be consistent with results of M. Wolf [25], who studied an algebraic version of this question in 1936. She proved that the algebra of symmetric free polynomials in d non-commuting variables is not finitely generated when $d > 1$, and is in fact isomorphic to the algebra of free polynomials in countably many variables. It follows that there is no polynomial map $\pi : \mathbb{M}^2 \to \mathbb{M}^d$, for any $d \in \mathbb{N}$, with the property that, for every symmetric free polynomial φ in 2 variables, there exists a free polynomial Φ such that $\varphi = \Phi \circ \pi$; otherwise, the components of π would constitute a finite basis for the algebra of symmetric free polynomials. Nevertheless, we show that there *is* a close parallel to the classical result in the context of free analysis on nc-manifolds. The main results of the paper are Theorem 10.1 and its unbounded analogue, Theorem 10.11.

Before stating a somewhat special case of the theorem, let us informally describe some key notions (precise definitions are in Sections 2 and 4). A subset of $\mathbb{M}^d$ is called an *nc set* if it is closed with respect to direct sums. A function f defined on some subset of $\mathbb{M}^d$ is called *graded* if $f(x) \in \mathbb{M}_n$ whenever $x \in \mathbb{M}_n^d$. A graded function on an nc set is called an *nc function* if it preserves direct sums and joint similarities. A graded function f on an arbitrary set $D \subseteq \mathbb{M}^d$ is called *conditionally nc* if it preserves joint similarities, and, in addition, there is a graded function $\hat{f}$ such that whenever x and $x \oplus y$ are in D, then $f(x \oplus y) = f(x) \oplus \hat{f}(y)$.

Theorem 1.2. *Let $\mathcal{S}$ be a symmetric nc set in $\mathbb{M}^2$ that is open in the free topology. There is subset $\mathcal{S}_{oo}$ of $\mathcal{S}$ and a polynomial map $\pi : \mathbb{M}^2 \to \mathbb{M}^3$ such that $\mathfrak{G}_{oo} = \pi(\mathcal{S}_{oo})$ is a topological nc-manifold in the Zariski-free topology, and such that there is a canonical isomorphism between bounded symmetric nc functions defined on $\mathcal{S}$ and bounded holomorphic functions on the manifold $\mathfrak{G}_{oo}$ that are conditionally nc.*

$$
\begin{array}{ccccc}
 & & \mathcal{S}_{oo} & \xrightarrow{\ \pi\ } & \mathfrak{G}_{oo} \xrightarrow{\ F_{oo}\ } \mathbb{M}^1 \\
 & \overset{f_{oo}}{\swarrow} & & \downarrow{\scriptstyle \subseteq} & \\
\mathbb{M}^1 & \xleftarrow{\ f\ } & \mathcal{S} & &
\end{array}
\tag{1.3}
$$

The map π is given by

$$\pi(x^1, x^2) = (\tfrac{1}{2}(x^1 + x^2), \tfrac{1}{4}(x^1 - x^2)^2, \tfrac{1}{8}(x^1 - x^2)(x^1 + x^2)(x^1 - x^2)). \quad (1.4)$$

It is generically 2-to-1 on $\mathbb{M}^2$, but there is a singular set on which it is many to one. This set is excluded in $\mathcal{S}_{oo}$, and both $\mathcal{S}_{oo}$ and $\mathfrak{G}_{oo}$ can be given the structure of topological nc-manifolds with respect to a topology called the Zariski-free topology, defined in Section 7. There is also an isomorphism between Zariski-freely holomorphic symmetric functions f_{oo} on $\mathcal{S}_{oo}$ and Zariski-freely holomorphic functions on $\mathfrak{G}_{oo}$ (with no assumption of boundedness needed)—this is Theorem 9.15.

The bulk of the paper comprises the establishment of a suitable notion of topological nc-manifold and construction of the manifolds $\mathfrak{G}_{oo}$. Topological nc-manifolds are defined in Section 4. The nc-universe $\mathbb{M}^d$ defined by equation (1.1) is unlike $\mathbb{R}^d$ and $\mathbb{C}^d$ in that it admits numerous natural topologies – a fact which adds an extra richness to the theory of topological nc-manifolds.

Section 5 describes the basic theory of free holomorphic functions of a single variable. In one variable such functions are determined by their action on scalars, and are given by the holomorphic functional calculus for matrices (Proposition 5.13).

In Section 6 we present a simple example of a one-dimensional free manifold – a non-commutative version of the Riemann surface of the square root function. This theory is an essential component of the construction of the manifold $\mathcal{G}$.

In Section 7 we define the Zariski-free topology, which is rather subtle, but seems necessary in order to avoid certain singularities in the map π. In Section 8 we construct a Zariski-free manifold $\mathcal{G}$, which is the manifold which would be $\mathfrak{G}_{oo}$ in Theorem 1.2 if $\mathcal{S}$ were all of $\mathbb{M}^2$. Of course bounded symmetric functions are not of interest when $\mathcal{S} = \mathbb{M}^2$, but there is a version of the theorem that does not require boundedness. This is Theorem 8.35.

Theorem 1.5. *There is a canonical bijection between*
    (i) *symmetric nc functions f that are freely holomorphic on $\mathbb{M}^2$, and*
    (ii) *holomorphic functions F_{oo} defined on the Zariski-free manifold $\mathcal{G}$ that are conditionally nc and have the property that for every $w \in \mathbb{M}^2$, there is a free neighborhood U of w such that F_{oo} is bounded on $\pi(U) \cap \mathcal{G}$.*

In Sections 9 and 10 we prove the main results of the paper, Theorems 10.1, 10.11 and 9.15. Finally, in Section 11, we give a non-commutative

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version of the classical Newton-Girard formulae for power sums in terms of elementary symmetric functions. There is a significant difference in the non-commutative context, since there is no longer a finite algebraic basis for the algebra of symmetric polynomials. We can, however, derive explicit iterative formulae for writing the symmetric sums

$$p_n \;=\; x^n + y^n,$$

where $n \in \mathbb{Z}$, as *rational* functions composed with the map π of equation (1.4).

2 Mappings and topologies on $\mathbb{M}^d$

2.1 Nc-functions

In this section we describe the basic objects of free analysis. Firstly, the *nc-sets* in the nc-universe $\mathbb{M}^d$ of equation (1.1) are the sets that are closed under direct sums. They are are the natural domains of definition of *nc-functions*, which are the $\mathbb{M}^1$-valued functions that are graded, preserve direct sums and respect similarity transformations. We now explain the precise meaning of these terms.

Let $\mathbb{N}$ denote the set of positive integers and, for $n \in \mathbb{N}$, let

$$\mathcal{I}_n = \{M \in \mathbb{M}_n \,|\, M \text{ is invertible}\}.$$

For $x_1 = (x_1^1, \dots, x_1^d) \in \mathbb{M}_{n_1}^d$ and $x_2 = (x_2^1, \dots, x_2^d) \in \mathbb{M}_{n_2}^d$, we define $x_1 \oplus x_2 \in \mathbb{M}_{n_1+n_2}^d$ by identifying $\mathbb{C}^{n_1} \oplus \mathbb{C}^{n_2}$ with $\mathbb{C}^{n_1+n_2}$ and direct summing x_1 and x_2 componentwise, that is,

$$x_1 \oplus x_2 = \left(x_1^1 \oplus x_2^1, \dots, x_1^d \oplus x_2^d \right).$$

Likewise, if $x = (x^1, \dots, x^d) \in \mathbb{M}_n^d$ and $S \in \mathbb{M}_n$ is invertible, we define $S^{-1}xS \in \mathbb{M}_n^d$ by

$$S^{-1}xS = (S^{-1}x^1 S, \dots, S^{-1}x^d S).$$

Definition 2.1. *If $D \subseteq \mathbb{M}^d$ we say that D is an* nc-set *if D is closed with respect to the formation of direct sums, that is:*
For all $n_1, n_2 \in \mathbb{N}$ and all $x_1 \in D \cap \mathbb{M}_{n_1}^d$, $x_2 \in D \cap \mathbb{M}_{n_2}^d$,

$$x_1 \oplus x_2 \in D \cap \mathbb{M}_{n_1+n_2}^d.$$

Definition 2.2. *An* nc-function in d variables *is a function f whose domain is an nc-set $D \subseteq \mathbb{M}^d$, whose codomain is $\mathbb{M}^1$, and which satisfies*

(1) *for all $n \in \mathbb{N}$ and all $x \in D \cap \mathbb{M}_n^d$*

$$f(x) \in \mathbb{M}_n,$$

(2) *for all $n_1, n_2 \in \mathbb{N}$ and all $x_1 \in D \cap \mathbb{M}_{n_1}^d$ and $x_2 \in D \cap \mathbb{M}_{n_2}^d$*

$$f(x_1 \oplus x_2) = f(x_1) \oplus f(x_2), \text{ and}$$

(3) *for all $n \in \mathbb{N}$, all $x \in D \cap \mathbb{M}_n^d$ and all $s \in \mathcal{I}_n$, if $s^{-1}xs \in D$ then*

$$f(s^{-1}xs) = s^{-1}f(x)s.$$

A function f that has property (1) is said to be graded.

Definition 2.3. *Let $D_1 \subseteq \mathbb{M}^{d_1}$ and $D_2 \subseteq \mathbb{M}^{d_2}$ be nc-sets and let $F : D_1 \to D_2$. We say that F is an* nc-mapping *if there exist nc-functions $f_1, f_2, \ldots, f_{d_2}$ on D_1 such that*

$$F(x) = (f_1(x), f_2(x), \ldots, f_{d_2}(x))$$

for all $x \in D_1$. If, in addition, F is a bijection and both F and F^{-1} are nc-mappings then we say that F is an nc-isomorphism.

J. Taylor proved [21] that properties (2) and (3) of Definition 2.2 are equivalent to the single property that f respects intertwinings, in the following sense:

(4) *for all $n_1, n_2 \in \mathbb{N}$, for $x_1 \in D \cap \mathbb{M}_{n_1}^d$, for $x_2 \in D \cap \mathbb{M}_{n_2}^d$ and all n_2-by-n_1 matrices L satisfying $Lx_1^j = x_2^j L$ for $j = 1, \ldots, d$,*

$$Lf(x_1) = f(x_2)L. \tag{2.4}$$

2.2 Topologies on the nc-universe

For each n, $\mathbb{M}_n$ carries the natural topology induced by the $n \times n$ matrix norm. This topology gives rise to the product topology on $\mathbb{M}_n^d$. Using equation (1.1), we can endow $\mathbb{M}^d$ with a topology whereby a set $D \subseteq \mathbb{M}^d$ is open if $D \cap \mathbb{M}_n^d$ is open in $\mathbb{M}_n^d$ for each $n \in \mathbb{N}$. This topology has been called the *finitely open topology,* the *disjoint union topology* and the *coproduct topology*. We shall call it the finitely open topology, abbreviated to f.o.

Definition 2.5. *Let τ be a topology on $\mathbb{M}^d$. We say that τ is an* nc topology *or equivalently an* admissible topology *if τ has a basis consisting of finitely open nc-sets.*

As we have mentioned, the nc-universe $\mathbb{M}^d$ admits several natural topologies [4]. In this paper we shall consider nc-manifolds based on $\mathbb{M}^d$ endowed with three different topologies, the fine, free and Zariski-free topologies. The *fine* topology is the topology generated by *all* finitely open nc-sets.

Definition 2.6. *A domain in $\mathbb{M}^d$ is a finely open set.*

Domains do not need to be nc-sets. For example the set

$$\mathcal{Q} = \{x \in \mathbb{M}^1 : \sigma(x) \cap \sigma(-x) = \emptyset\} \tag{2.7}$$

is a domain (see Proposition 6.5 below), indeed a free domain (see Definition 2.8), but is not an nc-set.

We now describe the free topology. It was introduced in [1] in the context of a non-commutative Oka-Weil approximation theorem.

For $\delta = [\delta_{ij}]$ an $I \times J$ matrix of free polynomials, let $B_\delta \subseteq \mathbb{M}^d$ be defined by

$$B_\delta = \{x \in \mathbb{M}^d \mid \|[\delta_{ij}(x)]\| < 1\}.$$

Observe that if δ_1 and δ_2 are matrices of free polynomials, then

$$B_{\delta_1} \cap B_{\delta_2} = B_{\delta_1 \oplus \delta_2}.$$

As a consequence of this fact, the collection of sets of the form B_δ is closed with respect to finite intersections and thus forms a basis for a topology.

Definition 2.8. *The* free topology *on $\mathbb{M}^d$ is the topology for which a basis is the collection of sets of the form B_δ where δ is a matrix of free polynomials in d variables. A set that is open in the free topology is a* free domain.

Since B_δ is an nc-set and is open in the f.o. topology, the free topology is admissible.

The free topology is much coarser than the f.o. topology, so much so that while the f.o. topology is Hausdorff, the topology induced on $\mathbb{M}_n^d$ by the free topology is not even T_1 for any $n \geq 2$ and $d \geq 1$. One way to see this is to observe that free open sets are invariant under unitary conjugation.

Proposition 2.9. *Any freely open nc-set is freely path-connected.*

Proof. Let U be a freely open nc-set in $\mathbb{M}^d$. Consider $x,\ y \in U$; then $x \oplus y \in U$. Define $h : [0,1] \to U$ by

$$
h(t) = \begin{cases} x & \text{if } 0 \le t < \frac{1}{2} \\[2mm] x \oplus y & \text{if } t = \frac{1}{2} \\[2mm] y & \text{if } \frac{1}{2} < t \le 1. \end{cases}
$$

h is constant on the intervals $[0, \frac{1}{2})$ and $(\frac{1}{2}, 1]$, and so freely continuous on those intervals. h is also freely continuous at the point $\frac{1}{2}$. For consider any basic free neighborhood B_γ of the point $h(\frac{1}{2}) = x \oplus y$ in U. Then we have $\|\gamma(x \oplus y)\| < 1$, and so $\|\gamma(x)\| < 1$ and $\|\gamma(y)\| < 1$, which is to say that $h(t) \in B_\gamma$ for all t in the neighborhood $[0,1]$ of $\frac{1}{2}$. Thus h is continuous at $\frac{1}{2}$, and so h is a continuous path in U such that $h(0) = x$ and $h(1) = y$. $\qquad \square$

Corollary 2.10. $\mathbb{M}^d$ *is connected and locally connected in the free topology.*

Proof. Path-connectedness implies connectedness. To say that $\mathbb{M}^d$ is locally connected means that every point in $\mathbb{M}^d$ has a neighborhood base of connected sets. The sets B_δ comprise such a base. $\qquad \square$

Remark 2.11. The proof of Proposition 2.9 shows slightly more. For any freely open set U and any points $x, y \in U$ such that $x \oplus y \in U$, there is a freely continuous path in U from x to y.

There is another nc-topology that we shall need, the Zariski-free topology. It is obtained by adjoining to the free topology sets that are locally complements of free varieties. Detailed definitions will be given in Section 7.

3 Holomorphy with respect to admissible topologies

The notion of holomorphy for a function on $\mathbb{M}^d$ depends on the chosen topology of $\mathbb{M}^d$.

Definition 3.1. *Let τ be an admissible topology on $\mathbb{M}^d$. We say that a graded function f defined on a set $D \subseteq \mathbb{M}^d$ is a τ-holomorphic function if*

(1) *D is open in the topology τ;*

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(2) f is τ-locally nc, *that is, for each $x \in D$ there exists an nc set $U \in \tau$ such that $x \in U \subseteq D$ and $f|U$ is an nc-function;*

(3) f is τ-locally bounded, *that is, for each $x \in D$, there exists a τ-neighborhood U of x such that $f|U$ is bounded.*

At first sight this is a surprising definition. It is at least partially justified by its relation to the following notion of analyticity.

Definition 3.2. *A graded function $f : D \to \mathbb{M}^1$, where $D \subseteq \mathbb{M}^d$, is* analytic *if D is finitely open and, for every positive integer n, the restriction of f to $D \cap \mathbb{M}_n^d$ is analytic in the usual sense of several complex variables.*

Proposition 3.3. *For any admissible topology τ on $\mathbb{M}^d$, every τ-holomorphic function f on a domain D in $\mathbb{M}^d$ is analytic.*

Proof. Consider a point $x \in D \cap \mathbb{M}_n^d$. Since φ is τ-holomorphic we may choose a τ-neighborhood U of x in D on which φ is an nc-function and another τ-neighborhood V on which φ is bounded. Since the topology τ is admissible, it is an nc-topology and so we may assume that V is an nc-set. Then $U \cap V$ is an nc-set and is a τ-neighborhood of x in D on which φ is a bounded nc-function. Theorem 4.6 of [1] asserts that under these hypotheses φ is analytic on $U \cap V \cap \mathbb{M}_n^d$. Thus φ is analytic on some neighborhood of an arbitrary point of $D \cap \mathbb{M}_n^d$, and therefore φ is analytic on $D \cap \mathbb{M}_n^d$. $\square$

Proposition 3.4. *Let τ be an admissible topology on $\mathbb{M}^d$, let D be a τ-open set and let f be a τ-holomorphic function on D. If D is an nc set then f is an nc function.*

Proof. Consider $z, w \in D$. Since $z \oplus w \in D$, $f(z \oplus w)$ is defined. By Definition 3.1(2) there is an nc set $U \in \tau$ such that $z \oplus w \in U \subseteq D$ and $f|U$ is an nc function. It follows that $f(z \oplus w) = f(z) \oplus f(w)$. $\square$

The following statement is routine to check.

Proposition 3.5. *Let τ be an admissible topology on $\mathbb{M}^d$ and let D be a τ-open set. The set of τ-holomorphic functions on D is an algebra with respect to pointwise operations. The set $H_\tau^\infty(D)$ of bounded τ-holomorphic functions on D is a Banach algebra with respect to pointwise operations and the supremum norm*

$$\|f\|_\infty = \sup_{x \in D} \|f(x)\|.$$

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The case where τ is the free topology will be of special interest here, and in this case we refer to τ-holomorphic functions as *free* (or *freely*) *holomorphic functions*. Such functions are particularly well behaved on account of the following theorem [1] (it is also proved in [6] and [3]).

Theorem 3.6. *Let $D \subseteq \mathbb{M}^d$ be a free domain. A function $f : D \subseteq \mathbb{M}^d \to \mathbb{M}^1$ is a free holomorphic function if and only if f can be locally uniformly approximated by free polynomials. That is, f is freely holomorphic if and only if for each $x \in D$ there exists a free domain U satisfying $x \in U \subseteq D$ with the property that for each $\varepsilon > 0$ there exists a free polynomial p such that*

$$\sup_{y \in U} \|f(y) - p(y)\| < \varepsilon.$$

To prove that freely holomorphic functions are freely continuous we need the following simple observation. In the Lemma and elsewhere the norm $\|\cdot\|$ on $\mathbb{M}^d_n$ is given by

$$\|x\| = \max\{\|x^1\|, \ldots, \|x^d\|\}$$

where $\|x^j\|$ is the standard C^* norm on $\mathbb{M}_n$.

Lemma 3.7. *A free polynomial is freely locally Lipschitz. That is, if p is a free polynomial in d variables and $z \in \mathbb{M}^d$ then there exist a free neighborhood G of z and a positive constant K such that*

$$\|p(x) - p(y)\| \leq K\|x - y\|$$

for all $x, y \in G$.

Proof. It is enough to prove the statement in the case that p is the monomial

$$p(x) = x^{r_1} x^{r_2} \ldots x^{r_k}$$

for some $k \in \mathbb{N}$ and $r_1, \ldots, r_k \in \{1, \ldots, d\}$. Let

$$\delta(x) = (1 + \|z\|)^{-1} \operatorname{diag}\{x^1, \ldots, x^d\}.$$

Then $x \in B_\delta$ if and only if $\|x^j\| < 1 + \|z\|$ for $j = 1, \ldots, d$. Thus $z \in B_\delta$. We have, for any $x, y \in \mathbb{M}^d$,

$$p(x) - p(y) = (x^{r_1} - y^{r_1})x^{r_2} \ldots x^{r_k} + y^{r_1}(x^{r_2} - y^{r_2})x^{r_3} \ldots x^{r_k} +$$
$$\cdots + y^{r_1} \ldots y^{r_{k-1}}(x^{r_k} - y^{r_k}).$$

Hence, for every $x, y \in B_\delta$,

$$\|p(x) - p(y)\| \le k(1 + \|z\|)^{k-1} \|x - y\|.$$

Thus p is freely locally Lipschitz. $\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Proposition 3.8. *Let U be a free domain in $\mathbb{M}^d$. A function $f : U \to \mathbb{M}^1$ is freely holomorphic if and only if f is a freely locally nc-function (as in Definition 3.1 (2)) and f is continuous when U and $\mathbb{M}^1$ are equipped with the free topologies.*

Proof. First assume that U is an nc-set, f is an nc-function and f is continuous when D and $\mathbb{M}^1$ are equipped with the free topologies. In the light of Definition 2.3 it suffices to show that f is locally bounded. Fix $z \in U$. As f is assumed continuous and U is open in the free topology, there exists a free matricial polynomial δ such that $z \in B_\delta \subseteq U$ and such that

$$f(B_\delta) \subseteq \{x \in \mathbb{M}^1 \mid \|x\| < \|f(z)\| + 1\}.$$

This proves that f is locally bounded.

Now assume that U is a free domain in $\mathbb{M}^d$ and f is a free holomorphic function on U. The continuity of f will follow if we can show that $f^{-1}(B_q)$ is freely open in $\mathbb{M}^d$ whenever q is a square matrix of polynomials in one variable.

Fix $z \in f^{-1}(B_q)$. By Theorem 3.6 there exists a square matrix δ of free polynomials such that f is bounded on B_δ and a sequence of free polynomials $p_1, p_2, \ldots$ such that $z \in B_\delta \subseteq U$ and

$$\lim_{k \to \infty} \sup_{x \in B_\delta} \|f(x) - p_k(x)\| = 0. \qquad\qquad (3.9)$$

It follows that the polynomials p_k are uniformly bounded on B_δ. By Lemma 3.7 q is locally Lipschitz on U, and hence

$$\lim_{k \to \infty} \sup_{x \in B_\delta} \|q(f(x)) - q(p_k(x))\| = 0. \qquad\qquad (3.10)$$

Fix a strictly decreasing sequence $t_1, t_2, \ldots$ with $t_k > 1$ for all k and $t_k \to 1$ as $k \to \infty$. Use the property (3.10) to construct inductively a sequence

$k_1, k_2, \ldots$ of positive integers such that

$$B_{t_1(q \circ f)} \cap B_\delta \subseteq B_{t_2(q \circ p_{k_1})} \cap B_\delta$$
$$\subseteq B_{t_3(q \circ f)} \cap B_\delta$$
$$\subseteq B_{t_4(q \circ p_{k_2})} \cap B_\delta$$
$$\subseteq B_{t_5(q \circ f)} \cap B_\delta$$
$$\ldots$$

The construction ensures that

$$B_{q \circ f} \cap B_\delta = \bigcup_{i=1}^{\infty} (B_{t_{2i-1}(q \circ f)} \cap B_\delta) = \bigcup_{\ell=1}^{\infty} (B_{t_{2\ell}(q \circ p_{k_\ell})} \cap B_\delta).$$

The last union in this formula is a union of basic freely open sets and so is a freely open set in $\mathbb{M}^d$. Hence $B_{q \circ f} \cap B_\delta$ is freely open. Since $B_{q \circ f} = f^{-1}(B_q)$ we have $z \in B_{q \circ f} \cap B_\delta \subseteq f^{-1}(B_q)$ and therefore $f^{-1}(B_q)$ is open in $\mathbb{M}^d$. $\square$

Similar results on analyticity of nc functions are known in several contexts, for example [17, Chapter 7].

