

FACTORIZATIONS INDUCED BY COMPLETE NEVANLINNA-PICK FACTORS

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ABSTRACT. We prove a factorization theorem for reproducing kernel Hilbert spaces whose kernel has a normalized complete Nevanlinna-Pick factor. This result relates the functions in the original space to pointwise multipliers determined by the Nevanlinna-Pick kernel and has a number of interesting applications. For example, for a large class of spaces including Dirichlet and Drury-Arveson spaces, we construct for every function f in the space a pluriharmonic majorant of $|f|^2$ with the property that whenever the majorant is bounded, the corresponding function f is a pointwise multiplier.

1. INTRODUCTION

Let Ω be a nonempty set and let \mathcal{E} be a separable Hilbert space. A function $\kappa : \Omega \times \Omega \rightarrow \mathcal{B}(\mathcal{E})$ is called positive definite, if whenever $n \in \mathbb{N}$ and $w_1, \dots, w_n \in \Omega$ and $x_1, \dots, x_n \in \mathcal{E}$, then $\sum_{i,j=1}^n \langle \kappa(w_j, w_i)x_i, x_j \rangle \geq 0$. If κ is positive definite, then we write $\kappa_w(z) = \kappa(z, w) \gg 0$. We note that such κ is positive definite, if and only if there is an auxiliary Hilbert space \mathcal{C} and a function $K : \Omega \rightarrow \mathcal{B}(\mathcal{C}, \mathcal{E})$ such that $\kappa_w(z) = K(z)K(w)^*$ for all $z, w \in \Omega$ (see [2], Theorem 2.62).

If $k_w(z)$ is a scalar-valued reproducing kernel and if $I_{\mathcal{E}}$ denotes the identity operator on \mathcal{E} , then $k_w(z)I_{\mathcal{E}}$ is a positive definite operator-valued kernel. It is the reproducing kernel for the space $\mathcal{H}_k(\mathcal{E})$ which consists of all functions $F : \Omega \rightarrow \mathcal{E}$ such that for each $x \in \mathcal{E}$ the function $F_x(z) = \langle F(z), x \rangle_{\mathcal{E}}$ is in \mathcal{H}_k and such that $\|F\|_{\mathcal{H}_k(\mathcal{E})}^2 = \sum_n \|F_{e_n}\|_{\mathcal{H}_k}^2 < \infty$, where $\{e_n\}$ is an orthonormal basis for \mathcal{E} . It is easy to show that

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the expression for $\|F\|_{\mathcal{H}_k(\mathcal{E})}$ is independent of the choice of orthonormal basis, and that for each $F \in \mathcal{H}_k(\mathcal{E})$, $x \in \mathcal{E}$, and $z \in \Omega$ one has $k_z x \in \mathcal{H}_k(\mathcal{E})$ and $\langle F(z), x \rangle_{\mathcal{E}} = \langle F, k_z x \rangle_{\mathcal{H}_k(\mathcal{E})}$. It follows that the set of finite linear combinations of functions of the form $k_z x$, $z \in \Omega$, $x \in \mathcal{E}$, is dense in $\mathcal{H}_k(\mathcal{E})$. Of course, the map: $f \otimes x \mapsto fx$ extends to be a Hilbert space isomorphism between $\mathcal{H}_k \otimes \mathcal{E}$ and $\mathcal{H}_k(\mathcal{E})$, but for this paper we prefer the standpoint of \mathcal{E} -valued functions.

If $k_w(z)$ and $s_w(z)$ are reproducing kernels on Ω and if \mathcal{C} and \mathcal{D} are separable Hilbert spaces, then $\text{Mult}(\mathcal{H}_s(\mathcal{D}), \mathcal{H}_k(\mathcal{C}))$ is the collection of functions $\Phi : \Omega \rightarrow \mathcal{B}(\mathcal{D}, \mathcal{C})$ such that $(M_\Phi F)(z) = \Phi(z)F(z)$ defines a bounded operator $M_\Phi : \mathcal{H}_s(\mathcal{D}) \rightarrow \mathcal{H}_k(\mathcal{C})$. It is easy to check that for $\Phi \in \text{Mult}(\mathcal{H}_s(\mathcal{D}), \mathcal{H}_k(\mathcal{C}))$ one has

$$(1.1) \quad M_\Phi^*(k_w x) = s_w \Phi(w)^* x$$

for all $x \in \mathcal{C}$ and $w \in \Omega$. Moreover, the multipliers Φ with $\|M_\Phi\| \leq A$ are characterized by

$$(1.2) \quad A^2 I_{\mathcal{C}} k_w(z) - \Phi(z) \Phi^*(w) s_w(z) \gg 0,$$

since it is equivalent to $\|M_\Phi^* f\|_{\mathcal{H}_s(\mathcal{D})} \leq A \|f\|_{\mathcal{H}_k(\mathcal{C})}$, for a dense subset of $\mathcal{H}_k(\mathcal{C})$.

