Nevanlinna-Pick Interpolation on the Bidisk

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Abstract

We solve the matrix-valued Nevanlinna-Pick problem in the space of bounded analytic functions on the bidisk, and give a description of all the interpolating functions. We also prove the Toeplitz-corona theorem on the bidisk.

1 Nevanlinna-Pick on the Disk

The classical Nevanlinna-Pick problem on the disk $\mathbb{D}$ is: given n distinct points $\lambda_1, \ldots, \lambda_n$ in $\mathbb{D}$ and n complex numbers $z_1, \ldots, z_n$, find the analytic function ϕ of smallest norm (in the space $H^\infty(\mathbb{D})$ of bounded analytic functions on $\mathbb{D}$ equipped with the supremum norm) that maps λ_i to z_i. The problem was first studied by G. Pick [12] who proved that there is a function of norm 1 interpolating the λ_i's to the z_i's if and only if the Pick matrix is positive definite, *i.e.*

$$\left(\frac{1 - \overline{z_i} z_j}{1 - \overline{\lambda_i} \lambda_j} \right)_{i,j=1}^{n} \geq 0. \tag{1.1}$$

The problem has been extensively studied since then, both by function theoretic and operator theoretic techniques. The optimal solution has certain qualitative properties that we shall consider (recall that a function in $H^\infty(\mathbb{D})$ is called inner if its boundary values have modulus one almost everywhere on the circle):

Theorem 1.2 *Suppose that* $\inf\{\|\phi\| : \phi(\lambda_i) = z_i,\ i = 1, \ldots, n\} = 1$. *Then*
(i) There is a unique ϕ *in* $H^\infty(\mathbb{D})$ *of norm one that interpolates the data.*
(ii) The function ϕ *is rational and inner.*

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See *e.g.* [14], [11] or [10] for proofs of this theorem. Condition (ii) can be phrased as saying that ϕ is a finite Blaschke product, but we write it this way to facilitate extension to higher dimensions.

On the bidisk, the analogue of Pick's theorem was proved by the first author [1]:

Theorem 1.3 *Let $z_1, \ldots, z_n$ be complex numbers. A necessary and sufficient condition for there to exist a function ϕ in $H^\infty(\mathbb{D}^2)$ with $\|\phi\| \leq 1$ and $\phi(\alpha_i, \beta_i) = z_i,\ i = 1, \ldots, n$, is that there exist positive n-by-n matrices Γ and Δ such that*

$$(1 - \overline{z_i} z_j) = (1 - \overline{\alpha_i}\alpha_j)\Gamma_{ij} + (1 - \overline{\beta_i}\beta_j)\Delta_{ij}$$

(For another treatment of the proof of this theorem and interpolation on the bidisk, see [9]). A problem with this theorem, however, is that even if Γ and Δ are known, one does not know what ϕ is, and cannot describe qualitative properties in the manner of Theorem 1.2.

Using operator theoretic ideas, we are able to describe all the solutions to the interpolation problem on the bidisk (Theorem 2.10). Our approach works equally well for matrix-valued interpolation, and gives a new proof of Theorem 1.3 even in the scalar case. Thus we have the first matrix-valued Nevanlinna-Pick theorem on the bidisk (Theorem 2.1).

On the bidisk, the solution of smallest norm to a Nevanlinna-Pick problem may or may not be unique, as easy examples show; but from our description of all solutions we can show that some optimal interpolant will be rational and inner (Corollary 2.13).

The corona problem on the disk is to decide, given two functions ϕ_1 and ϕ_2 in $H^\infty(\mathbb{D})$, whether there exist functions ψ_1 and ψ_2 in $H^\infty(\mathbb{D})$ so that $\psi_1\phi_1 + \psi_2\phi_2 = 1$. Carleson's corona theorem [8] asserts that the obvious necessary condition $|\phi_1| + |\phi_2| \geq \varepsilon > 0$ is sufficient; but the smallest possible norm of the pair ψ does not depend just on ε. The Toeplitz-corona theorem on the disk, proved in different ways by several authors [6], [17], [15], [13], does give optimal bounds for solving the corona problem, but at the expense of replacing a functional inequality by an operator inequality.

Let σ denote Lebesgue measure on the circle, and define the Hardy space $H^2(\sigma)$ to be the closure in $L^2(\sigma)$ of the analytic polynomials. For $f \in L^\infty(\sigma)$, the Toeplitz operator T_f is the operator on $H^2(\sigma)$ that sends a function g to $P(fg)$, where P is the orthogonal projection from $L^2(\sigma)$ onto $H^2(\sigma)$. The Toeplitz-corona theorem is this:

Theorem 1.4 *Let ϕ_1, ϕ_2 be in $H^\infty(\mathbb{D})$, and $\delta > 0$. Then the following are equivalent:*
(i) For any points $\lambda_1, \ldots, \lambda_n$ in the disk,

$$\left(\frac{\overline{\phi_1(\lambda_i)}\phi_1(\lambda_j) + \overline{\phi_2(\lambda_i)}\phi_2(\lambda_j) - \delta}{1 - \overline{\lambda_i}\lambda_j} \right)_{i,j=1}^{n} \geq 0$$