Definition 3.11. *Let τ be an admissible topology on $\mathbb{M}^{d_1}$ and let $D \subseteq \mathbb{M}^{d_1}$ be a τ-open set. We say that $F : D \to \mathbb{M}^{d_2}$ is a τ-holomorphic mapping if there exist τ-holomorphic functions $f_1, f_2, \ldots, f_{d_2}$ on D such that*

$$F(x) = (f_1(x), f_2(x), \ldots, f_{d_2}(x))$$

for all $x \in D$.

Let D_1, D_2 be τ-open sets in $\mathbb{M}^d$. We say that $F : D_1 \to D_2$ is a τ-biholomorphic mapping if F is a bijection and both F and F^{-1} are τ-holomorphic mappings.

Proposition 3.12. *Let $F = (f_1, \ldots, f_d)$ be a freely locally nc mapping defined on a free open set $D \subseteq \mathbb{M}^d$. The following statements are equivalent.*

(1) *F is a freely holomorphic map;*

(2) *F is freely locally bounded;*

(3) *F is freely continuous.*

In particular, F is a freely biholomorphic map if and only if F is a homeomorphism in the free topology.

This result is proved in much the same way as Proposition 3.8.

4 Nc-manifolds and topological nc-manifolds

4.1 Nc-manifolds

In this section we define nc-manifolds, topological nc-manifolds and free manifolds. The concept of an nc-manifold is the generalization of the concept of an nc-set as defined in Definition 2.1 and as such is purely algebraic in nature. The notions of charts, atlases and transition functions transfer directly from the classical theory, only in the new context a chart is a bijective map from a subset of an nc-manifold to a subset of $\mathbb{M}^d$; in the case of a *topological* nc-manifold the image of a chart is an open set in $\mathbb{M}^d$ with respect to some specified topology.

Definition 4.1. *If X is a set, then we say that α is a d-dimensional co-ordinate patch or chart on X if α is a bijection from a set $U_\alpha \subseteq X$ to a set $D_\alpha \subseteq \mathbb{M}^d$. If α and β are a pair of d-dimensional co-ordinate patches on X with $U_\alpha \cap U_\beta \neq \emptyset$, then we define the transition map $T_{\alpha\beta}$ by*

$$T_{\alpha\beta} : \alpha(U_\alpha \cap U_\beta) \to \beta(U_\alpha \cap U_\beta), \tag{4.2}$$

$$T_{\alpha\beta}(x) = \beta \circ \alpha^{-1}(x), \qquad x \in \alpha(U_\alpha \cap U_\beta). \tag{4.3}$$

If X is a set then we say that $\mathcal{A}$ is a d-dimensional nc-atlas for X if $\mathcal{A}$ is a collection of d-dimensional co-ordinate patches on X,

$$\bigcup_{\alpha \in \mathcal{A}} U_\alpha = X \tag{4.4}$$

and, for all $\alpha, \beta \in \mathcal{A}$,

(1) *$\alpha(U_\alpha \cap U_\beta)$ is a union of nc-sets, and*

(2) *for every nc-subset W of $\alpha(U_\alpha \cap U_\beta)$, the restriction of $T_{\alpha\beta}$ to W is an nc-mapping.*

A d-dimensional nc-manifold is an ordered pair $(X, \mathcal{A})$ where X is a set and $\mathcal{A}$ is a d-dimensional nc-atlas for X.

If $(X_1, \mathcal{A}_1)$ and $(X_2, \mathcal{A}_2)$ are nc-manifolds of dimensions d_1, d_2 respectively, then a map $f : X_1 \to X_2$ is an nc-map (or nc-mapping) if, for every $\alpha \in \mathcal{A}_1$ (with domain U_α and codomain D_α) and $\beta \in \mathcal{A}_2$ (with domain V_β and codomain E_β)

(1′) *the set*

$$W_{\alpha\beta} \overset{\text{def}}{=} \alpha(U_\alpha \cap f^{-1}(V_\beta))$$

is a union of nc-sets, and

(2′) *for every nc-subset W of $W_{\alpha\beta}$*

$$\beta \circ f \circ (\alpha^{-1}|W) : W \to E_\beta \subset \mathbb{M}^{d_2}$$

is an nc-map.

If $(X_1, \mathcal{A}_1)$ and $(X_2, \mathcal{A}_2)$ are nc-manifolds then an nc-isomorphism *from X_1 to X_2 is a bijective map $f : X_1 \to X_2$ such that both f and f^{-1} are nc-maps.*

Remark 4.5. 1. If $(X, \mathcal{A})$ is an nc-manifold then, for every $\alpha \in \mathcal{A}$, D_α is a union of nc-sets. This is a consequence of condition (1) in the definition of an nc-atlas above and the identity

$$D_\alpha = \bigcup_{\beta \in \mathcal{A}} \alpha(U_\alpha \cap U_\beta).$$

2. If $(X, \mathcal{A})$ is an nc-manifold then there is an nc-atlas $\mathcal{A}^*$ on X such that the identity map id_X is an nc-isomorphism from $(X, \mathcal{A})$ to $(X, \mathcal{A}^*)$ and, for every $\alpha \in \mathcal{A}^*$, the range of α is an nc-set (not merely a *union* of nc-sets). Indeed, we may define $\mathcal{A}^*$ to be the set of all maps $\alpha|V$ for some $\alpha \in \mathcal{A}$ and $V \subset U_\alpha$ such that $\alpha(V)$ is an nc-set. It would be possible to develop the theory of nc-manifolds with the assumption that the ranges of charts are always nc-sets, but in the topological context it is convenient to allow them to be merely *unions* of nc-sets.

To prove that $\mathrm{id}_X : (X, \mathcal{A}) \to (X, \mathcal{A}^*)$ is an nc-isomorphism, consider any $\alpha \in \mathcal{A}$ and $\gamma \in \mathcal{A}^*$, say $\gamma = \beta|V$ where $V \subset U_\beta$ and $\beta(V)$ is an nc-subset of D_β. Then $W_{\alpha\gamma} = \alpha(U_\alpha \cap V) \subset \alpha(U_\alpha \cap U_\beta) = T_{\beta\alpha}(\beta(V))$. Since $\beta(V)$ is an nc-set and $T_{\beta\alpha}$ is an nc-map, it follows that $W_{\alpha\gamma}$ is an nc-set. Moreover, for every nc-subset W of $W_{\alpha\gamma} = \alpha(U_\alpha \cap V)$,

$$\beta \circ \mathrm{id}_X \circ (\alpha^{-1}|W) = T_{\alpha\beta}$$

is an nc-map. Similarly, reversing the roles of $\mathcal{A}$ and $\mathcal{A}^*$, we obtain, for any nc-subset W of $W_{\gamma\alpha} = \beta(U_\alpha \cap V)$

$$\alpha \circ \mathrm{id}_X \circ (\gamma^{-1}|W) = T_{\beta\alpha}|W$$

is also an nc-map. Thus id_X is an nc-isomorphism with respect to $\mathcal{A}$ and $\mathcal{A}^*$.

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4.2   Topological nc-manifolds

We now consider the case where X, in addition to carrying the structure of an nc-manifold, is a topological space. X will be a topological nc-manifold if it is locally homeomorphic to an open set in some $\mathbb{M}^d$. Since there is no one "correct" topology to place on $\mathbb{M}^d$, we first fix a topology τ on $\mathbb{M}^d$. We assume that τ is an admissible topology in the sense of Definition 2.5.

If X is a set and $\mathcal{T}$ is a topology on X, then we say that α is a *topological d-dimensional nc-co-ordinate patch on $(X,\mathcal{T})$ with respect to τ* if α is a d-dimensional nc-co-ordinate patch on X and, in addition,

$$\alpha : U_\alpha \to D_\alpha \text{ is a homeomorphism.} \tag{4.6}$$

Here $U_\alpha \subseteq X$ is a $\mathcal{T}$-open set equipped with the relative topology induced by $\mathcal{T}$ and $D_\alpha \subseteq \mathbb{M}^d$ is a τ-open nc-set equipped with the relative topology induced by τ. Since τ is locally nc, D_α is a union of open nc-sets.

If $(X,\mathcal{T})$ is a topological space then we say that $\mathcal{A}$ is a *d-dimensional topological nc-atlas for $(X,\mathcal{T})$ with respect to τ* if $\mathcal{A}$ is a collection of topological d-dimensional nc-co-ordinate patches on $(X,\mathcal{T})$ and $\mathcal{A}$ is an nc-atlas for X with respect to τ in the sense of Definition 4.1.

Definition 4.7. *Let τ be an admissible topology on $\mathbb{M}^d$. A* topological d-dimensional nc-manifold with respect to τ *is an ordered triple $(X,\mathcal{T},\mathcal{A})$ where X is a set, $\mathcal{T}$ is a topology on X and $\mathcal{A}$ is a d-dimensional topological nc-atlas for $(X,\mathcal{T})$ with respect to τ.*

In the special case where τ is the free topology we say that X is a free manifold.

The manifolds studied in analysis usually have some smoothness property, such as C^∞ or analyticity, whereas the topological nc-manifolds introduced in Definition 4.7 are not assumed smooth. It is simple to extend the notion of topological nc-manifold further to bring in appropriate notions of smoothness.

Definition 4.8. *Let τ be an admissible topology on $\mathbb{M}^d$. A d-dimensional* holomorphic nc-manifold with respect to τ *is a topological d-dimensional nc-manifold $(X,\mathcal{T},\mathcal{A})$ with respect to τ such that, for all $\alpha,\beta \in \mathcal{A}$, the transition map $T_{\alpha\beta}$ is a τ-holomorphic mapping in the sense of Definitions 3.11 and 3.1.*

If $(X,\mathcal{T},\mathcal{A})$ is a d-dimensional holomorphic nc-manifold with respect to τ then a function $F : X \to \mathbb{M}^1$ is said to be holomorphic on X *if, for every*

$\alpha \in \mathcal{A}$, *the map*

$$F \circ \alpha^{-1} : \alpha(U_\alpha) \to \mathbb{M}^1$$

is a τ-holomorphic function on the τ-open set $\alpha(U_\alpha)$.

More generally, we can define the notion of a holomorphic map between two topological nc-manifolds.

Definition 4.9. *Let τ_1, τ_2 be admissible topologies on $\mathbb{M}^{d_1}, \mathbb{M}^{d_2}$ respectively, and let $(X_j, \mathcal{T}_j, \mathcal{A}_j)$ be a topological d_j-dimensional manifold with respect to τ_j for $j = 1, 2$. A map $f : X_1 \to X_2$ is said to be* holomorphic *at a point $x \in X_1$ if there exist $\alpha \in \mathcal{A}_1$ (with domain U_α and domain D_α) and $\beta \in \mathcal{A}_2$ (with domain V_β and range E_β) such that $x \in U_\alpha$, $f(x) \in V_\beta$ and the map*

$$\beta \circ f \circ \alpha^{-1} : \alpha(U_\alpha \cap f^{-1}(V_\beta)) \to E_\beta$$

is τ_1-holomorphic. We say that f is holomorphic on X_1 if f is holomorphic at every point of X_1.

Note that the definition is independent of τ_2, save for the requirement that τ_2 be admissible.

In the case that τ is the free topology, Proposition 3.12 tells us that continuity implies holomorphy. Accordingly, free manifolds are a precise noncommutative analog of complex manifolds. Indeed, they are the topological manifolds that are equipped with an atlas of homeomorphisms onto free domains in $\mathbb{M}^d$ with the property that the transition functions are freely biholomorphic maps. To see this fact, assume that $(X, \mathcal{T}, \mathcal{A})$ is a free manifold, $\alpha, \beta \in \mathcal{A}$, and $U_\alpha, U_\beta \in \mathcal{T}$ with $U_\alpha \cap U_\beta \neq \emptyset$. As $U_\alpha \cap U_\beta \in \mathcal{T}$, the hypothesis (4.6) implies that $\alpha(U_\alpha \cap U_\beta)$ and $\beta(U_\alpha \cap U_\beta)$ are open in the free topology. The hypothesis (4.6) also implies that

$$\beta \circ \alpha^{-1} : \alpha(U_\alpha \cap U_\beta) \to \beta(U_\alpha \cap U_\beta)$$

is a homeomorphism. Hence, by Proposition 3.12, $\beta \circ \alpha^{-1}$ is a free biholomorphic mapping. Conversely, if the transition functions are assumed to be free biholomorphic mappings, then Proposition 3.12 implies that the transition functions are free holomorphic mappings.

If $(X, \mathcal{T}, \mathcal{A})$ is a topological d-dimensional nc-manifold with respect to τ then the transition function $T_{\alpha\beta}$ is a composition of two homeomorphisms and hence is a homeomorphism between $\alpha(U_\alpha \cap U_\beta)$ and $\beta(U_\alpha \cap U_\beta)$ in their

respective τ topologies, as well as being an nc-isomorphism between nc-sets when restricted to any nc-subset of $\alpha(U_\alpha \cap U_\beta)$.

If $(X, \mathcal{T}, \mathcal{A})$ is a topological d-dimensional nc-manifold with respect to τ and τ' is a finer admissible topology on $\mathbb{M}^d$ then we define the topology $\mathcal{T}'$ on X to be the one for which a base is

$$\{\alpha^{-1}(V) : \alpha \in \mathcal{A}, V \subset D_\alpha \text{ and } V \in \tau'\}.$$

Then $(X, \mathcal{T}', \mathcal{A})$ is a d-dimensional topological nc-manifold with respect to τ'. The topology $\mathcal{T}'$ is finer than $\mathcal{T}$ and the map $\mathrm{id}_X : (X, \mathcal{T}', \mathcal{A}) \to (X, \mathcal{T}, \mathcal{A})$ is a holomorphic map of topological nc-manifolds.

In particular we may take τ' to be the fine topology on $\mathbb{M}^d$. Then the topology $\mathcal{T}'$ is the finest topology for which X is a topological nc-manifold.

5 Free holomorphic functions in one variable

In this section we show that a free holomorphic function in one variable is determined (via the functional calculus) by its restriction to $\mathbb{M}_1^1$.

Let $\mathbb{R}^+$ denote the set of positive real numbers. If $c \in \mathbb{C}^k$ and $r \in \mathbb{R}^{+k}$, we define $\Delta(c, r) \subseteq \mathbb{C}$ by

$$\Delta(c, r) = \bigcup_{j=1}^{k} \{z \in \mathbb{C} : |z - c_j| < r_j\}$$

$$= \bigcup_{j=1}^{k} (c_j + r_j \mathbb{D})$$

and define $D(c, r) \subseteq \mathbb{M}^1$ by

$$D(c, r) = \{x \in \mathbb{M}^1 : \sigma(x) \subseteq \Delta(c, r)\}$$

where $\sigma(x)$ denotes the spectrum of the matrix x. In the sequel we make the standing assumption that the radii $r_1, \ldots, r_k$ are so small that the discs $c_1 + r_1\mathbb{D}, \ldots, c_k + r_k\mathbb{D}$ are pairwise disjoint.

Proposition 5.1. $D(c, r)$ *is a free domain.*

Proof. Assume that $M \in D(c, r)$. We construct a polynomial δ satisfying

$$M \in B_\delta \subseteq D(c, r).$$

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Choose a polynomial q satisfying
$$\sigma(M) \subseteq \{z \mid |q(z)| < 1\} \subseteq \Delta(c, r).$$

As $\sigma(q(m)) = q(\sigma(m)) \subseteq \mathbb{D}$, there exists an integer N such that $\|q(M)^N\| < 1$. If we set $\delta = q^N$, it follows that $M \in B_\delta$.

It remains to show that $B_\delta \subseteq D(c, r)$. Fix $x \in B_\delta$ such that $\|q(x)^N\| < 1$. It follows that
$$q(\sigma(x))^N = \sigma(q(x)^N) \subseteq \mathbb{D}.$$
This implies that $q(\sigma(x)) \subseteq \mathbb{D}$. But then,
$$\sigma(x) \subseteq \{z \mid |q(z)| < 1\} \subseteq \Delta(c, r),$$
which implies that $x \in D(c, r)$. $\qquad\qquad\square$

If $c \in \mathbb{C}^k$ and $r \in \mathbb{R}^{+k}$ we let $[c, r] = \{[c, r]_1, \ldots, [c, r]_k\}$ denote the system of paths where, for $j = 1, \ldots, k$,
$$[c, r]_j(t) = c_j + r_j e^{it}, \qquad 0 \le t \le 2\pi.$$

If f is holomorphic on a neighborhood of $\Delta(c, r)$ and $x \in D(c, r)$ then we may employ the Riesz Functional Calculus to define $f^\wedge(x)$ by the formula
$$f^\wedge(x) = \int_{[c,s]} f(z)(z - x)^{-1} \, dz \tag{5.2}$$

where $s \in \mathbb{R}^{+k}$ is chosen so that $\sigma(x) \subseteq \Delta(c, s)$ and $\Delta(c, s)^- \subseteq \Delta(c, r)$.

Proposition 5.3. *If f is holomorphic on $\Delta(c, r)$, then $f^\wedge$ is freely holomorphic on $D(c, r)$.*

Proof. It is straightforward to verify that $f^\wedge$ is an nc-function. The proposition will follow if we can show that $f^\wedge$ is locally bounded. To that end, fix $M \in D(c, r)$. Chose $s \in \mathbb{R}^{+k}$ so that $\sigma(M) \subseteq \Delta(c, s)$ and $\Delta(c, s)^- \subseteq \Delta(c, r)$. Choose $C \in \mathbb{R}^+$ such that
$$C > \max_{z \in [c,s]} \|(z - M)^{-1}\|. \tag{5.4}$$

Choose a finite set $S \subseteq [c, s]$ such that, for all $z \in [c, s]$, there exists $w \in S$ such that
$$|z - w| < \frac{1}{2C}. \tag{5.5}$$

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Finally, define $G \subseteq \mathbb{M}^1$ by

$$G = \bigcap_{w \in S} \{x \in \mathbb{M}^1 \mid w \notin \sigma(x) \text{ and } \|(w-x)^{-1}\| < C\}. \qquad (5.6)$$

As $M \in G \cap D(c,s) \subseteq D(c,r)$, the proof of Proposition 5.3 will be complete if we can prove the following two claims.

Claim 1. $G \cap D(c,s)$ is a free domain.

Claim 2. f is bounded on $G \cap D(c,s)$.

To prove Claim 1, first notice that by Proposition 5.1, it suffices to show that G is a free domain. In the light of equation (5.6) and the fact that S is finite it will follow that G is a free domain if we can show that for each $w \in \mathbb{C}$ and each $C > 0$,

$$\{x \in \mathbb{M}^1 \mid w \notin \sigma(x) \text{ and } \|(w-x)^{-1}\| < C\} \text{ is a free domain.}$$

But by Theorem 10.1 in [1] for each fixed $w \in \mathbb{C}$, $g_w(x) = (w-x)^{-1}$ is a free holomorphic function on the free domain $\{x \in \mathbb{M}^1 \mid w \notin \sigma(x)\}$. As Proposition 3.8 guarantees that g_w is freely continuous, it follows that

$$\{x \in \mathbb{M}^1 \mid w \notin \sigma(x) \text{ and } \|(w-x)^{-1}\| < C\} = g_w^{-1}(\{y \in \mathbb{M}^1 \mid \|y\| < C\})$$

is a free domain.

To prove Claim 2, let $z \in [c,s]$ and let $x \in G$. As $z \in [c,s]$, inequality (5.5) guarantees that we may choose $w \in S$ such that

$$|w - z| < \frac{1}{2C}. \qquad (5.7)$$

As $x \in G$, equation (5.6) guarantees that $w \notin \sigma(x)$ and

$$\|(w-x)^{-1}\| < C. \qquad (5.8)$$

Now,

$$\begin{aligned}
(z-x)^{-1} &= \left((w-x) - (w-z)\right)^{-1} \\
&= (w-x)^{-1}\left(1 - \frac{w-z}{w-x}\right)^{-1}.
\end{aligned}$$

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But as inequalities (5.7) and (5.8) imply that

$$\left\| \frac{w-z}{w-x} \right\| < \frac{1}{2C} \, C = \frac{1}{2},$$

we can use the inequality $\|(1-y)^{-1}\| \leq (1 - \|y\|)^{-1}$ (valid for $\|y\| < 1$) to prove that

$$\left\| \left(1 - \frac{w-z}{w-x} \right)^{-1} \right\| \leq \frac{1}{1 - \|\frac{w-z}{w-x}\|} < 2.$$

Hence

$$\|(z-x)^{-1}\| \leq \|(w-x)^{-1}\| \left\| \left(1 - \frac{w-z}{w-x} \right)^{-1} \right\| < 2C.$$

To summarize, we have shown that if $z \in [c, s]$ and $x \in G$, then $\|(z - x)^{-1}\| \leq 2C$. Use the definition (5.2) to estimate $f^\wedge(x)$ to deduce that, if $x \in G \cap D(c, s)$, then

$$\|f^\wedge(x)\| \leq 2C \max_{z \in [c,s]} |f(z)| \, \text{length}([c, s]).$$

This completes the proof of both Claim 2 and the theorem. $\qquad \square$

Definition 5.9. *For any free domain in U in $\mathbb{M}^1$ we define U_1 to be $U \cap \mathbb{M}_1^1$. If f is a free holomorphic function on U we define a function f_1 on U_1 by*

$$f_1(z) = f([z]), \qquad z \in U_1.$$

In the sequel we make no distinction between $z \in \mathbb{C}$ and $[z] \in \mathbb{M}_1^1$, and in particular, view U_1 both as a subset of $\mathbb{C}$ and as a subset of $\mathbb{M}^1$.

Proposition 5.10. *If $U \subseteq \mathbb{M}^1$ is a free domain and $M \in U$, then $\sigma(M) \subseteq U_1$.*

Proof. Suppose that $M \in U$ and $z \in \sigma(M)$. We wish to show that $z \in U_1$ or equivalently, that $z \in U$.

Choose an $I \times J$ matrix of polynomials δ such that $M \in B_\delta \subseteq U$ and choose a unit vector $v \in \mathbb{C}^n$ such that $Mv = zv$. We view $\delta(M)$ as a linear transformation from $\mathbb{C}^J \otimes \mathbb{C}^n$ to $\mathbb{C}^I \otimes \mathbb{C}^n$. As $M \in B_\delta$, there exists $r < 1$ such that $\|\delta(M)\| \leq r$.

With the setup of the previous paragraph, if $c \in \mathbb{C}^J$,

$$
\begin{aligned}
\|\delta(z)c\| &= \|\delta(z)c\|\ \|v\| \\
&= \|(\delta(z)c) \otimes v\| \\
&= \|\delta(M)(c \otimes v)\| \\
&\leq r\|c \otimes v\| \\
&= r\|c\|\|v\| \\
&= r\|c\|.
\end{aligned}
$$

This proves that $\|\delta(z)\| \leq r < 1$. Hence $z \in B_\delta \subseteq U$, as was to be proved. $\quad\square$

Corollary 5.11. *A nonempty freely open set in $\mathbb{M}^1$ meets $\mathbb{M}^1_1$.*

Proposition 5.12. *If $U \subseteq \mathbb{M}^1$ is a free domain and f is a free holomorphic function on U, then f_1 is holomorphic on U_1.*

Proof. Fix $z_0 \in U_1$. As f is free holomorphic, by Theorem 3.6 f can be locally uniformly approximated by polynomials. Choose δ such that $z_0 \in B_\delta \subseteq U$ and f can be uniformly approximated by polynomials on B_δ. By continuity, there exists $\varepsilon > 0$ such that $\{z \in \mathbb{C} \,|\, |z - z_0| < \varepsilon\} \subseteq B_\delta$. It follows that f_1 can be uniformly approximated on a neighborhood in $\mathbb{C}$ by polynomials. Hence f_1 is holomorphic on a neighborhood of z_0. As $z_0 \in U_1$ was chosen arbitrarily, it follows that f_1 is holomorphic on U_1. $\quad\square$

Proposition 5.13. *If $U \subseteq \mathbb{M}^1$ is a free domain and f is a free holomorphic function defined on U, then*

$$
f(x) = f_1^\wedge(x)
$$

for all $x \in U$. Moreover $f(x)$ belongs to the algebra generated by x and the identity matrix of appropriate size.