As usual, $\text{Mult}(\mathcal{H}, \mathcal{H})$ is denoted by $\text{Mult}(\mathcal{H})$. Each $\varphi \in \text{Mult}(\mathcal{H}_k)$ defines a bounded operator on $\mathcal{H}_k(\mathcal{E})$, $(M_\varphi F)(z) = \varphi(z)F(z)$ and $\|\varphi\|_{\text{Mult}(\mathcal{H})} = \|M_\varphi\|_{\mathcal{B}(\mathcal{H}_k(\mathcal{E}))}$, the norm of M_φ acting on \mathcal{H}_k . We will refer to such $M_\varphi \in \mathcal{B}(\mathcal{H}_k(\mathcal{E}))$ as scalar multiplication operators.

The central role in this work is played by scalar reproducing kernels $s : \Omega \times \Omega \rightarrow \mathbb{C}$, of the form

$$(1.3) \quad s_w(z) = \frac{1}{1 - \sum_{n=1}^{\infty} u_n(z) \overline{u_n(w)}} = \frac{1}{1 - u(z)u^*(w)},$$

where $u_n : \Omega \rightarrow \mathbb{C}$, satisfy $u_n(z_0) = 0$ for each $n \in \mathbb{N}$, and some fixed point $z_0 \in \Omega$. Moreover, $u : \Omega \rightarrow \mathcal{B}(l^2, \mathbb{C})$ denotes the corresponding row operator-valued function. By a theorem of Agler and McCarthy [1] it follows that these are precisely the normalized ($s_{z_0} = 1$) reproducing kernels with the *complete Nevanlinna-Pick property*, such that the Hilbert space \mathcal{H}_s is separable. This class of kernels is well-known and extensively studied. A comprehensive treatment can be found in [2]. The standard definition based on finite interpolation problems is deferred to the preliminary Section 2. The normalization point z_0 will be fixed for the rest of the paper, and throughout we shall refer to such kernels as normalized CNP kernels. Clearly, the positivity of s implies that $u(z)$ is a strict contraction when $z \in \Omega$. The first examples that

come to mind are when $\Omega = \mathbb{B}_d$, the unit ball in \mathbb{C}^d , $d \in \mathbb{N}$, $z_0 = 0$, and $s_w(z) = \frac{1}{1-\langle z, w \rangle}$. If $d = 1$, \mathcal{H}_s is the standard Hardy space H^2 , while for $d > 1$, \mathcal{H}_s is called the Drury-Arveson space and is denoted by H_d^2 , but the list of examples is much larger. It includes weighted Dirichlet spaces in one variable, weighted Besov spaces in one or several variables (on \mathbb{B}_d), and even Sobolev spaces (see [2] for more examples).

The present paper is concerned with reproducing kernel Hilbert spaces $\mathcal{H}_k(\mathcal{E})$, with the property that the kernel k has a normalized CNP factor, i.e.

$$(1.4) \quad k = sg,$$

where s is a normalized CNP kernel and g is positive definite. It turns out that this condition is fulfilled for a large class of kernels, for example Hardy and weighted Bergman spaces on \mathbb{B}_d , or the polydisc \mathbb{B}_1^d , of course \mathcal{H}_s itself, or more generally, \mathcal{H}_{s^t} , $t \geq 1$. In Section 2 we shall show that whenever $\text{Mult}(\mathcal{H}_k)$ contains non-constant elements the kernel k has a nontrivial normalized CNP factor. Our purpose is to establish a factorization result for functions in $\mathcal{H}_k(\mathcal{E})$ which is motivated by a simple observation. If we assume that

$$k_w(z) = s_w(z)G(z)G^*(w), \quad s_w = \frac{1}{1 - u(z)u^*(w)},$$

with $z, w \in \Omega$, $G(z) \in \mathcal{B}(\mathcal{C}, \mathbb{C})$, $u(z) \in \mathcal{B}(l^2, \mathbb{C})$, it is not difficult to verify $G \in \text{Mult}(\mathcal{H}_s(\mathcal{C}), \mathcal{H}_k)$, $u \in \text{Mult}(\mathcal{H}_s(l^2), \mathcal{H}_s)$ and that both have multiplier norm at most 1. This leads to the factorization