(ii) The Toeplitz operators satisfy the inequality

$$T_{\phi_1}T_{\phi_1}^* + T_{\phi_2}T_{\phi_2}^* \geq \delta I.$$

(iii) There exist functions ψ_1, ψ_2 *in* $H^\infty(\mathbb{D})$ *such that*

$$\psi_1\phi_1 + \psi_2\phi_2 = 1$$

and

$$\sup_{z\in\mathbb{D}}[|\psi_1(z)|^2 + |\psi_2(z)|^2] \leq \frac{1}{\delta}.$$

We prove a bidisk version of this theorem (Theorem 3.1). Both the Toeplitz-corona theorem and the description of all solutions of the Nevanlinna-Pick problem rely heavily on a representation theorem of the first author describing bounded analytic matrix-valued functions on the bidisk [2]. As this theorem is essential for this paper, we include a proof (Theorem 4.1).

Remark: For notational convenience, we restrict ourselves to the bidisk. All the theorems and proofs in this paper go through, however, to the polydisk $\mathbb{D}^d$, provided the algebra $H^\infty(\mathbb{D}^2)$ is replaced by the algebra $H^\infty(\mathcal{F}_d)$, introduced in [1]. Here $\mathcal{F}_d$ is the collection of all d-tuples of commuting contractions, and $H^\infty(\mathcal{F}_d)$ is the set of all holomorphic functions on the polydisk whose norm, defined as

$$\|\phi\|_{d,\infty} = \sup\left\{\|\phi(T)\| \mid T \in \mathcal{F}_d\right\},$$

is finite. By von Neumann's inequality [18] and Andô's theorem [3], when $d = 1$ or 2 the algebras $H^\infty(\mathbb{D}^d)$ and $H^\infty(\mathcal{F}_d)$ are isometrically isomorphic; but this is no longer true for $d \geq 3$. As our methods are heavily operator theoretic, they extend naturally to $H^\infty(\mathcal{F}_d)$, but not to $H^\infty(\mathbb{D}^d)$.

Note: After writing this paper, we learned that most of the results have also been found independently by J. Ball and T. Trent [7].

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2 Solutions on the Bidisk

We shall first prove an extension of Theorem 1.3 to the matrix case. Let points λ_i on $\mathbb{D}^2$ be written as (α_i, β_i). Let I_k denote the k-by-k identity matrix, and let $H^\infty(\mathbb{D}^2, M_k)$ denote the bounded analytic functions on the bidisk taking values in the space of k-by-k matrices.

Theorem 2.1 *Let $z_1, \ldots, z_n$ be k-by-k matrices. A necessary and sufficient condition for there to exist a function ϕ in $H^\infty(\mathbb{D}^2, M_k)$ with $\|\phi\| \leq 1$ and $\phi(\alpha_i, \beta_i) = z_i$, $i = 1, \ldots, n$, is that there exist positive nk-by-nk matrices Γ and Δ such that, as i, j range from 1 to n and l, m range from 1 to k,*

$$(I_k - {z_i}^* z_j)_{l;m} = (1 - \overline{\alpha_i}\alpha_j)\Gamma_{i,l;j,m} + (1 - \overline{\beta_i}\beta_j)\Delta_{i,l;j,m}. \tag{2.2}$$

Proof: ($\Leftarrow$): Choose nk vectors $\{u_{i,l}\}_{i=1,\ldots,n;l=1,\ldots,k}$ in a Hilbert space $\mathcal{H}_1$ so that

$$\Gamma_{i,l;j,m} \;=\; \langle u_{j,m}, u_{i,l}\rangle.$$

Define a_i to be the matrix whose columns are $u_{i,1}$ through $u_{i,k}$; so a_i is a linear operator from $\mathbb{C}^k$ to $\mathcal{H}_1$ and

$$a_i^* a_j \;=\; [\langle u_{j,m}, u_{i,l}\rangle]_{l,m=1}^k$$

is the k-by-k submatrix of Γ corresponding to fixing i and j and letting l and m range from 1 to k.