Proof. First observe that the assertion of the proposition makes sense. If $x \in U$, then Proposition 5.10 implies that $\sigma(x) \subseteq U_1$. Also, if f is freely holomorphic on U, then Proposition 5.12 implies that f_1 is holomorphic on U_1. Thus, $f_1^\wedge(x)$ is well defined by equation (5.2) for all $x \in U$.

If p is a polynomial, then $p_1^\wedge = p_1$ and $p_1 = p$. Hence, the result holds in the special case where $f = p$. The general case then follows by approximation. $\quad\square$

6   The Riemann surface for $\sqrt{x}$

In this section we shall define $\sqrt{x}$ as a holomorphic function on a one-dimensional free nc-manifold. By gluing together locally defined branches of the inverse of the free holomorphic function $f(x) = x^2$ we construct a free nc-manifold in much the same way that elementary textbooks construct the Riemann surface for $\sqrt{z}$ by piecing together locally-defined function elements. We obtain a locally finitely-sheeted one-dimensional free nc-manifold which has properties analogous to the Riemann surface for $\sqrt{z}$.

The zero matrix in $\mathbb{M}_2$ has infinitely many square roots, but only one of them lies in the algebra generated by the zero and identity matrices. By Proposition 5.13, for any free holomorphic function f and any x for which $f(x)$ is defined, $f(x)$ lies in the algebra $\mathrm{alg}(x)$ generated by x and the identity. We shall therefore use the symbol $\sqrt{x}$ in the following way.

Definition 6.1. *The* free square root $\sqrt{x}$ *is the multivalued function on* $\mathbb{M}^1$ *given by*

$$\sqrt{x} = \{y \in \mathrm{alg}(x) | y^2 = x\} \subset \mathbb{M}^1.$$

Thus, if $x = 0_n$, the $n \times n$ zero matrix then $\sqrt{x} = \{0_n\}$ and if $x = I_n$, the $n \times n$ identity matrix, then $\sqrt{x} = \{I_n, -I_n\}$, whereas if $x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ then $\sqrt{x} = \emptyset$. We leave the proof of the following proposition to the reader.

Proposition 6.2. *The free square root of a matrix* $x \in \mathbb{M}^1$ *is empty if and only if the Jordan canonical form of* x *contains a nilpotent Jordan cell of type* $k \times k$ *for some* $k \geq 2$.

The set

$$\Xi \overset{\text{def}}{=} \{x \in \mathbb{M}^1 : \sqrt{x} \neq \emptyset\}$$

is not a freely open set: it contains $0_{2 \times 2}$, but any basic free neighborhood of Ξ contains a matrix $\begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}$ for some $t \neq 0$. For the purpose of constructing a Riemann surface we consider the restriction of the square root function to Ξ^o, the interior of Ξ in the free topology. We shall show (Proposition 6.6) that Ξ^o is the set of nonsingular matrices, and that the union of the sets $\{x\} \times \sqrt{x}$, as x ranges over all nonsingular matrices, can be given the structure of a free nc-manifold.

Let

$$\mathcal{I} = \{x \in \mathbb{M}^1 \,|\, x \text{ is nonsingular}\} \tag{6.3}$$

and

$$\mathcal{Q} = \{x \in \mathbb{M}^1 \,|\, \sigma(x) \cap \sigma(-x) = \emptyset\}. \tag{6.4}$$

Proposition 6.5. *The sets $\mathcal{I}$ and $\mathcal{Q}$ are open and connected in the free topology.*

Proof. Fix $M \in \mathcal{I}$. If q is the characteristic polynomial of M, then $q(M) = 0$ and $q(0) \neq 0$. If we set $\delta = q(0)^{-1}q$, then clearly, as $\|\delta(M)\| = 0 < 1$, $M \in B_\delta$. Also, if $x \in B_\delta$, then as $\|\delta(x)\| < 1$,

$$\delta(\sigma(x)) = \sigma(\delta(x)) \subseteq \mathbb{D}.$$

Since $\delta(0) = 1 \notin \delta(\sigma(x))$, it follows that $0 \notin \sigma(x)$, that is $x \in \mathcal{I}$. Summarizing, given $M \in \mathcal{I}$, we have constructed a free polynomial δ such that $M \in B_\delta \subseteq \mathcal{I}$. Hence $\mathcal{I}$ is open in the free topology.

$\mathcal{I}$ is clearly closed under direct sums, and so it is freely connected, by Proposition 2.9.

Now fix $M \in \mathcal{Q}$. Let $\sigma(M) = \{c_1, \ldots, c_k\}$ and choose $r_1, \ldots, r_k \in \mathbb{R}^+$ such that $\Delta(c, r) \cap \Delta(-c, r) = \emptyset$. Let q be the characteristic polynomial of M and define $\delta = \rho q$ where $\rho \in \mathbb{R}^+$ is chosen so large that

$$\{z \in \mathbb{C} \,|\, |\delta(z)| < 1\} \subseteq \Delta(c, r).$$

With these choices, $M \in B_\delta$ and if $x \in B_\delta$ then $\sigma(x) \subseteq \Delta(c, r)$. Hence, as then $\sigma(-x) \subseteq \Delta(-c, r)$ as well, $\sigma(x) \cap \sigma(-x) = \emptyset$, that is, $x \in \mathcal{Q}$. Summarizing, given $M \in \mathcal{Q}$, we have constructed a free polynomial δ such that $M \in B_\delta \subseteq \mathcal{Q}$. Hence $\mathcal{Q}$ is open in the free topology.

Consider any points $x, y \in \mathcal{Q}$. Although $\mathcal{Q}$ is not closed under $\oplus$, since $\sigma(x)$ can meet $\sigma(-y)$, it is nearly so. Choose $t \in (0, 1]$ such that $\sigma(tx)$ is disjoint from $\sigma(-y)$; then $tx \oplus y \in \mathcal{Q}$. Remark 2.11 shows that there is a freely continuous path in $\mathcal{Q}$ from tx to y, while there is an obvious freely continuous path in $\mathcal{Q}$ from x to tx. $\qquad\square$

Proposition 6.6. *The interior Ξ^o of the set Ξ in the free topology is $\mathcal{I}$.*

Proof. By Proposition 6.5, $\mathcal{I}$ is freely open, and it is clearly contained in Ξ. Suppose $\mathcal{I}$ is a proper subset of Ξ^o; then there is a singular matrix $M \in \Xi$ and a basic free neighborhood B_δ of M contained in Ξ. Since B_δ is invariant

under unitary conjugations, we may assume that M is upper triangular, and since M is singular, we may take M to have zero as its $(1,1)$ entry. Let $N = M \oplus M$; then $N \in B_\delta$, and for a suitable permutation matrix P, the matrix P^*NP in B_δ is upper triangular and has $0_{2\times 2}$ as a block in the $(1,1)$ position. For some complex $\zeta \neq 0$, B_δ contains the upper triangular matrix T differing from P^*NP only in that its $(1,2)$ entry is ζ. If $e_1, e_2, \ldots$ denotes the standard basis of $\mathbb{C}^n$, for the appropriate n, then a Jordan chain for T corresponding to the eigenvalue 0 is $e_2, \zeta e_1$. Hence the Jordan form of T has a nilpotent Jordan cell of type at least 2×2, and therefore, by Proposition 6.2, $\sqrt{T}$ is empty, contradicting the fact that $B_\delta \subset \Xi$. Hence $\Xi^o = \mathcal{I}$. □

We shall construct the Riemann surface for $\sqrt{x}$ by piecing together function elements over $\mathcal{I}$.

Definition 6.7. *By a free function element over $\mathcal{I}$ is meant a pair (f, U) where U is a free domain in $\mathcal{I}$ and f is a free holomorphic function on U. We say a function element (f, U) is a branch of $\sqrt{x}$ if $f(x)^2 = x$ for all $x \in U$.*

Lemma 6.8. *Let (f, U) and (g, V) be free function elements over $\mathcal{I}$, both assumed to be branches of $\sqrt{x}$. If $M \in U \cap V$ and $f(M) = g(M)$, then there exists a free domain W such that $M \in W \subseteq U \cap V$ and $f|W = g|W$.*

Proof. Let (f, U) and (g, V) be branches of $\sqrt{x}$ and assume that $M \in U \cap V$ and $f(M) = g(M)$. By Proposition 5.13 there exist holomorphic functions f_1 on U_1 and g_1 on V_1 such that $f = f_1^\wedge$ on U and $g = g_1^\wedge$ on V.

Now, since $M \in U \cap V$, by Proposition 5.10

$$\sigma(M) \subseteq U_1 \cap V_1.$$

Furthermore, since f and g are branches of $\sqrt{x}$ on U and V respectively,

$$f_1, g_1 \text{ are branches of } \sqrt{z} \text{ on } U_1 \cap V_1. \tag{6.9}$$

Let $\sigma(M) = \{c_1, \ldots, c_k\}$ and choose $r \in \mathbb{R}^{+k}$ so that $\Delta(c, r) \subseteq U_1 \cap V_1$. It follows from equation (6.9) that

$$f_1, g_1 \text{ are branches of } \sqrt{z} \text{ on } \Delta(c, r). \tag{6.10}$$

But since $f(M) = g(M)$, $f_1 = g_1$ on $\sigma(M)$. Since each component of $\Delta(c, r)$ meets $\sigma(M)$, it follows from equation (6.10) that $f_1 = g_1$ on $\Delta(c, r)$. Therefore, by Proposition 5.13,

$$f(x) = f_1^\wedge(x) = g_1^\wedge(x) = g(x)$$

for all $x \in D(c, r)$. Since Proposition 5.1 guarantees that $D(c, r)$ is a free domain, the lemma follows by the choice of $W = U \cap V \cap D(c, r)$. $\qquad \square$

The following definition expresses the Riemann surface for $\sqrt{x}$ as a union of graphs of function elements. This approach follows Chapter 8 of [5] quite closely. An alternative approach, based on cross-sections of a sheaf of germs of free holomorphic functions over $\mathcal{I}$, is also possible. However, in the simple special case we are considering, this latter approach would amount to little more than a change in notation.

Definition 6.11. *If (f, U) is a free function element, let*

$$\text{graph}(f, U) = \{(x, f(x)) \mid x \in U\}.$$

Let S denote the collection of all branches (f, U) of $\sqrt{x}$ where U is a basic free open set B_δ in $\mathcal{I}$ for some matricial free polynomial δ. Define $\mathcal{R}$ by

$$\mathcal{R} = \bigcup_{(f, U) \in S} \text{graph}\,(f, U)$$

and define $\mathcal{B}$ by

$$\mathcal{B} = \{\text{graph}(f, U) \mid (f, U) \in S\}.$$

Lemma 6.12. *There exists a unique topology $\mathcal{T}$ on $\mathcal{R}$ such that $\mathcal{B}$ is a basis for $\mathcal{T}$.*

Proof. The statement is that, if $(f, U), (g, V) \in S$ and

$$(M, N) \in \text{graph}(f, U) \cap \text{graph}(g, V),$$

then there exists $(h, W) \in S$ such that

$$(M, N) \in \text{graph}(h, W) \subseteq \text{graph}(f, U) \cap \text{graph}(g, V).$$

This assertion follows immediately from Lemma 6.8. $\qquad \square$

Definition 6.13. *Let $\mathcal{R}$ be equipped with the topology $\mathcal{T}$ of Lemma 6.12. For $(f, U) \in S$, define*

$$\alpha_{(f, U)} : \text{graph}(f, U) \to U$$

by the formula

$$\alpha_{(f, U)}(x, f(x)) = x, \qquad x \in U.$$

Let

$$\mathcal{A} = \{\alpha_{(f, U)} \mid (f, U) \in S\}.$$

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Theorem 6.14. $(\mathcal{R}, \mathcal{T}, \mathcal{A})$ *is a free manifold.*

Proof. The theorem follows from the following four facts each of which is a simple consequence of the previous constructions:

1. For each $(f, U) \in S$, $\text{graph}(f, U) \in \mathcal{T}$,

2. For each $(f, U) \in S$, U is a free domain,

3. For each $(f, U) \in S$, $\alpha_{f,U} : \text{graph}(f, U) \to U$ is a homeomorphism from $\text{graph}(f, U)$ equipped with the $\mathcal{T}$ topology to U equipped with the free topology.

4. If $(f, U), (g, V) \in S$ and $\text{graph}(f, U) \cap \text{graph}(g, V) \neq \emptyset$, then $\alpha_{(g,V)} \circ \alpha_{(f,U)}^{-1}$ is a free holomorphic function.

1. holds as $\text{graph}(f, U) \in \mathcal{B}$ if $(f, U) \in S$.

According to Definition 6.11, if $(f, U) \in S$ then U is a free domain. So 2. holds.

To see 3., assume that V is a free domain in U. Then

$$\alpha_{(f,U)}^{-1}(V) = \text{graph}(f|V, V) \in \mathcal{B} \subseteq \mathcal{T}.$$

Conversely, if $\text{graph}(V, g)$ is a basic $\mathcal{T}$-open set in $\text{graph}(f, U)$, then, as $V \subseteq U$ and $g = f|V$,

$$\alpha_{(f,U)}(\text{graph}(g, V)) = V \text{ is a free domain.}$$

4. follows from the fact that if $x \in \alpha(\text{graph}(f, U) \cap \text{graph}(g, V))$ then $\alpha_{(g,V)} \circ \alpha_{(f,U)}^{-1}(x) = x$. $\qquad\square$

As in the classical case, the point of the Riemann surface for a multivalued function f on a domain D is that f can be regarded as a single-valued holomorphic function on the Riemann surface, which is a holomorphic manifold lying over D. The following statement makes this notion precise in the context of the matricial square root. Consider a point $w \in \mathcal{R}$: then $w \in \text{graph}(f, U)$ for some function element (f, U), which is to say that $w = (x, f(x))$ for some $x \in U$. We shall say that w *lies over* x.

Theorem 6.15. *There is a holomorphic function F on the free holomorphic manifold $(\mathcal{R}, \mathcal{T}, \mathcal{A})$ such that, if $w \in \mathcal{R}$ lies over $x \in \mathcal{I}$, then $F(w)$ is a square root of x.*

Proof. Define F to be the restriction to $\mathcal{R}$ of the co-ordinate projection $(M^1, M^2) \mapsto M^2$. It is easy to check that F is holomorphic on $\mathcal{R}$. If w lies over $x \in \mathcal{I}$ then $w = (x, f(x))$ for some branch f of the square root on a neighborhood of x; then $F(w) = f(x)$, which is a square root of x. $\qquad\square$

It is interesting to observe that, in contrast to the commutative case, where the Riemann surface for $\sqrt{z}$ lies over $\mathbb{C} \setminus \{0\}$ as a 2-sheeted surface, in the noncommutative case there is no bound on the number of sheets in $\mathcal{R}$ that lie over a given point $M \in \mathcal{I}$.

Proposition 6.16. *Let $M \in \mathcal{I}$ and let $\sigma(M)$ have k elements. There exist exactly 2^k points $N \in \mathbb{M}^1$ such that $(M, N) \in \mathcal{R}$.*

Proof. The proof uses the simple observation from linear algebra that a matrix M has exactly 2^k square roots that lie in $\mathrm{alg}(M)$, the algebra generated by M. Fix $M \in \mathcal{I}$ and assume that $|\sigma(M)| = k$.

If $(M, N) \in \mathcal{R}$, then there exists $(f, U) \in \mathcal{S}$ such that $(M, N) \in \mathrm{graph}(f, U)$. Then

$$N = f(M) = f_1^{\wedge}(M) \in \mathrm{alg}(M).$$

Hence, there exist at most 2^k matrices N such that $(M, N) \in \mathcal{R}$.

To see that there exist at least 2^k matrices N such that $(M, N) \in \mathcal{R}$, let $\sigma = \{c_1, \ldots, c_k\}$ and choose $r \in \mathbb{R}^{+k}$ so that $\Delta(c, r)$ has k components and $0 \notin \Delta(c, r)$. Each of the 2^k distinct choices of square roots $b_1, \ldots, b_k$ for the points $c_1, \ldots, c_k$ gives rise to a distinct holomorphic branch h_b of $\sqrt{z}$ on $\Delta(c, r)$ satisfying $h_b(c_i) = b_i$, $i = 1, \ldots, k$. In turn each of these distinct holomorphic branches of h_b gives rise to a distinct function element $(h_b^{\wedge}, \Delta(c, r)) \in S$. As $\sigma(h_b^{\wedge}(M)) = \{b_1, \ldots, b_k\}$, this proves that there are at least 2^k matrices N such that $(M, N) \in \mathcal{R}$. $\qquad\square$

Equally interesting is to observe that despite the phenomenon described in the preceding proposition, $\mathcal{R}$ is isomorphic to a free domain in $\mathbb{M}^1$.

Proposition 6.17. *The map $\sigma : \mathcal{Q} \to \mathcal{R}$ defined by the formula*

$$\sigma(y) = (y^2, y) \qquad for\ y \in \mathcal{Q}$$

is a free biholomorphism from $\mathcal{Q}$ onto $\mathcal{R}$.

Proof. Clearly, σ is injective and onto. If $(f, U) \in S$, then

$$\left(\alpha_{(f,U)} \circ \sigma\right)(y) = \alpha_{(f,U)}(y^2, y) = y^2$$

is a free holomorphic function defined on $\{y \mid (y^2, y) \in \operatorname{graph}(f, U)\}$ and

$$\left(\sigma^{-1} \circ \alpha_{(f,U)}^{-1}\right)(x) = \sigma^{-1}(x, f(x)) = f(x)$$

is a free holomorphic function defined on $\{x \mid x \in U\}$. Therefore σ is a free biholomorphic mapping. $\qquad\square$

7 The Zariski-free topology

We come now to a modification of the free topology on $\mathbb{M}^d$ that will be needed for the construction of the topological nc-manifold $\mathcal{G}$ with properties described in Theorems 8.33 and 8.35.

7.1 Thin sets

Recall that a set T in a domain $U \subseteq \mathbb{C}^d$ is said to be *thin* if at each point $z \in U$ there exists an open neighborhood V of z in U and a nonconstant holomorphic function f on V such that $f = 0$ on $V \cap T$. Simple facts are that thin sets are nowhere dense, closures of thin sets are thin sets and finite unions of thin sets are thin sets.

A more subtle property of thin sets will be fundamental in later sections: if T is a thin set in U then every holomorphic function f on $U \setminus T$ that is locally bounded on U has a unique holomorphic extension to all of U [20, Theorem I.3.4]. Here the boundedness hypothesis is defined as follows.

Definition 7.1. *Let U be a domain in $\mathbb{C}^d$ and let $T \subset \mathbb{C}^d$. A function f on $U \setminus T$ is said to be* locally bounded on U *if, for every point $z \in U$ there is a neighborhood V of z in U such that f is bounded on $V \setminus T$.*

It is false that every holomorphic locally bounded function on $U \setminus T$, where T is thin and U is a domain in $\mathbb{C}^d$, has a holomorphic extension to U. An easy counterexample is $U = \mathbb{C}, T = \{0\}, f(z) = z^{-1}$. It is essential that f be locally bounded *on U* for a holomorphic extension to exist.

7.2 The Zariski-free topology

Let D be a free domain. By a *free variety in D* we mean a set $V \subseteq D$ that has the form

$$V = \{x \in D \mid f(x) = 0 \text{ for all } f \in S\} \tag{7.2}$$

for some set S of freely holomorphic functions on D. Note that the zero symbol in equation (7.2) stands simultaneously for square zero matrices of all orders. To make matters precise we define

$$\mathbf{0} \overset{\text{def}}{=} \{0_1, 0_2, 0_3, \dots, \}$$

where 0_n denotes the zero matrix in $\mathbb{M}_n$. Thus the definition (7.2) can equally be written

$$V = \{x \in D \mid f(x) \in \mathbf{0} \text{ for all } f \in S\}.$$

We shall be loose about distinguishing $\mathbf{0}$ and 0.

The following statement illustrates just one of the many surprises that result from the free topology not being Hausdorff. Free varieties in a free domain D are not necessarily relatively closed in D in the free topology.

Proposition 7.3. *For any freely holomorphic function p on a free domain $D \subset \mathbb{M}^1$, the free closure of the free variety*

$$T = \{x \in D : p(x) = 0\}$$

in D is the set

$$\{x \in D \mid p(x) \text{ is singular}\}.$$

Proof. Let $x \in D \cap \mathbb{M}_n$ be such that $p(x)$ is singular. We claim that x is in the closure T^- of T.

Since $p(x)$ is singular, there is an eigenvalue λ of x such that $p(\lambda) = 0$. Let $u \in \mathbb{C}^n$ be a corresponding eigenvector. Consider any basic free neighborhood B_δ of x in D, where δ is an $m \times m$ matrix of polynomials in one variable; then $\delta(x) \in \mathbb{M}_{mn}$ and $\|\delta(x)\|_{\mathbb{M}_{mn}} < 1$. For any choice of $\zeta = (\zeta_1, \dots, \zeta_m) \in \mathbb{C}^m$,

$$\delta_{ij}(x)\zeta_j u = \delta_{ij}(\lambda)\zeta_j u \quad \text{for } i, j = 1, \dots, m.$$

Let $\zeta \otimes u$ denote the $nm \times 1$ matrix $[\zeta_1 u^T \dots \zeta_m u^T]^T$, where the superscript T denotes transposition. The last equation shows that

$$
\delta(x)(\zeta \otimes u) = \begin{bmatrix} \delta_{11}(x) & \dots & \delta_{1m}(x) \\ \cdot & \dots & \cdot \\ \delta_{m1}(x) & \dots & \delta_{mm}(x) \end{bmatrix} \begin{bmatrix} \zeta_1 u \\ \vdots \\ \zeta_m u \end{bmatrix}
$$

$$
= \begin{bmatrix} \delta_{11}(\lambda) & \dots & \delta_{1m}(\lambda) \\ \cdot & \dots & \cdot \\ \delta_{m1}(\lambda) & \dots & \delta_{mm}(\lambda) \end{bmatrix} \begin{bmatrix} \zeta_1 u \\ \vdots \\ \zeta_m u \end{bmatrix}
$$

$$
= (\delta(\lambda)\zeta) \otimes u.
$$

Take norms of both sides in $\mathbb{C}^{mn}$:

$$
\|\delta(x)(\zeta \otimes u)\|_{\mathbb{C}^{mn}} = \|(\delta(\lambda)\zeta) \otimes u\|_{\mathbb{C}^{mn}}
$$
$$
= \|\delta(\lambda)\zeta\|_{\mathbb{C}^m} \, \|u\|_{\mathbb{C}^n}. \tag{7.4}
$$

Choose ζ to be a unit maximizing vector for $\delta(\lambda)$, so that

$$
\|\zeta\|_{\mathbb{C}^m} = 1 \quad \text{and} \quad \|\delta(\lambda)\zeta\|_{\mathbb{C}^m} = \|\delta(\lambda)\|_{\mathbb{M}_m}. \tag{7.5}
$$

Combining equations (7.4) and (7.5), we find that

$$
\|\delta(x)(\zeta \otimes u)\|_{\mathbb{C}^{mn}} = \|\delta(\lambda)\|_{\mathbb{M}_m} \, \|u\|_{\mathbb{C}^n}.
$$

Since $\|\delta(x)\| < 1$,

$$
\|\delta(\lambda)\|_{\mathbb{M}_m} \, \|u\|_{\mathbb{C}^n} \le \|\delta(x)\|_{\mathbb{M}_{mn}} \, \|\zeta \otimes u\|_{\mathbb{C}^{mn}}
$$
$$
< \|\zeta\|_{\mathbb{C}^m} \, \|u\|_{\mathbb{C}^n}
$$
$$
= \|u\|_{\mathbb{C}^n},
$$

and hence

$$
\|\delta(\lambda)\|_{\mathbb{M}_m} < 1.
$$

That is, $\lambda \in G_\delta$. Since $p(\lambda) = 0$, $\lambda \in T$. We have shown that every basic free neighborhood of x in D meets T, and so x is in the free closure of T.