$$k_w = \frac{\varphi}{1 - \psi},$$

where $\varphi(z) = G(z)G(w)^* \in \text{Mult}(\mathcal{H}_s, \mathcal{H}_k)$, $\psi(z) = u(z)u(w)^* \in \text{Mult}(\mathcal{H}_s)$ are multipliers with $\psi(z_0) = 0$, $\|\psi\|_{\text{Mult}(\mathcal{H}_s)} \leq 1$.

Our main theorem shows that this factorization continues to hold for arbitrary elements of $\mathcal{H}_k(\mathcal{E})$, and establishes a sharp estimate for the multiplier norms involved. Somewhat surprisingly, our factorization is unique for elements of unit norm.

Theorem 1.1. *Let Ω be a non-empty set, $z_0 \in \Omega$. Let $k_w(z)$ be a reproducing kernel on Ω that can be factored in the form $k_w(z) = s_w(z)g_w(z)$ where $s_w(z)$ is a normalized CNP kernel with $s_{z_0} = 1$ and $g_w(z) >> 0$.*

(i) For $F : \Omega \rightarrow \mathcal{E}$, the following are equivalent:

(a) $F \in \mathcal{H}_k(\mathcal{E})$ with $\|F\|_{\mathcal{H}_k(\mathcal{E})} \leq 1$,

(b) There is a $\psi \in \text{Mult}(\mathcal{H}_s)$ with $\psi(z_0) = 0$ and a $\Phi \in \text{Mult}(\mathcal{H}_s, \mathcal{H}_k(\mathcal{E}))$ such that

$$\|\psi h\|_{\mathcal{H}_s}^2 + \|\Phi h\|_{\mathcal{H}_k(\mathcal{E})}^2 \leq \|h\|_{\mathcal{H}_s}^2 \quad \text{for all } h \in \mathcal{H}_s,$$

and $F(z) = \frac{1}{1-\psi(z)}\Phi(z)$ for all $z \in \Omega$.

(ii) If $F \in \mathcal{H}_k(\mathcal{E})$ with $\|F\|_{\mathcal{H}_k(\mathcal{E})} = 1$, then the factorization given in (b) is unique. In fact, $s_z \in \text{Mult}(\mathcal{H}_k(\mathcal{E}))$, and if $V_F(z) = 2\langle F, s_z F \rangle_{\mathcal{H}_k(\mathcal{E})} - 1$, $z \in \Omega$, then $\text{Re } V_F \geq 0$ in Ω and (b) holds with

$$\psi = \frac{V_F - 1}{V_F + 1}, \quad \Phi = \frac{2}{V_F + 1}F.$$

From [4, Lemma 1] it follows that if ψ is the multiplier given in part (i) of the theorem then $1-\psi$ is cyclic, i.e. the multiplier-invariant subspace generated by this function equals \mathcal{H}_s . It will also turn out from the proof that if $\|F\|_{\mathcal{H}_k(\mathcal{E})} < 1$, the factorization is far from unique. In fact if (b) holds, the restriction imposed on $\|F\|_{\mathcal{H}_k(\mathcal{E})}$ is related only to the condition $\psi(z_0) = 0$. Our argument shows (see Proposition 3.5 below) that without this condition the factorization $F = \frac{1}{1-\psi}\Phi$ with Φ, ψ as above, holds true for an arbitrary $F \in \mathcal{H}_k(\mathcal{E})$.

In the case when $k = s$ and $\mathcal{E} = \mathbb{C}$, the factorization in (b) was recently proved in [4]. The result was previously shown for the Drury-Arveson space on \mathbb{B}_d , by Alpay, Bolotnikov and Kaptanoğlu [6, Theorem 10.3] while for the Dirichlet space, the corresponding question was posed in [24, Section 3] and at the end of [17]. The proof of the result in [4] relies on an appropriate version of Leech's theorem, and at the beginning of Section 3 we shall briefly indicate how one can use that method to extend the result to the setting considered here. However, our proof of Theorem 1.1 follows a different path, namely the one suggested by part (ii). The argument is based on an idea of Sarason (see [25, 26]) and further developments in [15] and will be presented in Section 3. It leads to a constructive approach which has a number of interesting applications given in Section 4. The function V_F defined in part (ii) of the main theorem plays a crucial role for our development, and it will be called *the Sarason function of F* . Its real part behaves similarly to the well known *least harmonic majorant* from the theory of Hardy spaces. We remark that in the case when $k = s$ and $\mathcal{E} = \mathbb{C}$, this argument also provides a more explicit construction of the factorization in the main result of [4].