Similarly, choose vectors $v_{i,l}$ in $\mathcal{H}_2$ so that

$$\Delta_{i,l;j,m} \;=\; \langle v_{j,m}, v_{i,l}\rangle,$$

and let

$$b_i \;=\; [v_{i,1} \ldots v_{i,k}].$$

Then (2.2) can be rewritten so that, for each i, j, the following k-by-k matrices are equal

$$[I_k + \overline{\alpha_i}\alpha_j a_i^* a_j + \overline{\beta_i}\beta_j b_i^* b_j] \;=\; [z_i^* z_j + a_i^* a_j + b_i^* b_j]. \tag{2.3}$$

Let $\{c_i\}_{i=1}^n$ be arbitrary vectors in $\mathbb{C}^k$, and define a linear transformation L on a subspace of $\mathbb{C}^k \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$ by

$$\begin{aligned} L:\ \mathrm{sp}\left\{ \begin{pmatrix} I_k \\ \alpha_i a_i \\ \beta_i b_i \end{pmatrix} c_i : 1 \leq i \leq n,\ c_i \in \mathbb{C}^k \right\} &\to \mathrm{sp}\left\{ \begin{pmatrix} z_i \\ a_i \\ b_i \end{pmatrix} c_i : 1 \leq i \leq n,\ c_i \in \mathbb{C}^k \right\} \\ \sum_{i=1}^n \begin{pmatrix} I_k \\ \alpha_i a_i \\ \beta_i b_i \end{pmatrix} c_i &\mapsto \sum_{i=1}^n \begin{pmatrix} z_i \\ a_i \\ b_i \end{pmatrix} c_i \end{aligned} \tag{2.3}$$

Then (2.3) says that L is an isometry. Extend L to a unitary U from all of $\mathbb{C}^k \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$ to $\mathbb{C}^k \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$. Let $\mathcal{M} = \mathcal{H}_1 \oplus \mathcal{H}_2$, and with respect to the decomposition of the space

$\mathbb{C}^k \oplus \mathcal{M}$, write U as

$$U \;=\; \begin{array}{c c} & \begin{array}{cc} \mathbb{C}^k & \mathcal{M} \end{array} \\ \begin{array}{c} \mathbb{C}^k \\ \mathcal{M} \end{array} & \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{array}. \tag{2.5}$$

For $\lambda = (\alpha, \beta)$ in $\mathbb{D}^2$, let E_λ be the operator on $\mathcal{M}$ that is multiplication by α on $\mathcal{H}_1$ and multiplication by β on $\mathcal{H}_2$. Define

$$\phi(\lambda) \;=\; A + BE_\lambda(1 - DE_\lambda)^{-1}C. \tag{2.6}$$

Note first that as $\lambda \mapsto E_\lambda$ is analytic and contraction valued, $(1 - DE_\lambda)$ is invertible for λ in $\mathbb{D}^2$, so ϕ is an analytic M_k-valued map. Moreover, using the fact that U is unitary, so $U^*U = I$, a calculation gives

$$1 - \phi(\lambda)^*\phi(\lambda) \;=\; ((1 - DE_\lambda)^{-1}C)^*(1 - E_\lambda^*E_\lambda)((1 - DE_\lambda)^{-1}C). \tag{2.7}$$

This is positive for λ in the bidisk, so ϕ is in $H^\infty(\mathbb{D}^2, M_k)$ and $\|\phi\| \leq 1$.

Let u_i be $a_i \oplus b_i$, a linear map from $\mathbb{C}^k$ to $\mathcal{M}$. In terms of A, B, C, D, (2.3) becomes

$$(A + BE_{\lambda_i}u_i)c_i \;=\; z_i c_i \tag{2.8}$$

$$(C + DE_{\lambda_i}u_i)c_i \;=\; u_i c_i \tag{2.9}$$

Solving, (2.9) gives

$$u_i \;=\; (1 - DE_{\lambda_i})^{-1}C,$$

and substituting this equation into (2.8) gives

$$z_i \;=\; A + BE_{\lambda_i}(1 - DE_{\lambda_i})^{-1}C \;=\; \phi(\lambda_i).$$

So ϕ interpolates, as desired.

($\Rightarrow$): If ϕ is any function in the ball of $H^\infty(\mathbb{D}^2, M_k)$, then there are Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and a unitary operator U on $\mathbb{C}^k \oplus (\mathcal{H}_1 \oplus \mathcal{H}_2)$ such that when U is written as in (2.5), then ϕ is given by (2.6). This was proved by the first author in [2]; as the result is central to this paper, we include a proof at the end of the paper (Theorem 4.1).

So suppose there exists a ϕ that interpolates the data, and write it in the above way. Let $u_i = (1 - DE_{\lambda_i})^{-1}C$, a linear operator from $\mathbb{C}^k$ to $\mathcal{H}_1 \oplus \mathcal{H}_2$, and decompose u_i as $a_i \oplus b_i$, where a_i maps $\mathbb{C}^k$ to $\mathcal{H}_1$ and b_i maps $\mathbb{C}^k$ to $\mathcal{H}_2$. Define the nk-by-nk positive matrix Γ by letting its $(i,j)^{th}$ k-by-k block be $a_i^*a_j$, and similarly define Δ by letting its $(i,j)^{th}$ block be $b_i^*b_j$.

Then the fact that ϕ interpolates means that (2.3) holds with U in place of L; and because U is unitary, this means (2.3) holds. So (2.2) holds as desired. $\square$

We can now describe all the interpolating functions, in both the scalar and matrix-valued case. Let $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ denote the bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$.