Conversely, suppose that $x \in D \cap \mathbb{M}_n$ and $p(x)$ is nonsingular. Then there is a basic free neighborhood B_δ of x in $\mathbb{M}^1$ that is disjoint from T. Indeed, we may choose δ to be the 1×1 polynomial $c(g \circ p)$ where g is the characteristic

polynomial of $p(x)$ and $c > 1/|\det p(x)|$. By the Cayley-Hamilton theorem, $\delta(x) = cg(p(x)) = 0$, so that $x \in B_\delta$. For any $m \in \mathbb{N}$ and $M \in T \cap \mathbb{M}_m$,

$$\delta(M) = cg(p(M)) = cg(0_m) = cg(0_1)1_m = c(\det p(x))1_m,$$

and therefore

$$\|\delta(M)\| = \|c(\det p(x))1_m\| = c|\det p(x)| > 1,$$

that is, $M \notin G_\delta$. Thus $D \cap B_\delta$ is a free neighborhood of x in D disjoint from T. Thus $x \notin T^-$. $\qquad\square$

Corollary 7.6. *If p is a nonconstant free polynomial in one variable then the free variety $p^{-1}(\mathbf{0})$ is not freely closed in $\mathbb{M}^1$.*

For we can easily write down a 2×2 diagonal matrix M such that $p(M)$ is singular but not 0_2.

Corollary 7.7. *For any free polynomial p in d variables, the free closure of the free variety*

$$T = \{x \in \mathbb{M}^d : p(x) = 0\}$$

in $\mathbb{M}^d$ is contained in the set

$$\{x \in \mathbb{M}^d \,|\, p(x) \text{ is singular}\}.$$

Proof. Let Sing denote the set of singular matrices in $\mathbb{M}^1$. By Proposition 7.3 and the free continuity of p, $p^{-1}(\text{Sing})$ is a freely closed set in $\mathbb{M}^d$. It clearly contains the set $T = p^{-1}(\mathbf{0})$. Hence it contains the free closure of T. $\qquad\square$

Many of the important results about varieties in commutative analysis depend critically on the fact that varieties are relatively closed. Accordingly the following modification of the free topology is natural and fruitful in the nc context, as we shall see in Subsection 8.5.

Definition 7.8. *A basic Zariski-free set in $\mathbb{M}^d$ is a set B that has the form*

$$B = D \setminus T$$

where D is a free domain in $\mathbb{M}^d$ and

$$T = \bigcup_{i=1}^{k} V_i$$

is a finite union of free varieties in D.

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In the definition, the set T is taken to be a finite union of free varieties. This is because, in contrast to the commutative case, a finite union of free varieties is not in general a free variety.

Proposition 7.9. *The collection of basic Zariski-free sets in $\mathbb{M}^d$ constitutes a base for a topology on $\mathbb{M}^d$.*

Proof. Clearly the collection covers $\mathbb{M}^d$. Consider any pair $D_j \setminus T_j$, $j = 1, 2$, of basic Zariski-free sets in $\mathbb{M}^d$. Observe that, for any freely open subset D of D_j, the intersection $D \cap T_j$ is a finite union of free varieties in D, since the restriction to D of a freely holomorphic function on D_j is freely holomorphic on D.

The set-theoretic identity

$$(D_1 \setminus T_1) \cap (D_2 \setminus T_2) = (D_1 \cap D_2) \setminus ((T_1 \cup T_2) \cap D_1 \cap D_2), \qquad (7.10)$$

thus implies that the collection of basic Zariski-free sets in $\mathbb{M}^d$ is closed with respect to finite intersections and hence forms a base for a topology. $\qquad \square$

Definition 7.11. *The Zariski-free topology on $\mathbb{M}^d$ is the topology that has as a base the collection of basic Zariski-free sets. A set that is open in the Zariski-free topology is a Zariski-free domain.*

Note that there is also a smaller base for the Zariski-free topology, consisting of the sets $B_\delta \setminus T$ where δ ranges over all matricial free polynomials and T ranges over all finite unions of free varieties in B_δ. This base has the additional feature that it consists of nc-sets. Consequently the topology is an nc topology. It is clearly coarser than the finitely open topology. We deduce, in the terminology of Definition 2.5:

Proposition 7.12. *The Zariski-free topology is an admissible topology on $\mathbb{M}^d$.*

An empty union is conventionally taken to be the empty set. Thus every freely open set in $\mathbb{M}^d$ is also Zariski-freely open.

Proposition 7.13. *The Zariski-free topology is finer than the free topology on $\mathbb{M}^d$.*

Any free variety, like a free open set, is preserved by unitary conjugations. It follows that Zariski-free open sets are also preserved by unitary conjugations.

We now explore some important relationships between free and Zariski-free domains and holomorphic functions.

Proposition 7.14. *If D_1 and D_2 are free domains and $f : D_1 \to D_2$ is a free holomorphic mapping then f is continuous in the Zariski-free topology.*

Proof. Let $D_1 \subset \mathbb{M}^{d_1}$, $D_2 \subset \mathbb{M}^{d_2}$. Fix a basic Zariski-free set $B = D \setminus T \subseteq \mathbb{M}^{d_2}$, where $D \subseteq D_2$ is a free domain and T is a finite union of free varieties in D. Suppose that, as in Definition 7.8, $T = V_1 \cup \cdots \cup V_k$ where $V_i = \{x \in D \mid g(x) = 0 \text{ for all } g \in S_i\}$ and S_i is a set of free holomorphic functions on D. Then

$$f^{-1}(V_i) = \{x \in f^{-1}(D) \mid g(f(x)) = 0 \text{ for all } g \in S_i\}.$$

Therefore, if we let $S_i^{\sim} = \{g \circ f \mid g \in S_i\}$, then

$$f^{-1}(V_i) = \{x \in f^{-1}(D) \mid h(x) = 0 \text{ for all } h \in S_i^{\sim}\}$$

is a free variety in $f^{-1}(D)$. Consequently

$$f^{-1}(T) = \bigcup_{i=1}^{k} \{x \in f^{-1}(D) \mid h(x) = 0 \text{ for all } h \in S_i^{\sim}\}$$

is a finite union of free varieties in $f^{-1}(D)$. The set

$$f^{-1}(B) = f^{-1}(D) \setminus f^{-1}(T)$$

is therefore Zariski-freely open. $\qquad\square$

Lemma 7.15. *Let φ be a Zariski-freely holomorphic function on $D \setminus T$, where D is a freely open nc set, T is a finite union of free varieties in D, and $D \setminus T$ is non-empty. Then φ extends to a unique freely holomorphic nc function $\hat{\varphi}$ on D.*

Proof. Let $z \in D \cap T$. Choose $w \in D \setminus T$. Then $z \oplus w \in D \setminus T$, and so $\varphi(z \oplus w)$ is defined. For each component of the d-tuple $z \oplus w$

$$\begin{bmatrix} 1 & 0 \end{bmatrix} z^j \oplus w^j = z^j \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Hence, by the intertwining property (2.4) of nc functions,

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \varphi(z \oplus w) = \varphi(z) \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Similarly, for each j,

$$z^j \oplus w^j \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} w^j$$

and therefore
$$\varphi(z \oplus w) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \varphi(w).$$

Hence
$$\varphi(z \oplus w) = \begin{bmatrix} a & 0 \\ 0 & \varphi(w) \end{bmatrix} \tag{7.16}$$

for some matrix a of the same type as the z^j. Likewise, if w_1 and w_2 are distinct elements of $D \setminus T$, then since
$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & w_1 \end{bmatrix} = \begin{bmatrix} z & 0 \\ 0 & w_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

the intertwining property (2.4) tells us that the value of a in equation (7.16) does not depend on the choice of w in $D \setminus T$. Hence we may define a graded function $\hat{\varphi}$ on $D \cap T$ by the relation
$$\varphi(z \oplus w) = \begin{bmatrix} \hat{\varphi}(z) & 0 \\ 0 & \varphi(w) \end{bmatrix} \tag{7.17}$$

for all $z \in D \cap T$ and $w \in D \setminus T$.

Extend $\hat{\varphi}$ to D by defining $\hat{\varphi}$ to agree with φ on $D \setminus T$. We claim that $\hat{\varphi}$ is a freely holomorphic extension of φ from $D \setminus T$ to D. That $\hat{\varphi}(z_1 \oplus z_2) = \hat{\varphi}(z_1) \oplus \hat{\varphi}(z_2)$ is immediate if both z_1 and z_2 are in $D \setminus T$, since φ is an nc function by Proposition 3.4. It is also immediate if just one of z_1, z_2 is in $D \setminus T$, by choice of the other as w. Assume therefore that both z_1 and z_2 are in $D \cap T$, and choose any $w \in D \setminus T$. The relation
$$\varphi \left(\begin{bmatrix} z_1 \oplus z_2 & 0 \\ 0 & w \end{bmatrix} \right) = \begin{bmatrix} \hat{\varphi}(z_1 \oplus z_2) & 0 \\ 0 & \varphi(w) \end{bmatrix}, \tag{7.18}$$

is true by the definition of $\hat{\varphi}$ if $z_1 \oplus z_2 \in T$, and by the fact that $\hat{\varphi}$ extends φ together with the nc property of φ if $z_1 \oplus z_2 \notin T$.

Since $z_2 \oplus w \in D \setminus T$,
$$\begin{aligned} \varphi \left(\begin{bmatrix} z_1 \oplus z_2 & 0 \\ 0 & w \end{bmatrix} \right) &= \varphi \left(\begin{bmatrix} z_1 & 0 \\ 0 & z_2 \oplus w \end{bmatrix} \right) \\ &= \begin{bmatrix} \hat{\varphi}(z_1) & 0 \\ 0 & \varphi(z_2 \oplus w) \end{bmatrix} \\ &= \begin{bmatrix} \hat{\varphi}(z_1) & 0 & 0 \\ 0 & \hat{\varphi}(z_2) & 0 \\ 0 & 0 & \varphi(w) \end{bmatrix}. \end{aligned} \tag{7.19}$$

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Comparison of equations (7.18) and (7.19) reveals that

$$\hat{\varphi}(z_1 \oplus z_2) = \hat{\varphi}(z_1) \oplus \hat{\varphi}(z_2).$$

To see that $\hat{\varphi}$ preserves similarity, consider $x \in D$ and $s \in \mathcal{I}$ such that $s^{-1}xs \in D$. If $x \notin T$ then also $s^{-1}xs \in D \setminus T$, and so

$$\hat{\varphi}(s^{-1}zs) = \varphi(s^{-1}zs) = s^{-1}\varphi(z)s = s^{-1}\hat{\varphi}(z)s.$$

Now suppose that $x \in D \cap T$; then $s^{-1}xs \in D \cap T$. Choose any $w \in D \setminus T$. We have

$$
\begin{aligned}
\hat{\varphi}(s^{-1}xs) \oplus \varphi(w) &= \varphi(s^{-1}xs \oplus w) && \text{by definition of}\hat{\varphi} \\
&= \varphi\left((s \oplus 1)^{-1}(x \oplus w)(s \oplus 1)\right) \\
&= (s \oplus 1)^{-1}\varphi(x \oplus w)(s \oplus 1) && \text{since } \varphi \text{ is nc} \\
&= (s \oplus 1)^{-1}(\hat{\varphi}(x) \oplus \varphi(w))(s \oplus 1) \\
&= s^{-1}\hat{\varphi}(x)s \oplus \varphi(w).
\end{aligned}
$$

Hence $\hat{\varphi}(s^{-1}xs) = s^{-1}\hat{\varphi}(x)s$.

It remains to show that $\hat{\varphi}$ is freely locally bounded, hence freely holomorphic. Let z be any point in D, and choose $w \in D \setminus T$. There is a Zariski-free basic neighborhood $B_\delta \setminus S$ containing $z \oplus w$ on which φ is bounded, by M say. For every $x \in B_\delta$,

$$\varphi(x \oplus z \oplus w) = \hat{\varphi}(x) \oplus \hat{\varphi}(z) \oplus \varphi(w)$$

is bounded by M, so $\hat{\varphi}$ is bounded on the free neighborhood B_δ of z.

To see that the extension is unique, it is sufficient to prove that if ψ is a free nc holomorphic function on D that vanishes on $D \setminus T$, then it is identically zero. Suppose not. Then there is some point $z \in T \cap D$ such that $\psi(z) \neq 0$. There is some point $w \in D \setminus T$. Then $z \oplus w$ is in $D \setminus T$, but $\psi(z \oplus w) = \psi(z) \oplus \psi(w) \neq 0$, a contradiction. $\square$

Lemma 7.20. *Let $\varphi : B_\delta \setminus T \to B_\gamma \setminus S$ be a Zariski-freely holomorphic nc map. Then φ extends to a unique freely holomorphic map $\hat{\varphi} : B_\delta \to B_\gamma$. Moreover, if φ is a Zariski-free homeomorphism, then $\hat{\varphi}$ is a free homeomorphism.*

Proof. Since B_γ is an nc set, we may apply Lemma 7.15 to each component of φ in turn to obtain a freely holomorphic nc map $\hat{\varphi}$ on B_γ. We must show

that the range of $\hat{\varphi}$ is contained in B_γ. This is automatic for points in $B_\delta \setminus T$, so consider a point $x \in T \cap B_\delta$. Suppose $\|\gamma(\hat{\varphi}(x))\| \geq 1$. Then there are unit vectors u, v such that

$$|\langle \gamma(\hat{\varphi}(x))u, v \rangle| \geq 1. \tag{7.21}$$

Let $w \in B_\delta \setminus T$. Then

$$|\langle \gamma(\hat{\varphi}(x))u, v \rangle| = |\langle \gamma(\varphi(x \oplus w))u \oplus 0, v \oplus 0 \rangle| < 1,$$

a contradiction.

The uniqueness of $\hat{\varphi}$ follows by Lemma 7.15.

Suppose $\psi : B_\gamma \setminus S \to B_\delta \setminus T$ is the inverse of φ. Let $w \in B_\delta \setminus T$. Then for all $z \in B_\delta$, we have $z \oplus w \in B_\delta \setminus T$, and

$$\begin{bmatrix} z & 0 \\ 0 & w \end{bmatrix} = \psi \circ \varphi \left(\begin{bmatrix} z & 0 \\ 0 & w \end{bmatrix} \right)$$

$$= \psi \left(\begin{bmatrix} \hat{\varphi}(z) & 0 \\ 0 & \varphi(w) \end{bmatrix} \right)$$

$$= \begin{bmatrix} \hat{\psi} \circ \hat{\varphi}(z) & 0 \\ 0 & \psi \circ \varphi(w) \end{bmatrix}.$$

Thus $\hat{\psi}$ is the inverse of $\hat{\varphi}$. $\qquad\qquad\square$

Lemma 7.22. *Let φ be a Zariski-freely holomorphic function on a Zariski-free open set $U \subseteq \mathbb{M}^d$. Suppose that U can be written as*

$$U = \bigcup_{\alpha \in A} (B_{\delta_\alpha} \setminus T_\alpha),$$

where T_α is a finite union of free varieties in B_{δ_α} such that
   (i) *whenever $B_{\delta_\alpha} \cap B_{\delta_\beta}$ is non-empty, so is $(B_{\delta_\alpha} \setminus T_\alpha) \cap (B_{\delta_\beta} \setminus T_\beta)$;*
   (ii) *the function φ is bounded and nc on each set $B_{\delta_\alpha} \setminus T_\alpha$ for $\alpha \in A$.*
   Then φ extends to a unique freely holomorphic function $\hat{\varphi}$ on

$$\hat{U} := \bigcup_{\alpha \in A} B_{\delta_\alpha}.$$

Proof. By Lemma 7.15, φ can be extended to a unique free function on each B_{δ_α}. To see that these extensions coincide on each $B_{\delta_\alpha} \cap B_{\delta_\beta}$, we observe that, by condition (i), if this intersection is non-empty, then there is a point w in $(B_{\delta_\alpha} \setminus T_\alpha) \cap (B_{\delta_\beta} \setminus T_\beta)$. We can use this point w in equation (7.17) to construct the extension of φ to both B_{δ_α} and B_{δ_β}, and so the extension will agree on the intersection. $\qquad\square$

The next result shows how close Zariski-free holomorphy is to free holomorphy.

Proposition 7.23. *A graded function on a free domain in $\mathbb{M}^d$ is freely holomorphic if and only if it is Zariski-freely holomorphic.*

Proof. Let U be a free domain in $\mathbb{M}^d$ and let $\varphi : U \to \mathbb{M}^1$ be a graded function. Suppose that φ is freely holomorphic – that is, it is freely locally nc and freely locally bounded on U. Since the Zariski-free topology is finer than the free topology, it is immediate that φ is Zariski-freely locally nc and Zariski-freely locally bounded. Thus φ is Zariski-freely holomorphic.

Conversely, suppose that φ is Zariski-freely holomorphic on U. Let $x \in U$. There is a Zariski-free open set $D \setminus T$ containing x on which φ is nc and bounded. Therefore there exists B_δ such that $B_\delta \setminus T$ contains x and φ is nc and bounded on $B_\delta \setminus T$. By Lemma 7.15, there is a bounded nc extension $\hat{\varphi}$ of $\varphi|_{B_\delta \setminus T}$ to B_δ. If $y \in B_\delta \cap T$, then

$$
\begin{array}{ccc}
\hat{\varphi}(x \oplus y) & = & \varphi(x \oplus y) \\
\shortparallel & & \shortparallel \\
\hat{\varphi}(x) \oplus \hat{\varphi}(y) & = & \varphi(x) \oplus \varphi(y).
\end{array}
$$

So $\hat{\varphi}$ agrees with φ on all of B_δ, including T. Therefore φ is freely locally bounded. $\qquad\square$

8 An nc-manifold for symmetrization

In this section we construct a two-dimensional topological nc-manifold $\mathcal{G}$ such that the algebra of holomorphic functions on $\mathcal{G}$ is canonically isomorphic (in a sense to be made precise in Theorem 8.35 below) to the algebra of symmetric free holomorphic functions in $\mathbb{M}^2$.

In the commutative case, if $\Omega \subset \mathbb{C}^2$ is a symmetric domain (that is, if $(z, w) \in \Omega$ implies that $(w, z) \in \Omega$) then the domain $\tilde{\Omega} \stackrel{\text{def}}{=} \{(z + w, zw) : (z, w) \in \Omega\}$ has the property that the symmetric holomorphic functions f on Ω are in bijective correspondence with the holomorphic functions F on $\tilde{\Omega}$ via the relation

$$
f(z, w) = F(z + w, zw) \quad \text{for all } (z, w) \in \Omega.
$$

We shall construct a topological nc-manifold $\mathcal{G}$ with properties analogous to those of $\tilde{\Omega}$ for the symmetric free domain $\Omega = \mathbb{M}^2$.

In [2] the algebra of symmetric free holomorphic functions on the biball B^2, the noncommutative analog of the bidisc, is analysed by means of operator-theoretic methods. The authors constructed a set $\Omega \subset \mathbb{M}^\infty$ and a map $\pi : B^2 \to \Omega$ such that every symmetric free holomorphic function φ on B^2 can be expressed as $\Phi \circ \pi$ for some holomorphic nc-function Φ on Ω. Here Ω is an infinite-dimensional set, but it can nevertheless be described in terms of only three noncommuting variables, provided that inverses and square roots are allowed. In this paper we adopt a more topological approach and obtain a 2-*dimensional* topological nc-manifold with properties analogous to Ω, but for symmetric functions on an arbitrary symmetric free domain. The theme that three variables suffice is reflected in the fact that the manifold we obtain in this section is presented as a subset of $\mathbb{M}^3$. However the topology of the manifold structure of $\mathcal{G}$ is not that induced by any natural topology of $\mathbb{M}^3$.

8.1   A geometric lemma

In this and the next two subsections we describe some simple combinatorial geometry and properties of free square roots, which play an essential part in the construction of $\mathcal{G}$.

Throughout Section 8 the symbol Δ will be reserved for sets in the plane that have the form

$$\Delta = \bigcup_{i=1}^{k}(c_i + r\mathbb{D}).$$

Alternatively, if γ denotes the finite set $\{c_1, c_2, \ldots c_k\}$,

$$\Delta = \gamma + r\mathbb{D}.$$

We refer to such sets as *simple sets with radius r*. We define the *separation* of a simple set $\Delta = \gamma + r\mathbb{D}$, denoted by $\operatorname{sep}\Delta$, by

$$\operatorname{sep}\Delta = \min\{|c - d| : c, d \in \gamma, c \neq d\}.$$

Definition 8.1. *A simple set Δ with radius r is t-isolated if $r < t \operatorname{sep}\Delta$. If Δ_1 and Δ_2 are simple sets, then Δ_2 is* subordinate *to Δ_1 if each disc in Δ_2 meets at most one disc in Δ_1.*

Lemma 8.2. *Let Δ_1 and Δ_2 be simple sets with radii r_1 and r_2 respectively. If Δ_1 and Δ_2 are $\frac{1}{4}$-isolated, then either Δ_1 is subordinate to Δ_2 or Δ_2 is subordinate to Δ_1.*

Proof. Suppose not, so that neither Δ_j is subordinate to the other. Since Δ_1 is not subordinate to Δ_2 there is a disc $c + r_1\mathbb{D}$ in Δ_1 that meets two discs $d_j + r_2\mathbb{D}$, $j = 1, 2$ in Δ_2, with $d_1 \neq d_2$. Pick points ζ_1, ζ_2 such that

$$\zeta_j \in (c + r_1\mathbb{D}) \cap (d_j + r_2\mathbb{D}), \qquad j = 1, 2.$$

Then

$$|\zeta_j - c| < r_1 \quad \text{and} \quad |\zeta_j - d_j| < r_2.$$

Since Δ_1, Δ_2 are $\frac{1}{4}$-isolated,

$$r_1 < \tfrac{1}{4}\operatorname{sep}\Delta_1 \quad \text{and} \quad r_2 < \tfrac{1}{4}\operatorname{sep}\Delta_2,$$

and so

$$|\zeta_j - c| < \tfrac{1}{4}\operatorname{sep}\Delta_1 \quad \text{and} \quad |\zeta_j - d_j| < \tfrac{1}{4}\operatorname{sep}\Delta_2.$$

Therefore,

$$|c - d_j| < \tfrac{1}{4}\operatorname{sep}\Delta_1 + \tfrac{1}{4}\operatorname{sep}\Delta_2, \qquad j = 1, 2.$$

In consequence,

$$
\begin{aligned}
\operatorname{sep}\Delta_2 &\leq |d_1 - d_2| \\
&\leq |c - d_1| + |c - d_2| \\
&< \tfrac{1}{2}\operatorname{sep}\Delta_1 + \tfrac{1}{2}\operatorname{sep}\Delta_2.
\end{aligned}
$$

Hence,

$$\operatorname{sep}\Delta_2 < \operatorname{sep}\Delta_1. \tag{8.3}$$

Since also Δ_2 is not subordinate to Δ_1, by repeating the above argument with Δ_1 and Δ_2 swapped, we deduce that $\operatorname{sep}\Delta_1 < \operatorname{sep}\Delta_2$, contradicting inequality (8.3). $\qquad\square$

8.2 Holomorphic square roots on the sets Δ_γ

Let Γ denote the set of finite subsets of $\mathbb{C} \setminus \{0\}$. For each $\gamma \in \Gamma$ we fix throughout the remainder of the section a simple set

$$\Delta_\gamma = \bigcup_{c \in \gamma} (c + r_\gamma \mathbb{D})$$

where r_γ is chosen so that

$$r_\gamma < \min \left\{ \min_{c \in \gamma} |c|, \tfrac{1}{4} \min_{\substack{c,d \in \gamma \\ c \neq d}} |c - d| \right\}.$$

This choice of r_γ guarantees that, for all $\gamma \in \Gamma$,

$$0 \notin \Delta_\gamma \tag{8.4}$$

and

$$\Delta_\gamma \text{ is } \tfrac{1}{4}\text{-isolated.} \tag{8.5}$$

Notice that the statement (8.5) implies that Δ_γ is a finite union of open discs whose closures are pairwise disjoint.