In Section 4 we show that multiplier-invariant subspaces in \mathcal{H}_k are generated by functions in $\text{Mult}(\mathcal{H}_s, \mathcal{H}_k)$, that extremal functions in $\mathcal{H}_k(\mathcal{E})$ belong to $\text{Mult}(\mathcal{H}_s, \mathcal{H}_k(\mathcal{E}))$, and derive a pointwise estimate for these

functions. These recover some results in [19] and in the recent paper [8], but apply to other situations as well. The Sarason function of an extremal element is constant equal to 1 in Ω , and motivated by this observation we continue the investigation of $\text{Mult}(\mathcal{H}_s, \mathcal{H}_k(\mathcal{E}))$ in terms of this object. It turns out that if the one-function corona theorem holds in $\text{Mult}(\mathcal{H}_s)$, then $F \in \text{Mult}(\mathcal{H}_s, \mathcal{H}_k(\mathcal{E}))$ whenever V_F is bounded in Ω . One of our main applications shows that for a large class of kernels k the same conclusion holds under the weaker assumption that the real part of the Sarason function is bounded. The corresponding class of spaces \mathcal{H}_k contains Bergman, Hardy and weighted Besov spaces on the ball or polydisc, in particular the Drury-Arveson spaces H_d^2 , $d \in \mathbb{N}$. In all of these cases, our Theorem 4.5 together with Corollary 3.3 show a surprising analogy to the classical H^2 -case:

The real part of the Sarason function of F is a majorant of $\frac{s_z(z)}{k_z(z)} \|F(z)\|_{\mathcal{E}}^2$ (in most cases pluriharmonic), such that whenever this majorant is bounded, F belongs to $\text{Mult}(\mathcal{H}_s, \mathcal{H}_k)$. The converse of this statement fails to be true. Based on the work of Shimorin ([28],[30]), we construct multipliers of the standard weighted Dirichlet spaces on the unit disc whose Sarason functions have unbounded real part.

Our theorem about multipliers applies also to Carleson embeddings. More precisely, in the special case when \mathcal{H}_k is a weighted Bergman space and $F = 1$ this extends to the general context a sufficient condition for such embeddings obtained recently in [12] for the Dirichlet space. As pointed out above, in Section 2 we gather some useful preliminary results.

2. PRELIMINARIES

2.1. The complete Nevanlinna-Pick property. Given a reproducing kernel s on the non-void set Ω , we say that s is a *complete Nevanlinna-Pick kernel* if \mathcal{H}_s has the following property: For every $r \in \mathbb{N}$ and every finite collection of points $z_1, \dots, z_n \in \Omega$ and matrices $W_1, \dots, W_n \in M_r(\mathbb{C})$, positivity of the $nr \times nr$ -block matrix

$$\left[s(z_i, z_j)(I_{\mathbb{C}^r} - W_i W_j^*) \right]_{i,j=1}^n$$

implies that there exists $u \in \text{Mult}(\mathcal{H}_s(\mathbb{C}^r))$ of norm at most 1 such that

$$u(z_i) = W_i \quad (i = 1, \dots, n).$$

Such kernels were characterized by a theorem of Quiggin [20] and McCullough [18]. Complete Nevanlinna-Pick kernels s can be normalized at any point provided that $s_w(z) \neq 0$ for all $z, w \in \Omega$ (see [2, Section

2.6]). However, in this paper we shall only use the form (1.3) which follows from the Agler-McCarthy theorem in [1]. For further purposes we record an elementary result whose proof is included for the sake of completeness.

Lemma 2.1. *If s is a normalized CNP kernel with*

$$s_w(z) = \frac{1}{1 - \sum_{n=1}^{\infty} u_n(z)\overline{u_n(w)}},$$

where $u_n : \Omega \rightarrow \mathbb{C}$ and $u_n(z_0) = 0$, then:

(i) $u_n \in \text{Mult}(\mathcal{H}_s)$ for each $n \in \mathbb{N}$ and if $h_n \in \mathcal{H}_s$, then

$$\left\| \sum_n u_n h_n \right\|^2 \leq \sum_n \|h_n\|^2.$$

(ii) $I - \sum_i M_{u_i} M_{u_i}^* = P_0$, where P_0 is the projection onto $s_{z_0} = 1$, i.e. $h - \sum_i M_{u_i} M_{u_i}^* h = h(z_0)$ for all $h \in \mathcal{H}_s$.