Theorem 2.10 *Suppose there is a function in $H^\infty(\mathbb{D}^2, M_k)$ of norm at most one that maps the points (α_i, β_i) to z_i, for $i = 1, \ldots, n$; so (2.2) holds. Choose operators a_i in $\mathcal{L}(\mathbb{C}^k, \mathcal{H}_1)$ and b_i in $\mathcal{L}(\mathbb{C}^k, \mathcal{H}_2)$, for some auxiliary Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, so that $a_i^* a_j = \Gamma_{ij}$ and $b_i^* b_j = \Delta_{ij}$. Let L be the isometry on the subspace of $\mathbb{C}^k \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$ given by*

$$L: \ \mathrm{sp}\left\{ \begin{pmatrix} I_k \\ \alpha_i a_i \\ \beta_i b_i \end{pmatrix} c_i : 1 \le i \le n,\ c_i \in \mathbb{C}^k \right\} \ \to \ \mathrm{sp}\left\{ \begin{pmatrix} z_i \\ a_i \\ b_i \end{pmatrix} c_i : 1 \le i \le n,\ c_i \in \mathbb{C}^k \right\}$$
$$\begin{pmatrix} I_k \\ \alpha_i a_i \\ \beta_i b_i \end{pmatrix} c_i \ \mapsto \ \begin{pmatrix} z_i \\ a_i \\ b_i \end{pmatrix} c_i$$

Let U be any unitary that extends L to all of $\mathbb{C}^k \oplus \mathcal{M}$, where $\mathcal{M} = \mathcal{H}_1 \oplus \mathcal{H}_2$. For $\lambda = (\alpha, \beta)$ a point of $\mathbb{D}^2$, let E_λ be the operator on $\mathcal{M}$ that is multiplication by α on $\mathcal{H}_1$ and multiplication by β on $\mathcal{H}_2$. Write U, with respect to the decomposition $\mathbb{C}^k \oplus \mathcal{M}$, as

$$U \ = \ \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{2.11}$$

Then

$$\phi(\lambda) \ = \ A + BE_\lambda(1 - DE_\lambda)^{-1}C \tag{2.12}$$

is a function in the unit ball of $H^\infty(\mathbb{D}^2)$ that interpolates the data. Moreover, all solutions of the interpolation problem arise in this way, for some choice of Γ, Δ, $\{a_i\}, \{b_i\}$ and some extension U.

PROOF: That one can choose a_i and b_i follows as in the proof of Theorem 2.1. The fact that L is an isometry is precisely the content of (2.2). Repeating the proof of Theorem 2.1 yields that ϕ defined by (2.12) is of norm at most one and interpolates.

Conversely, suppose ϕ is any function of norm at most one that interpolates. Using the representation theorem 4.1 again, just as in the proof of necessity of Theorem 2.1, one gets U, a_i and b_i (and hence also Γ and Δ) so that the theorem holds. $\square$

This description of the solutions allows us to show that there is always a rational inner solution. (An inner function in $H^\infty(\mathbb{D}^2)$ is one whose boundary values on $\mathbb{T}^2$, the distinguished boundary of the bidisk, have modulus one almost everywhere. A function in $H^\infty(\mathbb{D}^2, M_k)$ is called inner if its boundary values on $\mathbb{T}^2$ are unitary almost everywhere).

Corollary 2.13 *Suppose there exists some function in $H^\infty(\mathbb{D}^2, M_k)$ of norm less than or equal to 1 that interpolates the points λ_i to z_i. Then there is a rational inner function that interpolates.*

PROOF: Choose $\mathcal{H}_1$ and $\mathcal{H}_2$ to be finite dimensional (they can clearly always be chosen to have dimension nk at most), and then ϕ in (2.12) is rational.

As λ tends to a point on the distinguished boundary of the bidisk, $E_\lambda^* E_\lambda$ tends to the identity, so equation (2.7) shows that, as long as $(1 - DE_\lambda)$ remains invertible, $|\phi(\lambda)|$ tends to 1. Since the determinant of $(1 - DE_\lambda)$ is a polynomial, its zero set is of Lebesgue measure zero in the distinguished boundary, so $|\phi(\lambda)|$ tends to 1 almost everywhere. □

3 The Toeplitz-corona theorem on the bidisk

Let μ be any finite measure on $\mathbb{T}^2$. Let $H^2(\mu)$ denote the closure of the analytic polynomials in $L^2(\mu)$, and for any function f in $L^\infty(\mu)$ define the Toeplitz operator T_f^μ on $H^2(\mu)$ by

$$T_f^\mu g \ = \ P_{H^2(\mu)} f g.$$

In the statement of the folowing theorem, we shall let $\lambda = (\alpha, \beta)$ and $z = (a, b)$ be points on the bidisk. Saying $\Gamma(z, \lambda)$ is a positive semi-definite kernel means that for any points $\lambda_1, \dots, \lambda_n$, and any constants $c_1, \dots, c_n$, the sum

$$\sum_{i,j=1}^n c_i \overline{c_j} \Gamma(\lambda_i, \lambda_j) \ \geq \ 0.$$

Theorem 3.1 *Let ϕ_1 and ϕ_2 be in $H^\infty(\mathbb{D}^2)$, and $\delta > 0$. The following are equivalent:*
(i) There exist positive semi-definite kernels $\Gamma(z, \lambda)$ and $\Delta(z, \lambda)$ such that

$$\overline{\phi_1(\lambda)}\phi_1(z) + \overline{\phi_2(\lambda)}\phi_2(z) - \delta \ = \ (1 - \overline{\alpha} a)\Gamma(z, \lambda) + (1 - \overline{\beta} b)\Delta(z, \lambda). \tag{3.2}$$