For $\gamma \in \Gamma$ let $|\gamma|$ denote the cardinality of γ. Define $2^{|\gamma|}$ functions on Δ_γ in the following way. For each $\tau \in \{-1, 1\}^\gamma$, define $\iota_{\gamma\tau} : \Delta_\gamma \to \{-1, 1\}$ by the formula

$$\iota_{\gamma\tau}(z) = \tau(c) \text{ if } z \in c + r_\gamma \mathbb{D}.$$

The functions $\iota_{\gamma\tau}$ are holomorphic on Δ_γ, indeed, they precisely consist of the $2^{|\gamma|}$ holomorphic square roots of the constant function 1 on Δ_γ.

For each $\gamma \in \Gamma$, property (8.4) and the remark following statement (8.5) imply that there exists a branch of $\sqrt{z}$ defined on Δ_γ.

Convention 8.6. *For every $\gamma \in \Gamma$, fix a holomorphic branch s_γ of the square root function on Δ_γ.*

Once this s_γ is chosen, the other branches of $\sqrt{z}$ on Δ_γ can be described in terms of s_γ with the aid of the functions $\iota_{\gamma\tau}$; we define $s_{\gamma\tau}$ to be the holomorphic square root function $s_\gamma \iota_{\gamma\tau}$ on Δ_γ.

Lemma 8.7. *Assume $\gamma \in \Gamma$. A function f on Δ_γ is a holomorphic function satisfying $f(z)^2 = z$ for all $z \in \Delta_\gamma$ if and only if there exists $\tau \in \{-1, 1\}^\gamma$ such that $f = s_{\gamma\tau}$.*

8.3 Free square roots on the sets D_γ

For each $\gamma \in \Gamma$ define $D_\gamma \subseteq \mathbb{M}^1$ by

$$D_\gamma = \{x \in \mathbb{M}^1 \,|\, \sigma(x) \subseteq \Delta_\gamma\}.$$

Lemma 8.8.

$$\bigcup_{\gamma \in \Gamma} D_\gamma = \mathcal{I}$$

Proof. If $M \in D_\gamma$, then by definition, $\sigma(M) \subseteq \Delta_\gamma$. Hence property (8.4) implies that $0 \notin \sigma(M)$, that is, $M \in \mathcal{I}$. Conversely, if $M \in \mathcal{I}$, then $0 \notin \sigma(M)$. This implies that $\sigma(M) \in \Gamma$. Hence $M \in D_{\sigma(M)} \subseteq \cup_{\gamma \in \Gamma} D_\gamma$. $\square$

With the help of the functions s_γ and $\iota_{\gamma\tau}$ we may define free holomorphic functions on D_γ by means of the Riesz functional calculus. Since $x \in D_\gamma$ implies that $\sigma(x) \subseteq \Delta_\gamma$, we may use the formula (5.2) to define, for every $\gamma \in \Gamma$, every $\tau \in \{-1, 1\}^\gamma$ and every $x \in D_\gamma$,

$$S_\gamma(x) = s_\gamma^\wedge(x), \tag{8.9}$$

$$I_{\gamma\tau}(x) = \iota_{\gamma\tau}^\wedge(x), \tag{8.10}$$

$$S_{\gamma\tau}(x) = S_\gamma(x) I_{\gamma\tau}(x). \tag{8.11}$$

Since $s_\gamma(z)^2 = z$ and $\iota_{\gamma\tau}(z)^2 = 1$ for $z \in \Delta_\gamma$,

$$S_{\gamma\tau}(x)^2 = x \quad \text{and} \quad I_{\gamma\tau}(x)^2 = 1 \quad \text{for all } \gamma \in \Gamma \text{ and } x \in D_\gamma.$$

Notice that, by Proposition 5.3, $S_{\gamma\tau}$ and $I_{\gamma\tau}$ are free holomorphic functions on the free domain D_γ. Moreover, *every* square root of $x \in D_\gamma$ in the algebra generated by x has the form $S_{\gamma\tau}(x)$ for some $\tau \in \{-1, 1\}^\gamma$.

8.4 The Zariski-free domain $\mathcal{U}_\gamma$

For each $\gamma \in \Gamma$ we shall define a Zariski-free open set in $\mathbb{M}^2$ having a certain genericity property. Let

$$\mathcal{W}_\gamma = \{(u, x) \in \mathbb{M}^2 \,|\, x \in D_\gamma\}. \tag{8.12}$$

As D_γ is a free domain, so is $\mathcal{W}_\gamma$. Furthermore, for fixed $\gamma \in \Gamma$ and $\tau \in \{-1, 1\}^\gamma$, as $I_{\gamma\tau}$ is a free holomorphic function,

$$f_{\gamma\tau}(u, x) = u \, I_{\gamma\tau}(x) - I_{\gamma\tau}(x) \, u, \qquad (u, x) \in \mathcal{W}_\gamma,$$

defines a free holomorphic function on $\mathcal{W}_\gamma$. It follows that

$$V_{\gamma\tau} = \{(u, x) \in \mathcal{W}_\gamma \mid u\, I_{\gamma\tau}(x) = I_{\gamma\tau}(x)\, u\}$$

is a free variety in $\mathcal{W}_\gamma$.

In the constant cases, where either $\tau = 1$ (that is, $\tau(c) = 1$ for all $c \in \gamma$) or $\tau = -1$ (that is, $\tau(c) = -1$ for all $c \in \gamma$), we have either $I_{\gamma\tau} = 1$ or $I_{\gamma\tau} = -1$. Thus, in these two cases where τ is constant, $V_{\gamma\tau}$ is all of $\mathcal{W}_\gamma$. In the sequel, we express the condition that τ is not constant by writing $\tau \neq \pm 1$.

Since, for each $\tau \in \{-1, 1\}^\gamma$, $V_{\gamma\tau}$ is a free variety in $\mathcal{W}_\gamma$,

$$\mathcal{U}_\gamma = \mathcal{W}_\gamma \setminus \bigcup_{\substack{\tau \in \{-1,1\}^\gamma \\ \tau \neq \pm 1}} V_{\gamma\tau} \tag{8.13}$$

defines a set that is open in the Zariski-free topology of $\mathbb{M}^2$, though not, generally, in the free topology, since the varieties $V_{\gamma\tau}$ are typically not freely closed. We shall construct the Zariski-free manifold $\mathcal{G}$ by gluing together sheets in $\mathbb{M}^3$ that lie over the domains $\mathcal{U}_\gamma \subset \mathbb{M}^2$. We can express the definition of $\mathcal{U}_\gamma$ as follows:

$$\mathcal{U}_\gamma = \{(u, x) \in \mathbb{M}^2 : \ x \in D_\gamma \text{ and } u \text{ does not commute with } I_{\gamma\tau}(x)$$
$$\text{for any nonconstant } \tau \in \{-1, 1\}^\gamma\}. \tag{8.14}$$

The following statement is easy to see.

Proposition 8.15. *For any $\gamma \in \Gamma$, if $(u_1, x_1) \in \mathcal{U}_\gamma$ and $(u_2, x_2) \in \mathcal{W}_\gamma$ then $(u_1, x_1) \oplus (u_2, x_2) \in \mathcal{U}_\gamma$.*

8.5   The definition of $\mathcal{G}$

To motivate the ensuing definition of the set $\mathcal{G}$ we recount some ideas from [2]. It has long been known [25] that the algebra A of symmetric complex polynomials in two noncommuting variables w^1 and w^2 is not finitely generated. However, it was found in [2] that there is a close substitute for a finite basis. If we define

$$u = \tfrac{1}{2}(w^1 + w^2), \qquad v = \tfrac{1}{2}(w^1 - w^2)$$

then an algebraic basis of A is

$$u, \ v^2, \ vuv, \ vu^2v, \ldots, vu^jv, \ldots \tag{8.16}$$

Although this is an infinite basis of A, the relation

$$vu^jv = (vuv)(v^2)^{-1}(vuv)(v^2)^{-1}\ldots(v^2)^{-1}(vuv)$$

shows that every symmetric free polynomial in w^1, w^2 can be written as a *rational* expression in terms of the first three terms of the basis (8.16). Accordingly, it appeared that these three basic polynomials might have the potential to play the role that the elementary symmetric functions $w^1 + w^2$ and $w^1 w^2$ play in the scalar theory, though some extra complications result from the fact that rational expressions are required to represent polynomials.

Thus, the underlying idea is to study the image of $\mathbb{M}^2$ under the map (u, v^2, vuv). Here, we realise this approach. We write x for the variable v^2 (so that v is a square root of x), and then invoke the structure theory for free holomorphic square roots developed in Section 6. Our strategy for the construction of a topological nc-manifold with the desired 'universal symmetrization' property will be to apply the maps $(u, x, \sqrt{x}u\sqrt{x})$ to $\mathbb{M}^2$. The expectation is that the many branches of the square root lead to co-ordinate patches on the image set that can be pieced together to yield the required topological nc-manifold. Implementing this strategy runs into difficulties caused by singular behavior on certain subvarieties. The notion of the Zariski-free topology enables us to circumvent these difficulties – see Remark 8.28 below.

Let us first establish that the polynomials (8.16) do indeed constitute an algebraic basis for the algebra of free polynomials. We do not know whether this is a new observation. A closely related result of an analytic flavor is [2, Theorem 5.1].

Theorem 8.17. *Let x, y be non-commuting indeterminates and let*

$$u = \tfrac{1}{2}(x + y), \qquad v = \tfrac{1}{2}(x - y).$$

For any positive intger d, every free polynomial of total degree d in x, y can be written as a polynomial in the d elements $u, v^2, vuv, \ldots, vu^{d-2}v$.

Proof. It suffices to prove the result for *homogeneous* free polynomials. For $d \geq 1$ let P_d be the complex vector space of homogeneous polynomials of total degree d in x, y and let Sym_d be its subspace of symmetric polynomials. Clearly P_d has dimension 2^d, and therefore Sym_d has dimension 2^{d-1}.

Let Q_d be the space of polynomials in $u, v^2, vuv, \ldots, vu^{d-2}v$ that are homogeneous of degree d in u, v, and hence also in x, y. Then $Q_1 = \mathbb{C}u$ and

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$Q_d \subseteq \text{Sym}_d$. We claim that $\dim Q_d = 2^{d-1} = \dim \text{Sym}_d$, from which it will follow that $Q_d = \text{Sym}_d$, as required.

The claim is true when $d = 1$ since $\dim Q_1 = 1$. Let $d > 1$ and suppose the claim holds for $d - 1$. We have

$$Q_d = uQ_{d-1} \oplus v^2 Q_{d-2} \oplus vuvQ_{d-3} \oplus \cdots \oplus vu^{d-3}vQ_1 \oplus vu^{d-2}v\mathbb{C}.$$

By the inductive hypothesis,

$$\begin{aligned}
\dim Q_d &= \dim Q_{d-1} + \dim Q_{d-2} + \cdots + \dim Q_1 + 1 \\
&= 2^{d-2} + 2^{d-3} + \cdots + 1 + 1 \\
&= 2^{d-1}.
\end{aligned}$$

Hence $Q_d = \text{Sym}_d$ and the theorem follows. $\qquad\square$

For each $\gamma \in \Gamma$ and each $\tau \in \{-1, 1\}^\gamma$ we define a mapping $\Phi_{\gamma\tau} : \mathcal{U}_\gamma \to \mathbb{M}^3$ by the formula

$$\Phi_{\gamma\tau}(u, x) = (u, x, S_{\gamma\tau}(x) \ u \ S_{\gamma\tau}(x)), \qquad (u, x) \in \mathcal{U}_\gamma, \qquad (8.18)$$

where $S_{\gamma\tau}$ is the free holomorphic square root on D_γ defined in equation (8.11).

Let

$$\mathcal{G}_{\gamma\tau} = \text{ran}\, \Phi_{\gamma\tau}. \qquad (8.19)$$

Trivially, $\Phi_{\gamma\tau}$ is a bijection from $\mathcal{U}_\gamma$ to $\mathcal{G}_{\gamma\tau}$. Define $\mathcal{G} \subseteq \mathbb{M}^3$ by

$$\mathcal{G} = \bigcup_{\gamma \in \Gamma} \ \bigcup_{\tau \in \{-1,1\}^\gamma} \mathcal{G}_{\gamma\tau}. \qquad (8.20)$$

8.6   A Zariski-free atlas for $\mathcal{G}$

The set $\mathcal{G}$ defined in the previous subsection is just that, a set. It carries neither a topology nor an atlas of charts that would endow it with a manifold structure. In this section we shall topologize $\mathcal{G}$ and equip it with a Zariski-free atlas.

Definition 8.21. *Let*

$$\mathcal{B} = \{\Phi_{\gamma\tau}(U) \mid \gamma \in \Gamma, \tau \in \{-1, 1\}^\gamma, U \text{ is Zariski-freely open in } \mathcal{U}_\gamma\}.$$

Proposition 8.29 will imply that $\mathcal{B}$ is a base for a topology on $\mathcal{G}$. The proof hinges on the following technical statement.

Lemma 8.22. *Let* $(u, x) \in \mathcal{U}_{\gamma_1} \cap \mathcal{U}_{\gamma_2}$ *for some* $\gamma_1, \gamma_2 \in \Gamma$. *If, for some* $\tau_1 \in \{-1, 1\}^{\gamma_1}$ *and* $\tau_2 \in \{-1, 1\}^{\gamma_2}$,

$$S_{\gamma_1 \tau_1}(x) u S_{\gamma_1 \tau_1}(x) = S_{\gamma_2 \tau_2}(x) u S_{\gamma_2 \tau_2}(x) \tag{8.23}$$

then

$$either \quad S_{\gamma_1 \tau_1}(x) = S_{\gamma_2 \tau_2}(x) \quad or \quad S_{\gamma_1 \tau_1}(x) = -S_{\gamma_2 \tau_2}(x). \tag{8.24}$$

Proof. Since $(u, x) \in \mathcal{U}_{\gamma_1} \cap \mathcal{U}_{\gamma_2}$ we have

$$x \in D_{\gamma_1} \cap D_{\gamma_2},$$

or equivalently,

$$\sigma(x) \subseteq \Delta_{\gamma_1} \cap \Delta_{\gamma_2}. \tag{8.25}$$

In the light of property (8.5) and Lemma 8.2 we may assume that Δ_{γ_2} is subordinate to Δ_{γ_1}, that is, that each component of Δ_{γ_2} meets at most one component of Δ_{γ_1}. This implies that each component of Δ_{γ_2} contains at most one component of $\Delta_{\gamma_1} \cap \Delta_{\gamma_2}$. As both of the functions $s_{\gamma_1 \tau_1}$ and $s_{\gamma_2 \tau_2}$ are branches of $\sqrt{z}$ on $\Delta_{\gamma_1} \cap \Delta_{\gamma_2}$, they agree up to a factor ± 1 on each component of $\Delta_{\gamma_1} \cap \Delta_{\gamma_2}$. Hence there exists $\tau \in \{-1, 1\}^{\gamma_2}$ such that

$$s_{\gamma_1 \tau_1}(z) = \iota_{\gamma_2 \tau}(z) s_{\gamma_2 \tau_2}(z), \qquad z \in \Delta_{\gamma_1} \cap \Delta_{\gamma_2}. \tag{8.26}$$

Equations (8.25) and (8.23) imply that

$$S_{\gamma_1 \tau_1}(x) = I_{\gamma_2 \tau}(x) S_{\gamma_2 \tau_2}(x). \tag{8.27}$$

Substitution of this formula for $S_{\gamma_1 \tau_1}(x)$ into equation (8.23) yields

$$u = I_{\gamma_2 \tau}(x) u I_{\gamma_2 \tau}(x).$$

But $I_{\gamma_2 \tau}(x)$ is an involution. Therefore

$$f_{\gamma_2 \tau}(u, x) = u I_{\gamma_2 \tau}(x) - I_{\gamma_2 \tau}(x) u = 0,$$

that is, $(u, x) \in V_{\gamma_2 \tau}$. As $(u, x) \in \mathcal{U}_{\gamma_2}$, this implies that τ is constant on γ_2 and either $\tau = 1$ or $\tau = -1$. We deduce that either $\iota_{\gamma_2 \tau} \equiv 1$ or $\iota_{\gamma_2 \tau} \equiv -1$ on $\Delta_{\gamma_1} \cap \Delta_{\gamma_2}$. In conjunction with equation (8.27), this implies that equation (8.24) holds. $\square$

Remark 8.28. The foregoing technical lemma explains the introduction of the Zariski-free topology. It is only because of the exclusion of the varieties $V_{\gamma\tau}$, where a non-generic commutation relation holds, in the definition of the sets $\mathcal{U}_\gamma$ that the collection $\mathcal{B}$ of Definition 8.21 constitutes a base for a topology on $\mathcal{G}$.

Proposition 8.29. *If $\Omega_1, \Omega_2 \in \mathcal{B}$ and $\omega \in \Omega_1 \cap \Omega_2$, then there exists $\Omega_3 \in \mathcal{B}$ such that $\omega \in \Omega_3 \subseteq \Omega_1 \cap \Omega_2$.*

Proof. Fix $\Omega_1 = \Phi_{\gamma_1\tau_1}(U_1)$, $\Omega_2 = \Phi_{\gamma_2\tau_2}(U_2)$ and assume that $(u, x, y) \in \Omega_1 \cap \Omega_2$. As

$$(u, x) \in U_1 \cap U_2 \subseteq \mathcal{U}_{\gamma_1} \cap \mathcal{U}_{\gamma_2},$$

and

$$S_{\gamma_1\tau_1}(x)uS_{\gamma_1\tau_1}(x) = y = S_{\gamma_2\tau_2}(x)uS_{\gamma_2\tau_2}(x) \tag{8.30}$$

we may apply Lemma 8.22 to deduce that

$$\text{either } S_{\gamma_1\tau_1}(x) = S_{\gamma_2\tau_2}(x) \text{ or } S_{\gamma_1\tau_1}(x) = -S_{\gamma_2\tau_2}(x). \tag{8.31}$$

Let

$$\Omega_3 = \Phi_{\gamma_2\tau_2}(U_1 \cap U_2).$$

As $(u, x) \in U_1 \cap U_2$, equation (8.30) implies that

$$(u, x, y) \in \Omega_3.$$

Clearly

$$\Omega_3 = \Phi_{\gamma_2\tau_2}(U_1 \cap U_2) \subseteq \Phi_{\gamma_2\tau_2}(U_2) = \Omega_2.$$

To see that $\Omega_3 \subseteq \Omega_1$, fix $(u, x) \in U_1 \cap U_2$. Then, as $\sigma(x) \subseteq \Delta_1 \cap \Delta_2$, equation (8.31) implies that

$$\begin{aligned}
\Phi_{\gamma_2\tau_2}(u, x) &= (u, x, S_{\gamma_2\tau_2}(x) \, u \, S_{\gamma_2\tau_2}(x)) \\
&= (u, x, S_{\gamma_1\tau_1}(x) \, u \, S_{\gamma_1\tau_1}(x)) \\
&= \Phi_{\gamma_1\tau_1}(u, x) \\
&\in \Omega_1.
\end{aligned}$$

$\square$

Lemma 8.29 implies that $\mathcal{B}$ is a base for a unique topology $\mathcal{T}$ on $\mathcal{G}$. The topology $\mathcal{T}$ is defined so that the maps

$$\Phi_{\gamma\tau}^{-1} : \mathcal{G}_{\gamma\tau} \to \mathcal{U}_\gamma$$

are homeomorphisms when $\mathcal{G}_{\gamma\tau}$ carries the $\mathcal{T}$ topology and $\mathcal{U}_\gamma$ carries the Zariski-free topology. As $\mathcal{U}_\gamma$ is a Zariski-free domain, it follows that

$$\mathcal{A} = \{\Phi_{\gamma\tau}^{-1} \,|\, \gamma \in \Gamma, \tau \in \{-1, 1\}^\gamma\} \tag{8.32}$$

is a collection of topological nc-co-ordinate patches on $\mathcal{G}$ (see condition (4.6)).

In fact $\mathcal{A}$ is a Zariski-free atlas for $(\mathcal{G}, \mathcal{T})$. Since Definition (8.20) implies that the sets $\mathcal{G}_{\gamma\tau}$ cover $\mathcal{G}$, equation (4.4) holds. To see that the transition functions are Zariski-freely holomorphic, observe that

$$(\Phi_{\gamma_2\tau_2}^{-1} \circ \Phi_{\gamma_1\tau_1})(u, x) = (u, x)$$

for all $(u, x) \in \Phi_{\gamma_1\tau_1}^{-1}(\mathcal{G}_{\gamma_1\tau_1} \cap \mathcal{G}_{\gamma_2\tau_2})$, and $\Phi_{\gamma_1\tau_1}^{-1}(\mathcal{G}_{\gamma_1\tau_1} \cap \mathcal{G}_{\gamma_2\tau_2})$ is Zariski-freely open by Lemma 8.29.

We summarize the above observations.

Theorem 8.33. *If $\mathcal{G}$ is the set defined by equation (8.20), $\mathcal{T}$ is the topology on $\mathcal{G}$ generated by the set $\mathcal{B}$ in Definition 8.21 and $\mathcal{A}$ is given by equation (8.32) then $(\mathcal{G}, \mathcal{T}, \mathcal{A})$ is a Zariski-free manifold.*

Our original hope when we began this project was to construct a free nc-manifold $\mathcal{G}$ and a surjective free holomorphic map $\pi : \mathbb{M}^2 \to \mathcal{G}$ with the property that, for every symmetric free holomorphic function φ on $\mathbb{M}^2$, there exists a free holomorphic function Φ on $\mathcal{G}$ such that $\varphi = \Phi \circ \pi$. Our construction above falls short of this goal on at least two counts. Firstly, $\mathcal{G}$ is not a free manifold, but a *Zariski*-free manifold. Secondly, if we make the intended definition

$$\begin{aligned}\pi(w) &= (u, v^2, vuv) \\ &= (\tfrac{1}{2}(w^1 + w^2), \tfrac{1}{4}(w^1 - w^2)^2, \tfrac{1}{8}(w^1 - w^2)(w^1 + w^2)(w^1 - w^2))\end{aligned} \tag{8.34}$$

then a necessary condition that $\pi(w) \in \mathcal{G}$ is that $v^2 \in \mathcal{I}$, since we require (u, v^2) to belong to some $\mathcal{U}_\gamma$; thus π does not map $\mathbb{M}^2$ to $\mathcal{G}$. Nonetheless, there is a correspondence between φ and a suitable holomorphic function Φ on $\mathcal{G}$, but the correspondence is more subtle, as the next statement shows.

Theorem 8.35. *There is a canonical bijection between the classes of*
(i) *symmetric nc functions f that are freely holomorphic on $\mathbb{M}^2$, and*
(ii) *holomorphic functions F_{oo} defined on the Zariski-free manifold $\mathcal{G}$ that are conditionally nc and have the property that, for every $w \in \mathbb{M}^2$, there is a free neighborhood U of w such that F_{oo} is bounded on $\pi(U) \cap \mathcal{G}$.*

Theorem 8.35 will be proved in the next section, in greater generality – see Theorem 10.11. We define conditionally nc in Definition 9.25.