Proof. (i) From the identity

$$\left(1 - \sum_n u_n \overline{u_n(w)}\right) s_w = s_{z_0} = 1 \gg 0,$$

we see that (1.2) holds with $A = 1$ and $\Phi : \Omega \rightarrow \mathcal{B}(l^2, \mathbb{C})$, $\Phi(z)(x_n) = \sum_n u_n(z)x_n$. (ii) holds for reproducing kernels, hence it holds for all functions in \mathcal{H}_s because the span of reproducing kernels is dense in \mathcal{H}_s and the operator on the right hand side is bounded. The second part of (ii) is just a reformulation of the first. \square

2.2. Kernels with a CNP factor. The simplest examples of kernels k with the normalized CNP factor s are given by $k = s^t$, $t \geq 1$. Note that if $s = \frac{1}{1-uu^*}$ and $0 < t < 1$, then

$$s^t = \sum a_k(t)(uu^*)^k,$$

with $a_k(t) > 0$, hence s^t , $0 < t < 1$ is positive definite. Then s^t is positive definite for all $t > 0$ by the Schur product theorem.

The most general condition for a factorization of the form (1.4) is as follows.

Lemma 2.2. *Let s be a normalized CNP kernel with $s_w(z) = \frac{1}{1-u(z)\overline{u^*(w)}}$, with $u(z) \in \mathcal{B}(l^2, \mathbb{C})$. The kernel k can be factored as $k = sg$ with g positive definite if and only if $u \in \text{Mult}(\mathcal{H}_k(l^2), \mathcal{H}_k)$ with $\|M_u\| \leq 1$. In this case $\text{Mult}(\mathcal{H}_s(\mathcal{E}))$ is contractively contained in $\text{Mult}(\mathcal{H}_k(\mathcal{E}))$.*

Proof. To see the first part, use the identity

$$(I - u(z)u^*(w))k_w(z) = \frac{1}{s_w(z)}k_w(z),$$

to conclude that k/s is positive definite if and only if the left hand side is and then apply (1.2). For the second part, note that if $\varphi \in \text{Mult}(\mathcal{H}_s(\mathcal{E}))$ with $\|M_\varphi\| = 1$ then $(I_{\mathcal{E}} - \varphi(z)\varphi^*(w))s_w(z) \gg 0$, hence by the Schur product theorem

$$(I_{\mathcal{E}} - \varphi(z)\varphi^*(w))s_w(z)g_w(z) = (I_{\mathcal{E}} - \varphi(z)\varphi(w)^*)k_w(z) \gg 0.$$

□

It is interesting to note that the second part of the lemma can be used to characterize complete Nevanlinna-Pick kernels (see [9, Theorem 4.4]). With the result in hand we can list some further examples of kernels with a normalized CNP factor. Recall that a d -contraction on the Hilbert space \mathcal{H} is a commuting tuple (T_1, \dots, T_d) , $T_j \in \mathcal{B}(\mathcal{H})$, with $\sum_{j=1}^d T_j T_j^* \leq I$.

Corollary 2.3. (i) If $\text{Mult}(\mathcal{H}_k(l^2), \mathcal{H}_k)$ contains a nonzero element $u = (u_n)_{n \geq 1}$ of norm at most 1 with $u(z_0) = 0$ and $s_w(z) = \frac{1}{1 - \sum_{n=1}^{\infty} u_n(z)u_n(w)}$, then $k/s \gg 0$.

(ii) Let $\Omega = \mathbb{B}_d$, $d \in \mathbb{N}$ and assume that multiplication by the coordinates forms a d -contraction on \mathcal{H}_k . If $s = \frac{1}{1 - \langle z, w \rangle}$ then $k/s \gg 0$.

(iii) If the assumption in (ii) holds for $d = 1$, i.e. multiplication by the identity function is a contraction on \mathcal{H}_k , then for every normalized CNP kernel s such that \mathcal{H}_s consists of analytic functions, we have $k/s \gg 0$.