(ii) There exist functions ψ_1, ψ_2 in $H^\infty(\mathbb{D}^2)$ such that

$$\psi_1 \phi_1 + \psi_2 \phi_2 = 1 \tag{3.3}$$

and

$$\sup_{z \in \mathbb{D}^2} [|\psi_1(z)|^2 + |\psi_2(z)|^2] \leq \frac{1}{\delta}. \tag{3.4}$$

(iii) For all measures μ on $\mathbb{T}^2$,

$$T^{\mu}_{\phi_1} T^{\mu *}_{\phi_1} + T^{\mu}_{\phi_2} T^{\mu *}_{\phi_2} \geq \delta I. \tag{3.5}$$

Proof: We shall prove (i) $\Leftrightarrow$ (ii) and (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). Although logically redundant, we include the proof (ii) $\Rightarrow$ (i) because we think it will help the reader to understand the interconnections; see also Remark 1 after the proof.

(i) $\Rightarrow$ (ii): Choose vector-valued functions f_1 and f_2 so that

$$\begin{aligned} \langle f_1(z), f_1(\lambda)\rangle &= \Gamma(z,\lambda) \\ &\text{and} \\ \langle f_2(z), f_2(\lambda)\rangle &= \Delta(z,\lambda) \end{aligned}$$

(one can always do this: see *e.g.* [5]). Let

$$\begin{aligned} f(\lambda) &= \begin{pmatrix} f_1(\lambda) \\ f_2(\lambda) \end{pmatrix}, \\ E_\lambda f(\lambda) &= \begin{pmatrix} \alpha f_1(\lambda) \\ \beta f_2(\lambda) \end{pmatrix}, \\ &\text{and} \\ \phi(\lambda) &= \begin{pmatrix} \phi_1(\lambda) \\ \phi_2(\lambda) \end{pmatrix}. \end{aligned}$$

Then (3.2) says that there is an isometry

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \phi(\lambda) \\ \lambda f(\lambda) \end{pmatrix} \mapsto \begin{pmatrix} \delta^{\frac{1}{2}} \\ f(\lambda) \end{pmatrix}.$$

Solving, we get

$$\delta^{\frac{1}{2}} = [A + BE_\lambda(1 - DE_\lambda)^{-1}C]\phi(\lambda)$$

Because $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is an isometry, the 1-by-2 vector $[A + BE_\lambda(1 - DE_\lambda)^{-1}C]$ is of norm less than or equal to one. Let

$$\psi(\lambda) = (\psi_1(\lambda) \quad \psi_2(\lambda)) = \delta^{-\frac{1}{2}}[A + BE_\lambda(1 - DE_\lambda)^{-1}C].$$

Then (3.3) and (3.4) are satisfied.

(ii) $\Rightarrow$ (i): By hypothesis, $\delta^{1/2}\psi$ is in the ball of $H^\infty(\mathbb{D}^2, \mathcal{L}(\mathbb{C}^2, \mathbb{C}^1))$, the bounded analytic functions mapping the bidisk into the linear maps from $\mathbb{C}^2$ to $\mathbb{C}^1$. Therefore by 4.1 there is a Hilbert space $\mathcal{M} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and a unitary

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathbb{C}^2 \oplus \mathcal{M} \to \mathbb{C}^2 \oplus \mathcal{M},$$

such that

$$\delta^{\frac{1}{2}}\begin{pmatrix}\psi_1 & \psi_2\\ 0 & 0\end{pmatrix} \;=\; A+BE_\lambda(1-DE_\lambda)^{-1}C.$$

Define

$$f \;=\; (1-DE_\lambda)^{-1}C\begin{pmatrix}\phi_1(\lambda)\\ \phi_2(\lambda)\end{pmatrix},$$

and decompose f as $f_1\oplus f_2$ with respect to $\mathcal{H}_1\oplus\mathcal{H}_2$. Let $\Gamma(z,\lambda)=\langle f_1(z),f_1(\lambda)\rangle$, and $\Delta(z,\lambda)=\langle f_2(z),f_2(\lambda)\rangle$. Then equation (3.2), which we wish to establish, is the same as