9   Symmetric free holomorphic functions

In Section 8 we constructed an nc-manifold $\mathcal{G}$ for the representation of freely *entire* symmetric functions on $\mathbb{M}^2$. In this section we shall do likewise for symmetric free holomorphic functions on an arbitrary symmetric free domain.
Recall that a set $\mathcal{S} \subseteq \mathbb{M}^2$ is *symmetric* if

$$w = (w^1, w^2) \in \mathcal{S} \implies (w^2, w^1) \in \mathcal{S}.$$

In the sequel $\mathcal{S}$ is a fixed symmetric free domain in $\mathbb{M}^2$. We shall construct a Zariski-free manifold $\mathfrak{G}_{oo}(\mathcal{S})$ which is, roughly speaking, the restriction of $\mathcal{G}$ to $\mathcal{S}$.
All the notations and constructions in Section 8 are in effect in this section as well. For $w \in \mathbb{M}^2$ we shall frequently employ the change of variables
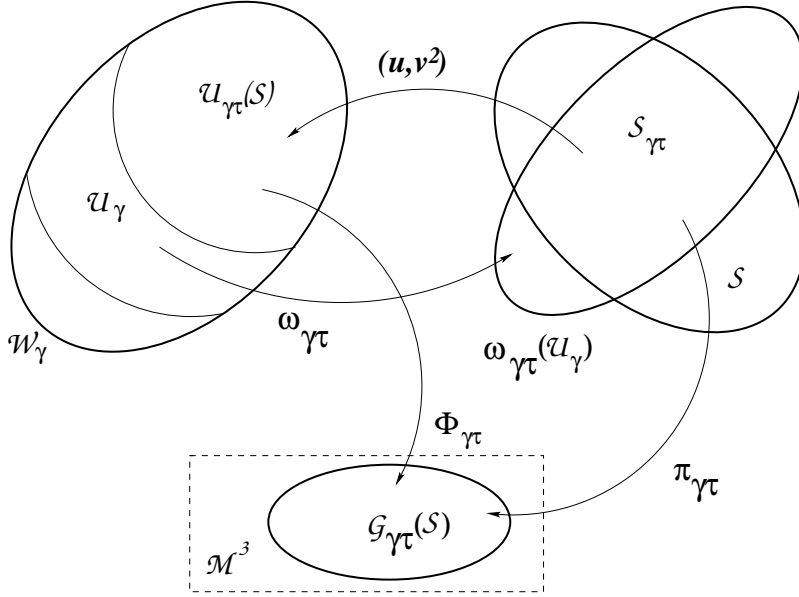
$$u = \tfrac{1}{2}(w^1 + w^2), \qquad v = \tfrac{1}{2}(w^1 - w^2), \tag{9.1}$$

or, equivalently,
$$w^1 = u + v, \qquad w^2 = u - v.$$

The operation of transposition of components will be denoted by $(w^1, w^2)^f = (w^2, w^1)$.
To construct $\mathcal{G}(\mathcal{S})$ we need to define several sets and mappings, as indicated schematically in the following figure.

Scheme for the construction of $\mathcal{G}(\mathcal{S})$

9.1 Local inverses of (u, v^2)

The purpose of this and the next two subsections is to define a submanifold of the Zariski-free manifold $\mathcal{G}$ corresponding to a symmetric free domain $\mathcal{S}$ in $\mathbb{M}^2$. First we introduce maps $\omega_{\gamma\tau}$ which are local inverses of the map $w \to (u, v^2)$, in the notation of equations (9.1).

Recall that, for each $\gamma \in \Gamma$, the set $\mathcal{W}_\gamma$ defined in equation (8.12) is freely open in $\mathbb{M}^2$. For each $\gamma \in \Gamma$ and each $\tau \in \{-1, 1\}^\gamma$ we define a mapping $\omega_{\gamma\tau} : \mathcal{W}_\gamma \to \mathbb{M}^2$ by the formula

$$\omega_{\gamma\tau}(u, x) = \Big(u + S_{\gamma\tau}(x) \ , \ u - S_{\gamma\tau}(x)\Big), \qquad (u, x) \in \mathcal{W}_\gamma \qquad (9.2)$$

where $S_{\gamma\tau}$ is the free holomorphic function on D_γ defined by equation (8.11).

Let $w = \omega_{\gamma\tau}(u, x)$, so that

$$u + S_{\gamma\tau}(x) = w^1 \qquad \text{and} \qquad u - S_{\gamma\tau}(x) = w^2.$$

Solution of these equations for u and x gives

$$u = \tfrac{1}{2}(w^1 + w^2) \qquad \text{and} \qquad x = S_{\gamma\tau}(x)^2 = \tfrac{1}{4}(w^1 - w^2)^2.$$

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Thus $\omega_{\gamma\tau}$ is injective and

$$\omega_{\gamma\tau}^{-1}(w) = \left(\tfrac{1}{2}(w^1 + w^2), \tfrac{1}{4}(w^1 - w^2)^2\right) \tag{9.3}$$
$$= (u, v^2)$$

for all $w \in \operatorname{ran} \omega_{\gamma\tau}$.

Proposition 9.4. *For each $\gamma \in \Gamma$ and $\tau \in \{-1, 1\}^\gamma$, $\omega_{\gamma\tau}$ is a free biholomorphic mapping from $\mathcal{W}_\gamma$ onto its range.*

Proof. We first show that $\operatorname{ran} \omega_{\gamma\tau}$ is open in the free topology. Equation (9.3) defines a free polynomial map F on all of $\mathbb{M}^2$. Hence, as this map is continuous in the free topology, $\operatorname{ran} \omega_{\gamma\tau} = F^{-1}(\mathcal{W}_\gamma)$ is open in the free topology.

Once it is known that both the domain and range of $\omega_{\gamma\tau}$ are freely open sets, that $\omega_{\gamma\tau}$ is a free biholomorphic mapping follows immediately from the formulas (9.2) and (9.3). $\square$

9.2 The sets $\mathcal{S}_{\gamma\tau}$ and $\mathcal{U}_{\gamma\tau}(\mathcal{S})$

Recall that, for each $\gamma \in \Gamma$, the set $\mathcal{U}_\gamma$ defined in equation (8.13) is a Zariski-free open subset of $\mathcal{W}_\gamma$. For each $\tau \in \{-1, 1\}^\gamma$, we define $\mathcal{S}_{\gamma\tau} \subseteq \mathcal{S}$ by

$$\mathcal{S}_{\gamma\tau} = \omega_{\gamma\tau}(\mathcal{U}_\gamma) \cap \mathcal{S} \tag{9.5}$$

and define $\mathcal{U}_{\gamma\tau}(\mathcal{S}) \subseteq \mathcal{U}_\gamma$ by

$$\mathcal{U}_{\gamma\tau}(\mathcal{S}) = \{(u, x) \in \mathcal{U}_\gamma \,|\, \omega_{\gamma\tau}(u, x) \in \mathcal{S}\}$$
$$= \omega_{\gamma\tau}^{-1}(\mathcal{S}_{\gamma\tau}). \tag{9.6}$$

Proposition 9.7. *$\mathcal{U}_{\gamma\tau}(\mathcal{S})$ and $\mathcal{S}_{\gamma\tau}$ are Zariski-free open sets and $\omega_{\gamma\tau}$ is a Zariski-free biholomorphic mapping from $\mathcal{U}_{\gamma\tau}(\mathcal{S})$ onto $\mathcal{S}_{\gamma\tau}$.*

Proof. By Proposition 9.4, $\omega_{\gamma\tau} : \mathcal{W}_\gamma \to \operatorname{ran} \omega_{\gamma\tau}$ is a free biholomorphic mapping. It follows by Proposition 7.14 that $\omega_{\gamma\tau}$ is a homeomorphism when $\mathcal{W}_\gamma$ and $\operatorname{ran} \omega_{\gamma\tau}$ are equipped with the Zariski-free topology. Hence equation (9.5) implies that $\mathcal{S}_{\gamma\tau}$ is Zariski-freely open and equation (9.6) implies that $\mathcal{U}_{\gamma\tau}(\mathcal{S})$ is Zariski-freely open.

That $\omega_{\gamma\tau}$ acting on $\mathcal{U}_{\gamma\tau}(\mathcal{S})$ and $\omega_{\gamma\tau}^{-1}$ acting on $\mathcal{S}_{\gamma\tau}$ are Zariski-free holomorphic mappings (that is, are Zariski-freely locally bounded) follow from the facts that $\omega_{\gamma\tau}$ acting on $\mathcal{W}_\gamma$ and $\omega_{\gamma\tau}^{-1}$ acting on $\operatorname{ran} \omega_{\gamma\tau}$ are free holomorphic mappings (and hence are freely locally bounded). $\square$

9.3 The sets $\mathcal{G}_{\gamma\tau}(\mathcal{S})$ and $\mathcal{G}(\mathcal{S})$

Recall that for each $\gamma \in \Gamma$ and each $\tau \in \{-1,1\}^{\gamma}$, $\Phi_{\gamma\tau}$ and $\mathcal{G}_{\gamma\tau}$ are defined by equations (8.18) and (8.19). For each γ and τ, we define

$$\mathcal{G}_{\gamma\tau}(\mathcal{S}) = \Phi_{\gamma\tau}(\mathcal{U}_{\gamma\tau}(\mathcal{S})),$$

and then, following the definition (8.20), we set

$$\boldsymbol{\mathfrak{G}}_{oo}(\mathcal{S}) = \bigcup_{\gamma \in \Gamma} \bigcup_{\tau \in \{-1,1\}^{\gamma}} \mathcal{G}_{\gamma\tau}(\mathcal{S}). \tag{9.8}$$

Recall that the topology $\mathcal{T}$ on $\mathcal{G}$ was chosen so that the maps $\Phi_{\gamma\tau} : \mathcal{U}_{\gamma} \to \mathcal{G}_{\gamma\tau}$ are homeomorphisms when $\mathcal{U}_{\gamma}$ carries the Zariski-free topology. Therefore $\boldsymbol{\mathfrak{G}}_{oo}(\mathcal{S})$ is an open subset of $\mathcal{G}$ and as such carries the structure of a Zariski-free manifold. A Zariski-free atlas for $\boldsymbol{\mathfrak{G}}_{oo}(\mathcal{S})$, $\mathcal{A}(\mathcal{S})$, can be obtained from the Zariski-free atlas $\mathcal{A}$ for $\mathcal{G}$ defined in equation (8.32) by simple restriction of the charts in $\mathcal{A}$ to the sets $\mathcal{G}_{\gamma\tau}(\mathcal{S})$, that is, by the definition

$$\mathcal{A}(\mathcal{S}) = \{\Phi_{\gamma\tau}^{-1} \,|\, \mathcal{G}_{\gamma\tau}(\mathcal{S}) : \gamma \in \Gamma, \tau \in \{-1,1\}^{\gamma}\}. \tag{9.9}$$

We summarize these observations in the following theorem.

Theorem 9.10. *Let $\mathcal{S}$ be a symmetric free domain in $\mathbb{M}^2$. If $\boldsymbol{\mathfrak{G}}_{oo}(\mathcal{S})$ is defined by equation (9.8), then $\boldsymbol{\mathfrak{G}}_{oo}(\mathcal{S})$ is an open subset of $\mathcal{G}$. Furthermore, if $\mathcal{T}(\mathcal{S})$ is the relativization of $\mathcal{T}$ to $\boldsymbol{\mathfrak{G}}_{oo}(\mathcal{S})$ and $\mathcal{A}(\mathcal{S})$ is defined by equation (9.9) then $(\boldsymbol{\mathfrak{G}}_{oo}(\mathcal{S}), \mathcal{T}(\mathcal{S}), \mathcal{A}(\mathcal{S}))$ is a Zariski-free manifold.*

9.4 A holomorphic cover of $\boldsymbol{\mathfrak{G}}_{oo}(\mathcal{S})$

In this subsection we shall construct a 2-to-1 holomorphic covering map from a Zariski-free subdomain of $\mathcal{S}$ onto $\boldsymbol{\mathfrak{G}}_{oo}(\mathcal{S})$. For each $\gamma \in \Gamma$ and each $\tau \in \{-1,1\}^{\gamma}$, define $\pi_{\gamma\tau} : \mathcal{S}_{\gamma\tau} \to \mathcal{G}_{\gamma\tau}(\mathcal{S})$ by

$$\pi_{\gamma\tau} = \Phi_{\gamma\tau} \circ \omega_{\gamma\tau}^{-1}.$$

Thus, for $w \in \mathcal{S}_{\gamma\tau}$,

$$\pi_{\gamma\tau}(w) = (u, v^2, S_{\gamma\tau}(v^2)uS_{\gamma\tau}(v^2)).$$

Lemma 9.11. *If $w \in \mathcal{S}_{\gamma_1\tau_1} \cap \mathcal{S}_{\gamma_2\tau_2}$, then $\pi_{\gamma_1\tau_1}(w) = \pi_{\gamma_2\tau_2}(w)$.*

Proof. As $w \in \mathcal{S}_{\gamma_1 \tau_1}$, there exists $(u_1, x_1) \in \mathcal{U}_{\gamma_1 \tau_1}$ such that $w = \omega_{\gamma_1 \tau_1}(u_1, x_1)$. Likewise, there exists $(u_2, x_2) \in \mathcal{U}_{\gamma_2 \tau_2}$ such that $w = \omega_{\gamma_2 \tau_2}(u_2, x_2)$. But

$$\omega_{\gamma_1 \tau_1}(u_1, x_1) = \omega_{\gamma_2 \tau_2}(u_2, x_2)$$

implies via equation (9.2) that

$$u_1 + S_{\gamma_1 \tau_1}(x_1) = u_2 + S_{\gamma_2 \tau_2}(x_2)$$

and

$$u_1 - S_{\gamma_1 \tau_1}(x_1) = u_2 - S_{\gamma_2 \tau_2}(x_2).$$

These equations imply that $u_1 = u_2$ and $S_{\gamma_1 \tau_1}(x_1) = S_{\gamma_2 \tau_2}(x_2)$. Noting that both $S_{\gamma_1 \tau_1}$ and $S_{\gamma_2 \tau_2}$ are branches of $\sqrt{x}$, we also have

$$x_1 = S_{\gamma_1 \tau_1}(x_1)^2 = S_{\gamma_2 \tau_2}(x_2)^2 = x_2.$$

Hence

$$\begin{aligned}
\pi_{\gamma_1 \tau_1}(w) &= \pi_{\gamma_1 \tau_1}(\omega_{\gamma_1 \tau_1}(u_1, x_1)) \\
&= \Phi_{\gamma_1 \tau_1}(u_1, x_1) \\
&= (u_1, x_1, S_{\gamma_1 \tau_1}(x_1) u_1 S_{\gamma_1 \tau_1}(x_1)) \\
&= (u_2, x_2, S_{\gamma_2 \tau_2}(x_2) u_2 S_{\gamma_2 \tau_2}(x_2)) \\
&= \Phi_{\gamma_2 \tau_2}(u_2, x_2) \\
&= \pi_{\gamma_2 \tau_2}(\omega_{\gamma_2 \tau_2}(u_2, x_2)) \\
&= \pi_{\gamma_2 \tau_2}(w).
\end{aligned}$$

$\square$

Armed with Lemma 9.11 we may define $\mathcal{S}_{oo} \subseteq \mathcal{S}$ by

$$\mathcal{S}_{oo} = \bigcup_{\gamma \in \Gamma} \bigcup_{\tau \in \{-1,1\}^\gamma} \mathcal{S}_{\gamma \tau}. \tag{9.12}$$

It is immediate that there is a globally well-defined mapping $\pi : \mathcal{S}_{oo} \to \mathfrak{G}_{oo}(\mathcal{S})$, locally defined by the formula,

$$\pi(w) = \pi_{\gamma \tau}(w) \text{ if } w \in \mathcal{S}_{\gamma \tau}.$$

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Theorem 9.13. π *is a holomorphic covering map from the Zariski-free domain* $\mathcal{S}_{oo}$ *onto the Zariski-free manifold* $\mathfrak{G}_{oo}(\mathcal{S})$. *Furthermore,* π *is given by the formula*

$$\pi(w) = (\tfrac{1}{2}(w^1 + w^2), \tfrac{1}{4}(w^1 - w^2)^2, \tfrac{1}{2}(w^1 - w^2)\tfrac{1}{2}(w^1 + w^2)\tfrac{1}{2}(w^1 - w^2)). \quad (9.14)$$

Finally, π *is 2-to-1 and* $\pi^{-1}(\pi(w)) = \{w, w^f\}$

Proof. By Proposition 9.7, $\mathcal{U}_{\gamma\tau}(\mathcal{S})$ and $\mathcal{S}_{\gamma\tau}$ are Zariski-free domains and

$$\omega_{\gamma\tau} : \mathcal{U}_{\gamma\tau}(\mathcal{S}) \to \mathcal{S}_{\gamma\tau}$$

is a Zariski-free homeomorphism. Also, by the comments following equation (9.8),

$$\Phi_{\gamma\tau} : \mathcal{U}_{\gamma\tau}(\mathcal{S}) \to \mathcal{G}_{\gamma\tau}(\mathcal{S})$$

is a Zariski-free homeomorphism. Therefore,

$$\pi|\mathcal{S}_{\gamma\tau} = \Phi_{\gamma\tau} \circ \omega_{\gamma\tau}^{-1}$$

is a Zariski-free homeomorphism. This proves that π is a local homeomorphism.

If $\alpha = \Phi_{\gamma\tau}^{-1} : \mathcal{G}_{\gamma\tau}(\mathcal{S}) \to \mathcal{U}_{\gamma\tau}(\mathcal{S})$ is a co-ordinate patch in $\mathcal{A}(\mathcal{S})$ (cf. equation (9.9)), then

$$\alpha \circ (\pi|\mathcal{S}_{\gamma\tau}) = \omega_{\gamma\tau}^{-1}$$

and

$$(\pi|\mathcal{S}_{\gamma\tau})^{-1} \circ \alpha^{-1} = \omega_{\gamma\tau}$$

are both holomorphic. Therefore π is biholomorphic on every $\mathcal{S}_{\gamma\tau}$.

As

$$\mathcal{G}_{\gamma\tau}(\mathcal{S}) = \mathrm{ran}\,\pi_{\gamma\tau} \subset \mathrm{ran}\,\pi,$$

for all $\gamma \in \Gamma$ and $\tau \in \{-1, 1\}^\gamma$, it follows from equation (9.8) that π is surjective.

If $(u, x) \in \mathcal{U}_\gamma$ and $w = \omega_{\gamma\tau}(u, x)$, then $u = \tfrac{1}{2}(w^1 + w^2)$, $S_{\gamma\tau}(x) = \tfrac{1}{2}(w^1 - w^2)$, and $x = \tfrac{1}{4}(w^1 - w^2)^2$. Hence,

$$\begin{aligned}
\pi_{\gamma\tau}(w) &= \pi_{\gamma\tau}(\omega_{\gamma\tau}(u, x)) \\
&= \Phi_{\gamma\tau}(u, x) \\
&= (u, x, S_{\gamma\tau}(x)uS_{\gamma\tau}(x)) \\
&= (\tfrac{1}{2}(w^1 + w^2), \tfrac{1}{4}(w^1 - w^2)^2, \tfrac{1}{2}(w^1 - w^2)\tfrac{1}{2}(w^1 + w^2)\tfrac{1}{2}(w^1 - w^2)).
\end{aligned}$$

This proves equation (9.14).

Assume that $\pi(w_1) = \pi(w_2)$ for some $w_1, w_2 \in \mathcal{S}_{oo}$. If $w_1 = \omega_{\gamma_1 \tau_1}(u_1, x_1)$ and $w_2 = \omega_{\gamma_2 \tau_2}(u_2, x_2)$ then the computation of the previous paragraph gives $u_1 = u_2$, $x_1 = x_2$ and

$$S_{\gamma_1 \tau_1}(x_1) u_1 S_{\gamma_1 \tau_1}(x_1) = S_{\gamma_2 \tau_2}(x_2) u_2 S_{\gamma_2 \tau_2}(x_2).$$

If we set $x_0 = x_1 = x_2$ and $u_0 = u_1 = u_2$, then this equation becomes equation (8.30) in the proof of Proposition 8.29. The argument in that proof shows that equation (8.31) holds with $x = x_0$, that is, either

$$S_{\gamma_1 \tau_1}(x_0) = S_{\gamma_2 \tau_2}(x_0) \quad \text{or} \quad S_{\gamma_1 \tau_1}(x_0) = -S_{\gamma_2 \tau_2}(x_0).$$

In the former case $w_2 = w_1$ and in the latter case $w_2 = w_1^f$.

Finally, to show that π is 2-to-1, observe that $w \neq w^f$ for $w \in \mathcal{S}_{\gamma \tau}$. Otherwise, suppose $w = \omega_{\gamma \tau}(u, x)$ for some $(u, x) \in \mathcal{U}_{\gamma \tau}$. Since $w = w^f$ we have $u + S_{\gamma \tau}(x) = u - S_{\gamma \tau}(x)$, and therefore $S_{\gamma \tau}(x) = 0$. Hence $x = S_{\gamma \tau}(x)^2 = 0$, contrary to the fact that $(u, x) \in \mathcal{U}_\gamma$. $\qquad \square$

We shall use formula (9.14) to extend π to a polynomial map $\mathbb{M}^2 \to \mathbb{M}^3$.

9.5 The representation of symmetric functions on $\mathcal{S}_{oo}$

Theorem 9.15. *Let $\mathcal{S}$ be a free symmetric domain in $\mathbb{M}^2$, $\mathfrak{G}_{oo}(\mathcal{S})$ the Zariski-free manifold described in Theorem 9.10 and $\pi : \mathcal{S}_{oo} \to \mathfrak{G}_{oo}(\mathcal{S})$ the covering map described in Theorem 9.13. If F_{oo} is a holomorphic function on $\mathfrak{G}_{oo}(\mathcal{S})$, then $f_{oo} = F_{oo} \circ \pi$ is a symmetric Zariski-free holomorphic function on $\mathcal{S}_{oo}$. Conversely, if f_{oo} is a symmetric Zariski-free holomorphic function on $\mathcal{S}_{oo}$, then there exists a unique holomorphic function F_{oo} on $\mathfrak{G}_{oo}(\mathcal{S})$ such that $f_{oo} = F_{oo} \circ \pi$.*

Proof. First assume that F_{oo} is Zariski-freely holomorphic on $\mathfrak{G}_{oo}(\mathcal{S})$ and let $f_{oo} = F_{oo} \circ \pi$. As F_{oo} and π are Zariski-freely holomorphic, so is $f_{oo} = F_{oo} \circ \pi$. As $\pi(w^f) = \pi(w)$ for all $w \in \mathcal{S}_{oo}$, so also $f_{oo}(w^f) = f_{oo}(w)$ for all $w \in \mathcal{S}_{oo}$. Thus, f_{oo} is a symmetric Zariski-freely holomorphic function.

Now assume that f_{oo} is a symmetric Zariski-freely holomorphic function on $\mathcal{S}_{oo}$. Attempt to define a function F_{oo} on $\mathfrak{G}_{oo}(\mathcal{S})$ by the formula

$$F_{oo}(\pi(w)) = f_{oo}(w), \qquad w \in \mathcal{S}_{oo}.$$

If $\pi(w^1) = \pi(w^2)$ and $w^1 \neq w^2$ then as $w^2 = (w^f)^1$ and f_{oo} is assumed symmetric, $F_{oo}(\pi(w^1)) = F_{oo}(\pi(w^2))$. This proves that F_{oo} is well defined, and clearly $f_{oo} = F_{oo} \circ \pi$. Also, as π is surjective, F_{oo} is defined on all of $\mathfrak{G}_{oo}(\mathcal{S})$. Since locally on the sets $\mathcal{G}_{\gamma\tau}(\mathcal{S})$, $F_{oo} = f_{oo} \circ \pi^{-1}$, a composition of Zariski-freely holomorphic functions, F_{oo} is Zariski-freely holomorphic. $\square$

We shall extend Theorem 9.15 in Theorems 10.1 and 10.11.