$$\delta+f^*f \;=\; \phi^*\phi+f^*E_\lambda^*E_\lambda f. \tag{3.6}$$

Using the fact that $U^*U=1$ as in (2.7) and the formula for f we get

$$\begin{aligned}
f^*(1-E_\lambda^*E_\lambda)f \;&=\; \phi^*((1-DE_\lambda)^{-1}C)^*(1-E_\lambda^*E_\lambda)((1-DE_\lambda)^{-1}C)\phi\\
&=\; \phi^*\left(1-\delta\begin{pmatrix}\overline{\psi_1} & 0\\ \overline{\psi_2} & 0\end{pmatrix}\begin{pmatrix}\psi_1 & \psi_2\\ 0 & 0\end{pmatrix}\right)\phi\\
&=\; \phi^*(1-\delta\psi^*\psi)\phi\\
&=\; \phi^*\phi-\delta
\end{aligned}$$

so (3.6) does hold, as desired.

(ii) $\Rightarrow$ (iii): Equation (3.4) gives that

$$\begin{pmatrix}T_{\psi_1}^{\mu*} & T_{\psi_2}^{\mu*}\end{pmatrix}\begin{pmatrix}T_{\psi_1}^{\mu}\\ T_{\psi_2}^{\mu}\end{pmatrix} \;\le\; \frac{1}{\delta}$$

for any measure μ supported on $\overline{\mathbb{D}^2}$. Therefore

$$\begin{pmatrix}T_{\psi_1}^{\mu}\\ T_{\psi_2}^{\mu}\end{pmatrix}\begin{pmatrix}T_{\psi_1}^{\mu*} & T_{\psi_2}^{\mu*}\end{pmatrix} \;\le\; \frac{1}{\delta}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$

and so

$$\begin{pmatrix}T_{\phi_1}^{\mu} & T_{\phi_2}^{\mu}\end{pmatrix}\begin{pmatrix}T_{\psi_1}^{\mu}\\ T_{\psi_2}^{\mu}\end{pmatrix}\begin{pmatrix}T_{\psi_1}^{\mu*} & T_{\psi_2}^{\mu*}\end{pmatrix}\begin{pmatrix}T_{\phi_1}^{\mu*}\\ T_{\phi_2}^{\mu*}\end{pmatrix} \;\le\; \frac{1}{\delta}(T_{\phi_1}^{\mu}T_{\phi_1}^{\mu*}+T_{\phi_2}^{\mu}T_{\phi_2}^{\mu*}).$$

But the left-hand side of this inequality is the identity by (3.3), so (3.5) is established.

(iii) $\Rightarrow$ (i): Let $K(z,\lambda)$ be any kernel such that

$$K(z,\lambda)(1-\overline{\alpha}a) \quad\ge\quad 0 \tag{3.7}$$

and

$$K(z,\lambda)(1-\overline{\beta}b) \quad\ge\quad 0 \tag{3.8}$$

Choose a vector valued function $f:\mathbb{D}^2\to\mathcal{H}$ such that

$$K(z,\lambda) \;=\; \langle f(z),f(\lambda)\rangle.$$

Let T_1 and T_2 be operators on $\mathcal{H}$ defined by

$$\begin{aligned} T_1 &: f(z) \mapsto af(z) \\ T_2 &: f(z) \mapsto bf(z) \end{aligned}$$

and extended to be linear on the span of the range of f (which we shall take to be all of $\mathcal{H}$). Then T_1 and T_2 commute, and by (3.7) and (3.8) they are contractions. Therefore by the commutant lifting theorem [16] they have extensions to commuting co-isometries S_1^* and S_2^*. Now (3.5) says that for all cyclic pairs of commuting isometries, and therefore for all commuting isometries S_1 and S_2, we have

$$\phi_1(S_1,S_2)\phi_1(S_1,S_2)^* + \phi_2(S_1,S_2)\phi_2(S_1,S_2)^* \ \geq \ \delta I. \tag{3.9}$$

We actually need a slight variant of (3.9). For ϕ in $H^\infty(\mathbb{D}^2)$, define $\breve{\phi}(z) := \overline{\phi(\overline{z})}$. We introduce this notation because $\breve{\phi}(S_1,S_2)^* = \phi(S_1^*,S_2^*)$ is an extension of $\phi(T_1,T_2)$.

As (3.5) holds for all μ, it also holds if ϕ_1 and ϕ_2 are replaced by $\breve{\phi}_1$ and $\breve{\phi}_2$, because if $\breve{\mu}$ is the measure defined by $\breve{\mu}(E) = \mu(\overline{E})$, with $\overline{E}$ here the complex conjugate of E, then

$$\langle T^{\breve{\mu}}_{\breve{\phi}} T^{\breve{\mu}*}_{\breve{\phi}} \breve{p}, \breve{p}\rangle_{L^2(\breve{\mu})} \ = \ \langle T^{\mu}_{\phi} T^{\mu*}_{\phi} p, p\rangle_{L^2(\mu)}.$$

Therefore we also have

$$\breve{\phi}_1(S_1,S_2)\breve{\phi}_1(S_1,S_2)^* + \breve{\phi}_2(S_1,S_2)\breve{\phi}_2(S_1,S_2)^* \ \geq \ \delta I. \tag{3.10}$$

Now

$$\begin{aligned} &\langle f(z), \ \ [\phi_1(T_1,T_2)^*\phi_1(T_1,T_2) + \phi_2(T_1,T_2)^*\phi_2(T_1,T_2) - \delta] f(\lambda)\rangle \\ &= \qquad [\phi_1(z)\overline{\phi_1(\lambda)} + \phi_2(z)\overline{\phi_2(\lambda)} - \delta] K(z,\lambda). \end{aligned}$$

So from (3.10), we have that

$$[\phi_1(z)\overline{\phi_1(\lambda)} + \phi_2(z)\overline{\phi_2(\lambda)} - \delta] K(z,\lambda) \geq 0 \tag{3.11}$$

whenever K is a kernel such that (3.7) and (3.8) hold.

(Notice that if K is only defined on a subset $S \times S$ of $\mathbb{D}^2 \times \mathbb{D}^2$, and if (3.7) and (3.8) hold on $S \times S$, then the preceding argument shows that (3.10) implies that (3.11) holds on $S \times S$).

We claim first that (3.2) holds on any *finite* set. For, fix a finite set $F = \{\lambda_1, \ldots, \lambda_n\}$ of distinct points in $\mathbb{D}^2$, and write each λ_i as (α_i, β_i). Let Φ be the n-by-n self-adjoint matrix with (i,j) entry

$$\Phi_{ij} \ = \ \phi_1(\lambda_j)\overline{\phi_1(\lambda_i)} + \phi_2(\lambda_j)\overline{\phi_2(\lambda_i)} - \delta.$$

Let $\mathcal{T}$ be the set of all n-by-n self-adjoint matrices T representable in the form

$$T_{ij} \;=\; (1-\alpha_i\overline{\alpha_j})\Gamma_{ij} + (1-\beta_i\overline{\beta_j})\Delta_{ij},$$

where Γ and Δ are positive. As $\mathcal{T}$ is a closed wedge, if Φ were not in $\mathcal{T}$, there would be a real linear functional on the space of all n-by-n self-adjoint matrices that was positive on $\mathcal{T}$ and strictly negative on Φ.

Any such linear functional must be of the form $T \mapsto tr(WT)$ for some self-adjoint matrix W. As $\mathcal{T}$ contains the set of all positive matrices (let $\Delta = 0$ and Γ be the Schur product of the positive matrix $1/(1-\overline{\alpha_i}\alpha_j)$ with an arbitrary positive matrix), W must be positive. Let K be the transpose of W. Then

$$\sum_{i,j=1}^{n} c_i\overline{c_j}K_{ij}(1-\alpha_i\overline{\alpha_j}) \;=\; tr(WT),$$

where

$$T_{ij} \;=\; (1-\alpha_i\overline{\alpha_j})c_i\overline{c_j}$$

is in $\mathcal{T}$. So K satisfies (3.7), and similarly also (3.8). Therefore, by the remark after (3.11), $\Phi \cdot K$ is positive, so

$$tr(W\Phi) \;=\; \sum_{i,j=1}^{n} K(\lambda_i,\lambda_j)\Phi_{ij} \;\geq 0.$$

Therefore Φ is in $\mathcal{T}$, proving the claim.

Given that (3.2) holds on all finite sets F, we wish to find one Γ and Δ that work for the whole bidisk. This follows from general topology. For each finite set F, let S_F denote the set of pairs of positive semi-definite kernels Γ and Δ on F that satisfy (3.2). Then each S_F is a non-empty compact set, and for any superset G of F there is a restriction map from S_G to S_F. Therefore by Kurosh's theorem [4, p.75], the projective limit of this system is non-empty, which means precisely that there are kernels Γ and Δ defined on the whole disk for which (3.2) holds. □

Remark 1: In the proof that (ii) $\Rightarrow$ (i) the construction produces an analytic function f, so it follows that if (3.2) holds for any Γ and Δ, then $\Gamma(z,\lambda)$ and $\Delta(z,\lambda)$ can be chosen to be analytic in z and conjugate-analytic in λ.

Remark 2: It is obvious that the theorem extends from two to n functions for any n. It is also true that the functions can be taken to be in $H^\infty(\mathbb{D}^2, M_k)$, with the above proof working *mutatis mutandis* (note that f_1 and f_2 will then take values in the space of linear operators from $\mathbb{C}^k$ to an auxiliary Hilbert space).

Remark 3: One could combine the preceding theorems into one general result:

Theorem 3.12 *Let $S \subseteq \mathbb{D}^2$ and let $\phi_1, \dots, \phi_p: S \to M_{n\times k}$ and $\psi_1, \dots, \psi_q: S \to M_{m\times k}$ be functions. Then the following are equivalent:*

(i) There exist positive kernels $\Gamma, \Delta: S \times S \to M_{k\times k}$ such that

$$\sum_{i=1}^{q} \psi_i(\lambda)^* \psi_i(z) - \sum_{j=1}^{p} \phi_j(\lambda)^* \phi_j(z) = (1-\overline{\alpha}a)\Gamma(z,\lambda) + (1-\overline{\beta}b)\Delta(z,\lambda)$$

for all $z = (a,b)$, $\lambda = (\alpha, \beta)$ in S.