9.6 The representation of symmetric functions on $\mathcal{S}$

For any symmetric free domain $\mathcal{S}$ in $\mathbb{M}^2$ we define $\mathcal{S}_o \subseteq \mathcal{S}$ by

$$\mathcal{S}_o = \bigcup_{\gamma \in \Gamma} \bigcup_{\tau \in \{-1,1\}^\gamma} \omega_{\gamma\tau}(\mathcal{W}_\gamma) \cap \mathcal{S}. \tag{9.16}$$

Proposition 9.4 implies that $\mathcal{S}_o$ is a free domain and equations (9.5) and (9.12) imply that $\mathcal{S}_{oo} \subseteq \mathcal{S}_o$. An alternative, intrinsic characterization of $\mathcal{S}_o$ in terms of the set $\mathcal{Q}$ defined by equation (6.4) is given in the following proposition.

Proposition 9.17. $\mathcal{S}_o = \{w \in \mathcal{S} \,|\, w^1 - w^2 \in \mathcal{Q}\}$

Proof. First assume that $w = \omega_{\gamma\tau}(u, x) \in \mathcal{S}_o$, so that $\frac{1}{2}(w^1 - w^2) = S_{\gamma\tau}(x)$ and $x \in D_\gamma$ is nonsingular. The spectrum of $S_{\gamma\tau}(x)$ is obtained by the application of $s_{\gamma\tau}$ to $\sigma(x)$, which process does not produce two eigenvalues λ and $-\lambda$. Hence $S_{\gamma\tau}(x) \in \mathcal{Q}$, and therefore $w \in \mathcal{Q}$. Conversely, suppose that $w \in \mathcal{S}$ and $\frac{1}{2}(w^1 - w^2) \in \mathcal{Q}$. If we set $x = \frac{1}{4}(w^1 - w^2)^2$ and $\gamma = \sigma(x)$, then there exists $\tau \in \{-1, 1\}$ such that $S_{\gamma\tau}(x) = \frac{1}{2}(w^1 - w^2)$. If $u = \frac{1}{2}(w^1 + w^2)$, then $w = S_{\gamma\tau}(u, x) \in \mathcal{S}_o$. $\square$

Let $\mathcal{S}$ be a symmetric free domain in $\mathbb{M}^2$, and define $\mathcal{S}_o$ by equation (9.16) and $\mathcal{S}_{oo}$ by equation (9.12), where π is the polynomial map (9.14). Write $\mathfrak{G}_{oo}$ for $\mathfrak{G}_{oo}(\mathcal{S})$, and define $\mathfrak{G}_o = \pi(\mathcal{S}_o)$ and $\mathfrak{G} = \pi(\mathcal{S})$. Consider the following diagram:

$$\tag{9.18}$$

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In Theorem 9.15 we showed that $f_{oo} : \mathcal{S}_{oo} \to \mathbb{M}^1$ is a Zariski-free holomorphic map if and only if there exists a Zariski-free holomorphic map $F_{oo} : \mathfrak{G}_{oo} \to \mathbb{M}^1$ such that $F_{oo} \circ \pi = f_{oo}$.

Suppose $F : \mathfrak{G} \to \mathbb{M}^1$ is a free holomorphic map, by which we mean that at every point λ of $\mathfrak{G}$ there is a free neighborhood B_δ of λ in $\mathbb{M}^3$ and a bounded free holomorphic function g on B_δ such that g agrees with F on $B_\delta \cap \mathfrak{G}$. Then $F \circ \pi$ is a symmetric free holomorphic function on $\mathcal{S}$.

Question 9.19. *Is the converse true?*

In this generality, the answer to Question 9.19 is no, as the following example, suggested to us by James Pascoe, shows.

Example 9.20. Let $\mathcal{S}$ be the *nc-bidisc*, that is, the set of pairs of strict contractions in $\mathbb{M}^2$:

$$\mathcal{S} = \{w \in \mathbb{M}^2 : \|w^1\|, \|w^2\| < 1\}.$$

Let

$$
\begin{aligned}
f(w) &= (w^1 - w^2)(w^1 + w^2)^2(w^1 - w^2) \\
&= 16vu^2v.
\end{aligned}
$$

The natural choice for an F such that $f = F \circ \pi$ is $F(z) = 16z^3(z^2)^{-1}z^3$, but this function is not freely holomorphic on a neighborhood of 0.

Indeed, there is no free holomorphic function F defined on a neighborhood B_δ of 0 in $\mathbb{M}^3$ such that $F \circ \pi = f$. For suppose that F is such a function. Replacing $\delta(z)$ by $t(\delta(z) - \delta(0))$, where $t \geq 1/(1 - \|\delta(0)\|)$, we can assume that $\delta(0) = 0$. By [1], $F(z)$ can be represented by a convergent series on B_δ whose terms are non-commutative polynomials in the entries $\delta_{ij}(z)$. If $F \circ \pi = f$, then

$$F(u, v^2, uvu) = 16vu^2v. \tag{9.21}$$

Expand F in a power series in $\mathbb{D}$, and group terms by homogeneity. Then the left-hand side of equation (9.21) is a linear combination of

$$u^2v^2, uv^2u, v^2u^2.$$

No linear combination of these three elements is equal to vu^2v.

One can show more: there is no function F at all satisfying $F \circ \pi = f$ on $\mathcal{S}$. We shall show this by giving two points w and W in $\mathcal{S} \cap \mathbb{M}_4^2$ that are identified by π but not by f. Let

$$2u = w^1 + w^2, \quad 2U = W^1 + W^2, \quad 2v = w^1 - w^2, \quad 2V = W^1 - W^2.$$

Let r be a small positive number. Let

$$v = \begin{bmatrix} \begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix} \end{bmatrix}, \quad V = \begin{bmatrix} \begin{bmatrix} 0 & -r \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix} \end{bmatrix}$$

Let $u = U$ have the $(2,3)$ and $(4,1)$ entries equal to 0, but the $(2,3)$ element of their square not equal to 0. Then $\pi(w) = (u, v^2, vuv) = \pi(W)$, but

$$f(w) = 16vu^2v \neq 16VU^2V = f(W),$$

since the $(1,4)$ entry in vu^2v is r^2 times the $(2,3)$ entry of u^2, and in VU^2V it is the negative of this quantity.

The fact that v is not invertible is crucial in the example.

Lemma 9.22. *Let $\mathcal{S}$ be a symmetric freely open set in $\mathbb{M}^2$ and f a symmetric free holomorphic function on $\mathcal{S}$.*

(1) *f can be approximated locally uniformly on $\mathcal{S}$ in the free topology by a sequence of symmetric free polynomials;*

(2) *Suppose that (i) $w_1, w_2 \in \mathcal{S}$ (ii) $\pi(w_1) = s^{-1}\pi(w_2)s$ for some $s \in \mathcal{I}$ (iii) $w_1^1 - w_1^2$ is invertible and (iv) $w_1 \oplus w_2 \oplus w_1^f \oplus w_2^f \in \mathcal{S}$; then $f(w_1) = s^{-1}f(w_2)s$.*

Proof. (i) Consider any point $w_0 \in \mathcal{S}$. Since f is freely holomorphic on $\mathcal{S}$, by Theorem 3.6 there is a basic free neighborhood B_δ of w_0 in $\mathcal{S}$ and a sequence of free polynomials p_n that approximates f uniformly on B_δ. Replacing $\delta(w)$ by $\delta(w) \oplus \delta(w^f)$ if necessary, we can assume that B_δ is symmetric, and on replacing $p_n(w)$ by $\frac{1}{2}(p_n(w) + p_n(w^f))$, we can assume that p_n is symmetric on B_δ.

(ii) Let

$$u_1 = \tfrac{1}{2}(w_1^1 + w_1^2), \quad v_1 = \tfrac{1}{2}(w_1^1 - w_1^2), \quad u_2 = \tfrac{1}{2}(w_2^1 + w_2^2), \quad v_2 = \tfrac{1}{2}(w_2^1 - w_2^2).$$

Every p_n, being symmetric, can be written as a finite linear combination of terms $1, u$, and $vu^j v$ for $j = 0, 1, 2, \ldots$.

By hypothesis, $\pi(w_1) = s^{-1}\pi(w_2)s$, which means that $u_1 = s^{-1}u_2 s$ and $v_1 u_1^j v_1 = s^{-1}v_2 u_2^j v_2 s$ for $j = 0, 1$. But for $j \geq 2$, we have

$$v_1 u_1^j v_1 = v_1 u_1^{j-1} v_1 (v_1)^{-2} v_1 u_1 v_1,$$

and so by induction we deduce that $p_n(w_1) = s^{-1}p_n(w_2)s$. Since $p_n(w_1)$ converges to $f(w_1)$ and $p_n(w_2)$ converges to $f(w_2)$, it follows that $f(w_1) = s^{-1}f(w_2)s$. $\qquad\square$

Lemma 9.23. *The set of points $w \in \mathbb{M}^2$ such that $\pi^{-1}(\pi(w))$ has cardinality 2 is dense in the finitely open topology.*

Proof. Let $2u = w^1 + w^2$ and $2v = w^1 - w^2$. The set of $w \in \mathbb{M}_n^2$ for which v has n distinct non-zero eigenvalues is dense, and the subset where furthermore u has no non-zero entries when expressed with respect to the basis of eigenvectors of v is still dense.

For such a w, if $\pi(w_1) = \pi(w)$, then $u = u_1$, $(v)^2 = (v_1)^2$ and $vuv = v_1 u v_1$. The second equation says v_1 has the same eigenvectors as v, with each eigenvalue $\pm$ the corresponding eigenvalue for v. If the ith is positive and the jth is negative, then the (i,j) entry of vuv is minus the (i,j) entry of $v_1 u v_1$, and is non-zero since u_{ij} is non-zero.

Thus $v_1 = \pm v$, and hence w_1 is either w or w^f. $\qquad\square$

9.7 Conditionally nc functions

We shall call a set $D \subset \mathbb{M}^d$ *reductive* if, whenever $x \oplus y \in D$, then x and y are in D.

Proposition 9.24. *Freely open sets are reductive.*

Proof. Let D be a freely open set. Consider z, w such that $z \oplus w \in D$. Then there is a matrix δ of free polynomials such that $z \oplus w \in B_\delta \subseteq D$. Since $\delta(z \oplus w) = \delta(z) \oplus \delta(w)$, it follows that $z, w \in B_\delta$, and hence $z, w \in D$. $\qquad\square$

Zariski-freely open sets, however, need not be reductive.

Definition 9.25. *Let D be a subset of $\mathbb{M}^d$ and f be a mapping from D to $\mathbb{M}$. We say that f is* conditionally nc *if f is a graded function and*

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(i) *if* $x, s^{-1}xs \in D$, *then* $f(s^{-1}xs) = s^{-1}f(x)s$;

(ii) *there exists a graded function* $\hat{f}$ *defined on the set*

$$\hat{D} \overset{\text{def}}{=} \{y \in \mathbb{M}^d : \text{ there exists } x \in D \text{ such that } x \oplus y \in D\} \qquad (9.26)$$

such that, for all $x \in D$ *such that* $x \oplus y \in D$,

$$f\left(\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\right) = \begin{bmatrix} f(x) & 0 \\ 0 & \hat{f}(y) \end{bmatrix}. \qquad (9.27)$$

Example 9.28. (1) If $D = \{3 \oplus 2 \oplus 1\}$ then $\hat{D}$ is empty and the conditionally nc functions on D are the functions f on D given by

$$f(3 \oplus 2 \oplus 1) = A$$

for some diagonal matrix $A \in \mathbb{M}_3^1$. The fact that A is diagonal ensures that condition (i) of Definition 9.25 is satisfied, while condition (ii) is vacuous.

(2) If $D = \{3 \oplus 2 \oplus 1, 3\}$ then $\hat{D} = \{2 \oplus 1\}$ and the conditionally nc functions on D are the functions f on D given by

$$f(3 \oplus 2 \oplus 1) = a \oplus b \oplus c, \qquad f(3) = a$$

for some $a, b, c \in \mathbb{C}$. The unique graded function $\hat{f}$ on $\hat{D}$ such that condition (ii) holds is $\hat{f}(2 \oplus 1) = b \oplus c$.

The following properties of conditionally nc functions are straightforward to verify.

Proposition 9.29. *Let* f *be a conditionally nc function on* $D \subset \mathbb{M}^d$ *and let* $D_1 \subseteq D$.

(1) *There is a* unique *graded function* $\hat{f}$ *on* $\hat{D}$ *such that equation* (9.27) *holds.*

(2) *If* $\|f(x)\| \leq M$ *for all* $x \in D$ *then* $\|\hat{f}(y)\| \leq M$ *for all* $y \in \hat{D}$.

(3) $\hat{f}$ *agrees with* f *on* $D \cap \hat{D}$.

(4) $\hat{D}_1 \subseteq \hat{D}$.

(5) *If* $f_1 = f|D_1$ *then* f_1 *is conditionally nc and* $\hat{f}_1 = \hat{f}|\hat{D}_1$.

Remark 9.30. A bounded conditionally nc function f on a free domain U is freely holomorphic. For any $x \in U$ we may choose a basic neighborhood B_δ of x contained in U; then f is conditionally nc on B_δ, hence is nc on B_δ. Thus f is freely locally nc.

On the other hand, we do not know if a freely holomorphic function on a free domain U must be conditionally nc, since being conditionally nc is not a local property. Such a function does preserve direct sums and similarities on basic free neighborhoods, but we do not assert that it preserves similarities on all of U.

On a reductive set D, a function f is conditionally nc if and only if it preserves similarities and satisfies $f(x \oplus y) = f(x) \oplus f(y)$ whenever $x \oplus y \in D$. If D is both nc and reductive, then a conditionally nc function on D is the same as an nc function.

10 Nc Waring-Lagrange theorems

Any polynomial in d commuting variables can be expressed as a polynomial in the d elementary symmetric functions in the variables. In an interesting historical survey [10, pp. 364-365], H. G. Funkhauser attributes this fundamental fact about symmetric functions to Edward Waring, Lucasian Professor at Cambridge [24] in 1770 and independently to Joseph Lagrange [18] in 1798. Lagrange's account is later, but is somewhat more explicit, so we shall call the statement the *Waring-Lagrange theorem*. It is likely that Euler had the result around the same time.

We now come to our nc version of the theorem, which asserts (to oversimplify somewhat) that symmetric nc functions in two variables can be factored through the map π given by the definition (8.34). The precise statement is as follows.

In the theorem we do not assume that $\mathcal{S}$ is an nc set.

Theorem 10.1. *Let $\mathcal{S}$ be a symmetric free domain in $\mathbb{M}^2$. Assume that, for all $w_1, w_2 \in \mathcal{S}$,*

$$\text{if } \pi(w_1) \text{ is similar to } \pi(w_2) \text{ then } w_1 \oplus w_2 \in \mathcal{S}. \qquad (10.2)$$

Define $\mathcal{S}_o$ by equation (9.16) and $\mathcal{S}_{oo}$ by equation (9.12), and let $\mathfrak{G}_{oo}$ be the Zariski-free manifold $\mathfrak{G}_{oo}(\mathcal{S})$ defined in equation (9.8). Let $\mathfrak{G}_o = \pi(\mathcal{S}_o)$.

There are canonical bijections between the following five sets of graded functions.

(i) Bounded symmetric conditionally nc functions f defined on $\mathcal{S}$.

(ii) Bounded symmetric conditionally nc functions f_o defined on $\mathcal{S}_o$.

(iii) Bounded symmetric Zariski-freely holomorphic functions f_{oo} defined on $\mathcal{S}_{oo}$ that are conditionally nc.

(iv) Bounded holomorphic functions F_{oo} defined on the Zariski-free manifold $\mathfrak{G}_{oo}$ that are conditionally nc (when $\mathfrak{G}_{oo}$ is viewed as a subset of $\mathbb{M}^3$).

(v) Bounded functions F_o defined on $\mathfrak{G}_o$ that are conditionally nc.

Moreover, if functions f, f_o, f_{oo}, F_o and F_{oo} correspond under these canonical bijections, then the following diagram commutes when the two copies of $\mathbb{M}^1$ are identified.

$$
\begin{array}{ccc}
& \mathcal{S}_{oo} \xrightarrow{\pi} \mathfrak{G}_{oo} & \\
f_{oo} \swarrow \quad \downarrow{\subseteq} \qquad \downarrow{\subseteq} \quad \searrow F_{oo} & \\
\mathbb{M}^1 \xleftarrow{f_o} \mathcal{S}_o \xrightarrow{\pi} \mathfrak{G}_o \xrightarrow{F_o} \mathbb{M}^1 & \\
\nwarrow{f} \quad \downarrow{\subseteq} & \\
\mathcal{S} &
\end{array} \qquad (10.3)
$$

Proof. Starting with f, one can define f_o and f_{oo} by restriction. By Proposition 9.29, f_o and f_{oo} are conditionally nc. By Remark 9.30, f is freely holomorphic, and so f_{oo} is Zariski-freely holomorphic, being the restriction of the freely holomorphic function f to a Zariski-freely open set.

$(f_o \to f)$ First note that at each level n, the set $\mathcal{S} \setminus \mathcal{S}_o$ is thin, in the sense of Subsection 7.1. Indeed, to be in $\mathcal{S} \setminus \mathcal{S}_o$, by Proposition 9.17, we must have $w^1 - w^2 \notin \mathcal{Q}$. A matrix M is not in $\mathcal{Q}$ if and only if $p_1(\lambda) = \det(\lambda - M)$ and $p_2(\lambda) = \det(\lambda + M)$ have a common zero. This happens if and only if their resultant, which is a polynomial in the entries of M, vanishes. So $\mathcal{S} \setminus \mathcal{S}_o$ is the intersection of $\mathcal{S}$ with the zero set of a non-constant polynomial, hence is thin.

Since f_o is bounded and $\mathcal{S} \setminus \mathcal{S}_o$ is thin, the function f_o has an extension by continuity at each level n to a bounded function f on $\mathcal{S}$. By continuity, the function f is symmetric and preserves direct sums, in the sense that if $w \oplus y \in \mathcal{S}$ then $w, y \in \mathcal{S}$ (since $\mathcal{S}$ is a free domain) and $f(w \oplus y) = f(w) \oplus f(y)$.

Suppose $w = s^{-1} y s$ for some $w, y \in \mathcal{S}$ and $s \in \mathcal{I}$. Let $w_n \in \mathcal{S}_o$ converge to w, and let $y_n := s w_n s^{-1}$. For n large enough, $y_n \in \mathcal{S}$, and since w_n is in

$\mathcal{S}_o$, so is y_n by Proposition 9.17. Then $f_o(w_n) = s^{-1} f_o(y_n) s$, and so in the limit we obtain $f(w) = s^{-1} f(y) s$.

$(f_{oo} \to f_o)$ Starting with f_{oo}, construct f_o as follows.

First show that $\mathcal{S}_o \subseteq \widehat{\mathcal{S}_{oo}}$. Consider $w \in \mathcal{S}_o$ and let

$$u = \tfrac{1}{2}(w^1 + w^2), \quad v = \tfrac{1}{2}(w^1 - w^2),$$

so that v is nonsingular, by Proposition 9.17. Let γ be the spectrum of v^2; then $0 \notin \gamma$ and $(u, v^2) \in \mathcal{W}_\gamma$ by equation (8.12). Since v is a square root of v^2, by Subsection 8.3 there exists $\tau \in \{-1, 1\}^\gamma$ such that $v = S_{\gamma\tau}(v^2)$. By the definition (9.2) of $\omega_{\gamma\tau}$,

$$\omega_{\gamma\tau}(u, v^2) = (u + S_{\gamma\tau}(v^2), u - S_{\gamma\tau}(v^2)) = (u + v, u - v) = (w^1, w^2) = w.$$

Thus $w \in \omega_{\gamma\tau}(\mathcal{W}_\gamma)$.

Choose $\lambda \in \omega_{\gamma\tau}(\mathcal{U}_\gamma)$. Then by Proposition 8.15, $\lambda \oplus w \in \omega_{\gamma\tau}(\mathcal{U}_\gamma) \subseteq \mathcal{S}_{oo}$. Thus $\lambda, \lambda \oplus w \in \mathcal{S}_{oo}$, which is to say that $w \in \widehat{\mathcal{S}_{oo}}$. Thus

$$\mathcal{S}_{oo} \subseteq \mathcal{S}_o \subseteq \widehat{\mathcal{S}_{oo}}.$$

We claim that

$$\widehat{\mathcal{S}_o} = \widehat{\mathcal{S}_{oo}}. \tag{10.4}$$

The inclusion $\widehat{\mathcal{S}_o} \supseteq \widehat{\mathcal{S}_{oo}}$ is immediate. Consider any $y \in \widehat{\mathcal{S}_o}$. Then there exists $w \in \mathcal{S}_o$ such that $w \oplus y \in \mathcal{S}_o$. By the above construction there exists $\gamma \in \Gamma$ and $\tau \in \{-1, 1\}^\gamma$ such that $w \oplus y \in \omega_{\gamma\tau}(\mathcal{W}_\gamma)$. Let $\lambda \in \omega_{\gamma\tau}(\mathcal{U}_\gamma)$; then, again by Proposition 8.15, $\lambda \oplus w$ and $\lambda \oplus w \oplus y$ belong to $\mathcal{S}_{oo}$. Hence $y \in \widehat{\mathcal{S}_{oo}}$.

Define f_o to be the restriction of $\widehat{f_{oo}}$ to $\mathcal{S}_o$. By Proposition 9.29(3), f_o is an extension of f_{oo}.

Since f_{oo} is conditionally nc, there is a function $\widehat{f_{oo}}$ on $\widehat{\mathcal{S}_{oo}}$ such that

$$f_{oo}\left(\begin{bmatrix} \lambda & 0 \\ 0 & w \end{bmatrix}\right) = \begin{bmatrix} f_{oo}(\lambda) & 0 \\ 0 & \widehat{f_{oo}}(w) \end{bmatrix}. \tag{10.5}$$

As f_{oo} is bounded, so is f_o, by Proposition 9.29(2) and, since f_{oo} is symmetric, so is f_o.

To show that f_o is conditionally nc we must exhibit a function $\widehat{f_o}$ on $\widehat{\mathcal{S}_o}$ such that, for all $y \in \widehat{\mathcal{S}_o}$ and all $w \in \mathcal{S}_o$ such that $w \oplus y \in \mathcal{S}_o$,

$$f_o\left(\begin{bmatrix} w & 0 \\ 0 & y \end{bmatrix}\right) = \begin{bmatrix} f_o(w) & 0 \\ 0 & \widehat{f_o}(y) \end{bmatrix}. \tag{10.6}$$

In view of equation (10.4), we may define $\widehat{f_o}$ to be $\widehat{f_{oo}}$. Then we must show that

$$\widehat{f_{oo}}\left(\begin{bmatrix} w & 0 \\ 0 & y \end{bmatrix}\right) = \begin{bmatrix} \widehat{f_{oo}}(w) & 0 \\ 0 & \widehat{f_{oo}}(y) \end{bmatrix} \tag{10.7}$$

whenever $w, w \oplus y \in \mathcal{S}_o$. As above there exists $\lambda \in \mathcal{S}_{oo}$ such that $\lambda \oplus w \oplus y \in \mathcal{S}_{oo}$, which means that $\lambda \oplus w$ is also in $\mathcal{S}_{oo}$. Thus

$$
\begin{aligned}
\begin{bmatrix} f_{oo}(\lambda) & 0 \\ 0 & \widehat{f_{oo}}\left(\begin{bmatrix} w & 0 \\ 0 & y \end{bmatrix}\right) \end{bmatrix} &= f_{oo}\left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & w & 0 \\ 0 & 0 & y \end{bmatrix}\right) \\
&= \begin{bmatrix} f_{oo}\left(\begin{bmatrix} \lambda & 0 \\ 0 & w \end{bmatrix}\right) & 0 \\ 0 & \widehat{f_{oo}}(y) \end{bmatrix} \\
&= \begin{bmatrix} f_{oo}(\lambda) & 0 & 0 \\ 0 & \widehat{f_{oo}}(w) & 0 \\ 0 & 0 & \widehat{f_{oo}}(y) \end{bmatrix}.
\end{aligned}
$$

It follows that equation (10.7) holds, as required.