(ii) There exists a holomorphic function $h: \mathbb{D}^2 \to \mathrm{Ball}(M_{np\times mq})$ such that

$$h(z) \begin{pmatrix} \psi_1(z) \\ \vdots \\ \psi_q(z) \end{pmatrix} = \begin{pmatrix} \phi_1(z) \\ \vdots \\ \phi_p(z) \end{pmatrix}$$

for all $z \in S$.

The proof of Theorem 3.12, and the representation of h as

$$h(\lambda) \;=\; A + BE_\lambda(1-DE_\lambda)^{-1}C$$

for an appropriate unitary $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, require no new ideas.

4 A proof of the representation theorem for $H^\infty(\mathbb{D}^2, M_k)$

For completeness, we include a proof of the representation theorem from [2].

Theorem 4.1 *[2] Let ϕ be in $H^\infty(\mathbb{D}^2, M_k)$ of norm less than or equal to 1. Then there exist auxiliary Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and a unitary operator U on $\mathbb{C}^k \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$ such that, if U is written as*

$$U \;=\; \begin{array}{c c} & \begin{array}{cc} \mathbb{C}^k & \mathcal{H}_1\oplus\mathcal{H}_2 \end{array} \\ \begin{array}{c} \mathbb{C}^k \\ \mathcal{H}_1\oplus\mathcal{H}_2 \end{array} & \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{array} \tag{4.2}$$

then

$$\phi(\lambda) \;=\; A + BE_\lambda(1-DE_\lambda)^{-1}C. \tag{4.3}$$

Here, for $\lambda = (\alpha, \beta)$ in $\mathbb{D}^2$, E_λ is the operator of multiplication by α on $\mathcal{H}_1$ and multiplication by β on $\mathcal{H}_2$.

PROOF: First, we shall give a proof in the scalar case $k = 1$.

By Andô's theorem [3], if T_1 and T_2 are any commuting contractions, then $\|\phi(T_1, T_2)\| \leq 1$. Let $K(z, \lambda)$ be any positive definite kernel on $\mathbb{D}^2$. Choose a vector valued function f so that $K(z, \lambda) = \langle f(z), f(\lambda)\rangle$. Throughout the proof we shall write $z = (a, b)$ and $\lambda = (\alpha, \beta)$. Let $T_1 : f(\lambda) \mapsto \alpha f(\lambda)$ and $T_2 : f(\lambda) \mapsto \beta f(\lambda)$.

Then T_1 and T_2 are contractions if and only if

$$K(z, \lambda)(1 - \overline{\alpha} a) \geq 0 \tag{4.4}$$

and

$$K(z, \lambda)(1 - \overline{\beta} b) \geq 0 \tag{4.5}$$

The operator $\phi(T_1, T_2)$ is a contraction if and only if

$$K(z, \lambda)(1 - \overline{\phi(\lambda)}\phi(z)) \geq 0. \tag{4.6}$$

So by Andô's theorem, the positivity of (4.4) and (4.5) implies the positivity of (4.6). Just as in the proof of Theorem 3.1 (iii) $\Rightarrow$ (i), this means that if F is any finite set in $\mathbb{D}^2$, there exist positive kernels Γ and Δ on F such that

$$1 - \overline{\phi(\lambda)}\phi(z) = (1 - \overline{\alpha} a)\Gamma(z, \lambda) + (1 - \overline{\beta} b)\Delta(z, \lambda). \tag{4.7}$$

Again, a projective limit argument implies that there is a choice of Γ and Δ such that (4.7) holds on the whole bidisk. Factor Γ and Δ in the now familiar way to get the desired representation of ϕ.

(Matrix Case). The above proof works in essentially the same way. Let $K(z, \lambda)$ be an M_k-valued positive definite kernel, and as in the proof of Theorem 2.1, factor K as $f(\lambda)^* f(z)$ where $f(\lambda) \in \mathcal{L}(\mathbb{C}^k, \mathcal{H})$. Define T_1 and T_2 on $\mathcal{H}$ by, for $c \in \mathbb{C}^k$, $T_1 : f(\lambda)c \mapsto \alpha f(\lambda)c$ and $T_2 : f(\lambda)c \mapsto \beta f(\lambda)c$. Then repeating the proof of the scalar case, one gets

$$I_k - \phi(z)\overline{\phi(\lambda)} = (1 - \overline{\alpha} a)\Gamma(z, \lambda) + (1 - \overline{\beta} b)\Delta(z, \lambda),$$

and from this, as in the proof of Theorem 2.1, one gets the desired representation of ϕ. $\square$

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