To see that f_o preserves similarities, we argue as follows. Let $w, s^{-1}ws = y \in \mathcal{S}_o$. By assumption (10.2) we have $w \oplus y \in \mathcal{S}$. As w and y have the same spectrum, By Proposition 9.17, $w \oplus y \in \mathcal{S}_o$. Hence there exists $\lambda \in \mathcal{S}_{oo}$ such that $\lambda \oplus w \oplus y$ is in $\mathcal{S}_{oo}$, as are $\lambda \oplus w$ and $\lambda \oplus y$. Since f_{oo} preserves similarities and

$$\begin{bmatrix} 1 & 0 \\ 0 & s^{-1} \end{bmatrix}\begin{bmatrix} \lambda & 0 \\ 0 & w \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & s^{-1}ws \end{bmatrix},$$

we find that $f_o(s^{-1}ws) = s^{-1}f_o(w)s$. Thus f_o is conditionally nc.

$(f_o \to F_o)$ Consider any bounded symmetric conditionally nc function f_o on $\mathcal{S}_o$. By the previous construction, f_o has a bounded symmetric conditionally nc extension f to $\mathcal{S}$. By Remark 9.30, f is freely holomorphic.

We claim that if $\pi(w_1) = \pi(w_2)$ for some pair w_1, w_2 in $\mathcal{S}_o$, then $f(w_1) = f(w_2)$. Indeed, by assumption (10.2), $w_1 \oplus w_2 \in \mathcal{S}$. By the symmetry of $\mathcal{S}$, $(w_1 \oplus w_2)^f \in \mathcal{S}$, which is to say that $w_1^f \oplus w_2^f \in \mathcal{S}$. By the symmetry of π, $\pi(w_1 \oplus w_2) = \pi((w_1 \oplus w_2)^f)$. Again by assumption (10.2), $w_1 \oplus w_2 \oplus w_1^f \oplus w_2^f \in \mathcal{S}$. Furthermore, since $w_1 \in \mathcal{S}_o$, it follows from Proposition 9.17 that $w_1^1 - w_1^2$ is invertible. We may therefore apply Lemma 9.22 to conclude that $f(w_1) = f(w_2)$. Hence $f_o(w_1) = f_o(w_2)$.

Thus, at the level of set theory, we can define F_o by $f_o = F_o \circ \pi$. To see that F_o is conditionally nc, suppose first that $\pi(w)$ and $\pi(w) \oplus \pi(y)$ are in $\mathfrak{G}_o$, for some $w, w \oplus y \in \mathcal{S}_o$. Then $y \in \mathcal{S}_o$ also, by Proposition 9.17. Thus

$$
\begin{aligned}
F_o \left(\begin{bmatrix} \pi(w) & 0 \\ 0 & \pi(y) \end{bmatrix} \right) &= f_o \left(\begin{bmatrix} w & 0 \\ 0 & y \end{bmatrix} \right) \\
&= \left(\begin{bmatrix} f_o(w) & 0 \\ 0 & f_o(y) \end{bmatrix} \right) \\
&= \left(\begin{bmatrix} F_o(\pi(w)) & 0 \\ 0 & F_o(\pi(y)) \end{bmatrix} \right).
\end{aligned}
$$

To show that F_o preserves similarities, consider similar triples z_1 and $z_2 = sz_1s^{-1}$ in $\mathfrak{G}_o$. Then there exist $w_1, w_2 \in \mathcal{S}_o$ such that $z_1 = \pi(w_1), z_2 = \pi(w_2)$. We have

$$
\begin{aligned}
z_2 &= sz_1s^{-1} \\
&= s\pi(w_1)s^{-1} = \pi(sw_1s^{-1}).
\end{aligned}
\tag{10.8}
$$

If it happens that sw_1s^{-1} is in $\mathcal{S}_o$, then

$$
\begin{aligned}
F_o(z_2) &= F_o \circ \pi(sw_1s^{-1}) \\
&= f_o(sw_1s^{-1}) \\
&= sf_o(w_1)s^{-1} \quad \text{since } f_o \text{ is conditionally nc} \\
&= sF_o \circ \pi(w_1)s^{-1} \\
&= sF_o(z_1)s^{-1}.
\end{aligned}
\tag{10.9}
$$

Not knowing that $sw_1s^{-1} \in \mathcal{S}_o$, we use Lemma 9.22. By Assumption (10.2), $w_1 \oplus w_2 \oplus w_1^f \oplus w_2^f \in \mathcal{S}$, and since $w_1 \in \mathcal{S}_o$, the lemma tells us that $f(w_2) = s^{-1}f(w_1)s$, and so $F_o(z_2) = s^{-1}F_o(z_1)s$, as required.

$(F_o \to f_o)$ Define f_o to be $F_o \circ \pi$. Then f_o is bounded, symmetric, and conditionally nc.

$(F_o \to F_{oo})$ By restriction.

$(F_{oo} \to f_{oo})$ Define f_{oo} to be $F_{oo} \circ \pi$.

$(F_{oo} \to F_o)$ By the composition $F_{oo} \to f_{oo} \to f_o \to F_o$. $\qquad\square$

The set $\mathcal{S}_o$ is not closed with respect to direct sums, that is, it is not an nc set. If $\mathcal{S}$ in Theorem 10.1 is assumed to be a symmetric freely open nc set, then assumption (10.2) is automatically satisfied, and the conditionally nc functions on $\mathcal{S}$ are the same as the nc functions. This yields the following corollary.

Corollary 10.10. *Let $\mathcal{S}$ be a symmetric freely open nc set in $\mathbb{M}^2$. Define $\mathcal{S}_o$ by equation (9.16), and let $\mathfrak{G}_o = \pi(\mathcal{S}_o)$. Then there is a canonical bijection between the bounded symmetric nc functions f on $\mathcal{S}$ and the bounded conditionally nc functions F_o defined on $\mathfrak{G}_o$ such that $F_o = (f|\mathcal{S}_o) \circ \pi$.*

If we drop the assumption that f be bounded, and require only that f be freely locally bounded, then there are corresponding changes made to the requirements of the other functions. Recall that in Definition 7.1 we defined a function φ defined on $U \setminus T$ to be locally bounded on U if, for every point z in U, there is a neighborhood V of z such that φ is bounded on $V \setminus T$. The theorem becomes:

Theorem 10.11. *Let $\mathcal{S}$ be a symmetric free domain in $\mathbb{M}^2$. Assume condition (10.2).*

There are canonical bijections between the following five sets of graded functions.

(i) Symmetric conditionally nc functions f that are freely holomorphic on $\mathcal{S}$.

(ii) Symmetric conditionally nc functions f_o defined on $\mathcal{S}_o$ that are freely locally bounded on $\mathcal{S}$.

(iii) Symmetric Zariski-freely holomorphic functions f_{oo} defined on $\mathcal{S}_{oo}$ that are conditionally nc and freely locally bounded on $\mathcal{S}$.

(iv) Holomorphic functions F_{oo} defined on the Zariski-free manifold $\mathfrak{G}_{oo}$ that are conditionally nc and have the property that for all w in $\mathcal{S}$, there is a free neighborhood U of w such that F_{oo} is bounded on $\pi(U) \cap \mathfrak{G}_{oo}$.

(v) Bounded graded functions F_o defined on $\mathfrak{G}_o$ that are conditionally nc and have the property that for all w in $\mathcal{S}$, there is a free neighborhood U of w such that F_o is bounded on $\pi(U) \cap \mathfrak{G}_o$.

Moreover, if functions f, f_o, f_{oo}, F_o and F_{oo} correspond under these canonical bijections, then the diagram (10.3) commutes when the two copies of $\mathbb{M}^1$ are identified.

11 Nc Newton-Girard formulae

Instances of the Waring-Lagrange theorem are furnished by a series of formulae for power sums in terms of elementary symmetric functions. Classically such formulae were first given in 1629 by Albert Girard [11], though they were subsequently often attributed to Newton [19]. We are only concerned

with polynomials in *two* variables x and y. When these variables commute, the Newton-Girard formulae express the power sums $p_n(x, y) \overset{\text{def}}{=} x^n + y^n$ in terms of the elementary symmetric functions

$$e_1(x, y) = x + y, \qquad e_2(x, y) = xy.$$

The first four formulae are

$$p_1 = e_1$$
$$p_2 = e_1^2 - 2e_2$$
$$p_3 = e_1^3 - 3e_1 e_2$$
$$p_4 = e_1^4 - 4e_1^2 e_2 + 2e_2^2.$$

Further formulae are obtained from the recursion $p_{n+2} = e_1 p_{n+1} - e_2 p_n$.

In the case of non-commuting indeterminates x, y we retain the notation p_n for the nth power sum, but now there is no finite set of 'elementary symmetric functions' in terms of which all p_n can be written as polynomials. However, the foregoing nc Waring-Lagrange theorems tell us that it *is* possible to express p_n as a *rational* expression in the variables

$$\alpha = u, \quad \beta = v^2, \quad \gamma = vuv \tag{11.1}$$

where

$$u = \tfrac{1}{2}(x + y), \qquad v = \tfrac{1}{2}(x - y).$$

In this section we shall show how to construct such expressions explicitly, in the spirit of Girard and Newton. Since we are obliged to work with rational expressions, it is natural to allow the n in p_n to be an arbitrary integer, positive, negative or zero. We first express p_n and the antisymmetric rational function

$$q_n \overset{\text{def}}{=} x^n - y^n$$

in terms of u and v. We have $p_0 = 2$, $q_0 = 0$. For any integer n,

$$p_n = xx^{n-1} + yy^{n-1} = (u + v)x^{n-1} + (u - v)y^{n-1}$$
$$= u(x^{n-1} + y^{n-1}) + v(x^{n-1} - y^{n-1})$$
$$= up_{n-1} + vq_{n-1}. \tag{11.2}$$

Similarly,

$$q_n = vp_{n-1} + uq_{n-1}. \tag{11.3}$$

We may write equations (11.2) and (11.3) in the matrix form

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = T \begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix} \qquad \text{for all } n \in \mathbb{Z} \tag{11.4}$$

where

$$T = \begin{bmatrix} u & v \\ v & u \end{bmatrix}. \tag{11.5}$$

Define the free polynomial s^n_{even} in u, v for $n \geq 0$ to be the sum of all monomials in u, v of total degree n and of even degree in v. When regarded as a polynomial in x, y, s^n_{even} is symmetric. Likewise, s^n_{odd} is defined to be the sum of all monomials in u, v of total degree n and odd degree in v. Thus s^n_{odd} is antisymmetric as a polynomial in x, y.

By induction, for $n \geq 1$,

$$T^n = \begin{bmatrix} s^n_{\text{even}} & s^n_{\text{odd}} \\ s^n_{\text{odd}} & s^n_{\text{even}} \end{bmatrix}. \tag{11.6}$$

By iteration of equation (11.4),

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = T^n \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = T^n \begin{pmatrix} 2 \\ 0 \end{pmatrix}. \tag{11.7}$$

From equation (11.6) we obtain, for $n \geq 0$,

$$p_n = 2s^n_{\text{even}} \tag{11.8}$$
$$q_n = 2s^n_{\text{odd}}. \tag{11.9}$$

Thus

$$p_1 = 2s^1_{\text{even}} = 2u = 2\alpha$$
$$p_2 = 2s^2_{\text{even}} = 2(u^2 + v^2) = 2(\alpha^2 + \beta)$$
$$p_3 = 2s^3_{\text{even}} = 2(u^3 + uv^2 + vuv + v^2u) = 2(\alpha^3 + \alpha\beta + \gamma + \beta\alpha)$$
$$p_4 = 2s^4_{\text{even}} = 2(u^4 + u^2v^2 + uvuv + vu^2v + uv^2u + vuvu + v^2u^2 + v^4)$$
$$= 2(\alpha^4 + \alpha^2\beta + \alpha\gamma + \gamma\beta^{-1}\gamma + \alpha\beta\alpha + \gamma\alpha + \beta\alpha^2 + \beta^2).$$

In general, any monomial in which v occurs with even degree can be written as a monomial in α, β, γ and β^{-1}. Indeed, starting at one end of the monomial, replace all the initial u's by α's. The first v must be followed by another

(since the number of v's is even). If it is immediately following, replace v^2 by β. If there are k u's between the first and second v's, replace $vu^k v$ by $(\gamma\beta^{-1})^{k-1}\gamma$. Continue in this way until all u's and v's have been replaced.

We have shown the following.

Theorem 11.10. (An nc Newton-Girard theorem) *For every positive integer n there exists a rational function P_n in three nc variables such that $p_n = P_n \circ \pi$. Moreover $P_n(\alpha, \beta, \gamma)$ can be expressed as a free polynomial in α, β, γ and β^{-1}. The functions P_n can be calculated from the equation (11.7) or recursively from the relations*

$$P_0(\alpha, \beta, \gamma) = 2, \qquad Q_0(\alpha, \beta, \gamma) = 0$$

and, for $n \geq 0$,

$$P_{n+1}(\alpha, \beta, \gamma) = \beta P_n + Q_n$$
$$Q_{n+1}(\alpha, \beta, \gamma) = \beta P_n + \gamma\beta^{-1}Q_n.$$

The first four P_n are given by

$$P_1 = 2\alpha$$
$$P_2 = 2(\alpha^2 + \beta)$$
$$P_3 = 2(\alpha^3 + \alpha\beta + \gamma + \beta\alpha)$$
$$P_4 = 2(\alpha^4 + \alpha^2\beta + \alpha\gamma + \gamma\beta^{-1}\gamma + \alpha\beta\alpha + \gamma\alpha + \beta\alpha^2 + \beta^2). \qquad (11.11)$$

Now consider sums of negative powers. By equation (11.8),

$$\begin{pmatrix} p_{-1} \\ q_{-1} \end{pmatrix} = T^{-1} \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}. \qquad (11.12)$$

Take inverses of both sides of the identity

$$T = \begin{bmatrix} 1 & 0 \\ vu^{-1} & 1 \end{bmatrix} \begin{bmatrix} u & 0 \\ 0 & u - vu^{-1}v \end{bmatrix} \begin{bmatrix} 1 & u^{-1}v \\ 0 & 1 \end{bmatrix},$$

to get

$$\begin{aligned}
T^{-1} &= \begin{bmatrix} 1 & -u^{-1}v \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u^{-1} & 0 \\ 0 & (u - vu^{-1}v)^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -vu^{-1} & 1 \end{bmatrix} \\
&= \begin{bmatrix} u^{-1} + u^{-1}v(u - vu^{-1}v)^{-1}vu^{-1} & -u^{-1}v(u - vu^{-1}v)^{-1} \\ -(u - vu^{-1}v)^{-1}vu^{-1} & (u - vu^{-1}v)^{-1} \end{bmatrix}.
\end{aligned}$$

This expression can be simplified, using the identities

$$
\begin{aligned}
u^{-1}v(u - vu^{-1}v)^{-1} &= (u - vu^{-1}v)^{-1}vu^{-1} \\
u^{-1} + u^{-1}v(u - vu^{-1}v)^{-1}vu^{-1} &= (u - vu^{-1}v)^{-1} \\
-u^{-1}v(u - vu^{-1}v)^{-1} &= (v - uv^{-1}u)^{-1},
\end{aligned}
$$

to obtain the formula

$$
T^{-1} = \begin{bmatrix} (u - vu^{-1}v)^{-1} & (v - uv^{-1}u)^{-1} \\ (v - uv^{-1}u)^{-1} & (u - vu^{-1}v)^{-1} \end{bmatrix}. \tag{11.13}
$$

From this formula and equation (11.12) we deduce that

$$
\begin{pmatrix} p_{-1} \\ q_{-1} \end{pmatrix} = T^{-1} \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} (u - vu^{-1}v)^{-1} \\ (v - uv^{-1}u)^{-1} \end{pmatrix}.
$$

Hence

$$
p_{-1} = 2(u - vu^{-1}v)^{-1} = 2(\alpha - \beta\gamma^{-1}\beta)^{-1}. \tag{11.14}
$$

Although q_{-1} is not expressible as a rational function of α, β, γ the product vq_{-1} is.

$$
vq_{-1} = -\beta\gamma^{-1}\beta(\alpha - \beta\gamma^{-1}\beta)^{-1} \tag{11.15}
$$

This suggests the following statement.

Lemma 11.16. *For every positive integer n*

$$
T^{-n} = \begin{bmatrix} f_n(u, v) & g_n(u, v) \\ g_n(u, v) & f_n(u, v) \end{bmatrix}
$$

where f_n, g_n are rational functions such that $f_n(u, v)$ and $vg_n(u, v)$ are expressible as rational functions of α, β, γ.

Proof. The statement is true when $n = 1$, as is easily seen from equations (11.14) and (11.15).

Suppose it true for some $n \geq 1$. From the relation $T^{-(n+1)} = T^{-1}T^{-n}$ we have the equations

$$
\begin{aligned}
f_{n+1}(u, v) &= (u - vu^{-1}v)^{-1}f_n(u, v) + (v - uv^{-1}u)^{-1}g_n(u, v), \\
g_{n+1}(u, v) &= (v - uv^{-1}u)^{-1}g_n(u, v) + (u - vu^{-1}v)^{-1}f_n(u, v). \tag{11.17}
\end{aligned}
$$

The assertion follows by induction and the relations

$$
\begin{aligned}
f_{n+1} &= (u - vu^{-1}v)^{-1}f_n + (v - uv^{-1}u)^{-1}v^{-1}vg_n \\
&= (\alpha - \beta\gamma^{-1}\beta)^{-1}f_n + (\beta - \gamma\beta^{-1}\alpha)^{-1}vg_n, \\
vg_{n+1} &= v(u - vu^{-1}v)^{-1}g_n + v(v - uv^{-1}u)^{-1}f_n \\
&= (vuv^{-1} - v^2u^{-1})^{-1}vg_n + (1 - uv^{-1}uv^{-1})^{-1}f_n \\
&= (\gamma\beta^{-1} - \beta\alpha^{-1})^{-1}vg_n + (1 - \alpha\beta^{-1}\gamma\beta^{-1})^{-1}f_n.
\end{aligned}
$$

$\square$

Theorem 11.18. (A Newton-Girard theorem for negative powers)
For every non-negative integer n there exists a rational function P_{-n} in three non-commuting variables such that $p_{-n} = P_{-n} \circ \pi$. The functions P_{-n} are given recursively by the formulae

$$
P_0(\alpha, \beta, \gamma) = 2, \qquad Q_0(\alpha, \beta, \gamma) = 0
$$

and

$$
\begin{aligned}
P_{-(n+1)}(\alpha, \beta, \gamma) &= (\alpha - \beta\gamma^{-1}\beta)^{-1}P_{-n} + (\beta - \gamma\beta^{-1}\alpha)^{-1}Q_{-n} \\
Q_{-(n+1)}(\alpha, \beta, \gamma) &= \beta(\beta - \alpha\beta^{-1}\gamma)^{-1}P_{-n} + \beta(\gamma - \beta\alpha^{-1}\beta)^{-1}Q_{-n} \qquad (11.19)
\end{aligned}
$$

for $n \geq 0$.

Proof. Replace n by $-n$ in equation (11.7) to obtain

$$
\begin{pmatrix} p_{-n} \\ q_{-n} \end{pmatrix} = 2T^{-n} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
$$

Thus

$$
p_{-n} = 2f_n(u, v)
$$

in the notation of Lemma 11.16. By that lemma, p_{-n} is expressible as a rational function of the variables α, β, γ. $\square$

One can readily calculate the first two P_{-n} from these formulae; we expect that P_{-2} can be further simplified.

$$
P_{-1}(\alpha, \beta, \gamma) = 2(\alpha - \beta\gamma^{-1}\beta)^{-1} \qquad (11.20)
$$

$$
\begin{aligned}
P_{-2}(\alpha, \beta, \gamma) = 2 \big(&\alpha^2 + \beta - \alpha\beta(\beta^{-1}\gamma + \gamma^{-1}\beta^2)^{-1} \\
&- \gamma(\gamma^{-1}\beta\gamma + \beta^2)^{-1}\gamma - \alpha\beta(\gamma\beta^{-1}\gamma + \beta^2)^{-1}\beta\alpha \\
&-(\beta^{-1}\gamma\beta^{-1} + \beta\gamma^{-1})^{-1}\alpha\big)^{-1}. \qquad (11.21)
\end{aligned}
$$

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There are some minor subtleties concerning the interpretation of the foregoing Newton-Girard formulae for p_n. When $n \geq 0$, since P_n is a free polynomial in $\alpha, \beta, \gamma, \beta^{-1}$, the statement

$$x^n + y^n = P_n(\alpha, \beta, \gamma)$$

is meaningful and valid whenever x and y are matrices of the same order such that $x - y$ is nonsingular. It can also be interpreted as an identity in the *free field*, which is the smallest universal division ring containing the ring of free polynomials in x, y. When $n < 0$ the issue is less immediate, since then the structure of P_n is more complicated. In this paper we are concerned with functions of tuples of matrices. In this context, a noncommutative rational expression is called *non-degenerate* if its domain in $\mathbb{M}^d$ is non-empty. The domain will always be Zariski open at every level n (restrictions on the domain come about when there is an inverse in the expression, as whatever needs to be inverted must be non-singular), but this does not imply Zariski-freely open in our sense. Different nondegenerate noncommutative rational expressions may have different domains where they can be evaluated, but agree on the intersection of these domains. Such expressions are called equivalent, and a noncommutative rational function is formally an equivalence class of nondegenerate noncommutative rational expressions. See for example [23] for a discussion. For the expressions P_{-n} it is easy to see that they are non-degenerate; from equation (11.19) we see that the functions can be evaluated as long as all four of the expressions

$$\alpha - \beta\gamma^{-1}\beta, \quad \beta - \gamma\beta^{-1}\alpha, \quad \beta - \alpha\beta^{-1}\gamma, \quad \gamma - \beta\alpha^{-1}\beta$$

are invertible. In particular, choosing x, y to be the scalar matrices 4 and 2 respectively gives the values $\alpha = 3$, $\beta = 1$, $\gamma = 3$, and one finds that the above four expressions do evaluate to invertible matrices.

It is interesting to compare the identities in Theorems 11.10 and 11.18, thought of as equations in the algebra of rational functions in x and y, with the statements about nc functions contained in our main theorems in Sections 8 and 9. When $n \geq 0$ the symmetric free polynomial p_n is freely holomorphic on $\mathbb{M}^2$. Theorem 8.35 applies to yield a holomorphic function P_n on the Zariski-free manifold $\mathcal{G}$, having a certain local boundedness property and satisfying $p_n = P_n \circ \pi$ on a suitable subset of $\mathbb{M}^2$. When $n < 0$ we must take the domain $\mathcal{S}$ of p_n to be the set $\{(x, y) \in \mathbb{M}^2 : x \in \mathcal{I}, y \in \mathcal{I}\}$. Theorem 8.35 no longer applies, so we appeal to Theorem 10.1. Notice that here $\mathcal{S}$

is an nc set, so that Assumption (10.2) is automatically satisfied. We again deduce that there is a holomorphic function P_n, this time on the Zariski-free manifold $\mathfrak{G}_{oo}(\mathcal{S})$, satisfying a version of the relation $p_n = P_n \circ \pi$.

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