Proof. (i) and (ii) follow directly from Lemma 2.2. (iii) Let s be a normalized CNP kernel with $s_w(z) = \frac{1}{1 - \sum_{n=1}^{\infty} u_n(z)u_n(w)}$, where each u_n is analytic. Write $k/s = (k/s_0)(s_0/s)$, where s_0 is the Szegő kernel. By the Schur product theorem it suffices to show that $s_0/s \gg 0$, since $s/s_0 \gg 0$ follows from the assumption. This is obviously equivalent to the operator inequality

$$\sum_{n=1}^{\infty} M_{u_n} M_{u_n}^* \leq I$$

in H^2 . But in this space we have

$$\sum_{n=1}^{\infty} M_{u_n} M_{u_n}^* \leq \sum_{n=1}^{\infty} M_{u_n}^* M_{u_n}$$

and the right hand side is just the Toeplitz operator with symbol $\sum_{n=1}^{\infty} |u_n|^2 \leq 1$, i.e.

$$\sum_{n=1}^{\infty} M_{u_n}^* M_{u_n} \leq I.$$

□

Other examples are provided by the following result.

Proposition 2.4. *Let s be a normalized CNP kernel on Ω and k be a reproducing kernel with $k/s \gg 0$. If \mathcal{M} is a closed subspace of \mathcal{H}_k which is invariant for $\text{Mult}(\mathcal{H}_s)$, then the reproducing kernel $k^{\mathcal{M}}$ of \mathcal{M} satisfies $k^{\mathcal{M}}/s \gg 0$.*

Proof. If $s_w(z) = \frac{1}{1 - \sum_{n=1}^{\infty} u_n(z)u_n(w)}$ then by Lemma 2.1 and Lemma 2.2 we have that

$$\sum_{n=1}^{\infty} \|u_n h_n\|_{\mathcal{H}_k}^2 \leq \sum_{n=1}^{\infty} \|h_n\|_{\mathcal{H}_k}^2$$

if $h_n \in \mathcal{H}_k$, in particular if $h_n \in \mathcal{M}$. Thus $M_u : \mathcal{M}(l^2) \rightarrow \mathcal{M}$ is a contraction and the result follows by another application of Lemma 2.2. □

3. PROOF OF THE MAIN RESULT

Before we give the actual proof we shall discuss briefly some related ideas as well as the motivation for our approach.

3.1. (a) \Rightarrow (b) via Leech's theorem. As pointed out in the Introduction, this part of Theorem 1.1, that is, the representation $F = \frac{1}{1-\psi}\Phi$, where $\|F\|_{\mathcal{H}_s(\mathcal{E})} \leq 1$, $\psi \in \text{Mult}(\mathcal{H}_s)$ with $\psi(z_0) = 0$ and $\Phi \in \text{Mult}(\mathcal{H}_s, \mathcal{H}_k(\mathcal{E}))$ with

$$\|\psi h\|_{\mathcal{H}_s}^2 + \|\Phi h\|_{\mathcal{H}_k(\mathcal{E})}^2 \leq \|h\|_{\mathcal{H}_s}^2 \quad \text{for all } h \in \mathcal{H}_s,$$

can be proved with the method in [4] and we shall describe briefly this approach.

In [4] the factorization theorem is proved in the scalar case and when $k = s$ with help of an appropriate version of Leech's theorem, which is the implication (i) \Rightarrow (ii) of [2, Theorem 8.57]. This can easily be adapted for normalized CNP kernels. A careful inspection of the argument shows that it extends to the vector-valued case as well. Moreover, even if the approach is not constructive, the method gives the multiplier-norm estimates when $k = s$.

The general case of a kernel k with $k/s \gg 0$, where s is a normalized CNP kernel, can be deduced as follows. Let

$$(3.1) \quad k_w(z) = s_w(z)G(z)G^*(w),$$

with $G : \Omega \rightarrow \mathcal{B}(\mathcal{C}, \mathbb{C})$ and define $\tilde{G} : \Omega \rightarrow \mathcal{B}(\mathcal{C} \otimes \mathcal{E}, \mathcal{E})$ by

$$\tilde{G}(z)(x \otimes y) = (G(z)x)y.$$

Observe first that \tilde{G} is a contractive multiplier from $\mathcal{H}_s(\mathcal{C} \otimes \mathcal{E})$ into $\mathcal{H}_k(\mathcal{E})$. Indeed, $\tilde{G}(z)^*y = G(z)^* \otimes y$, hence from (3.1) it follows that the map defined on a spanning set by $M_{\tilde{G}}^*(k_z y) = s_z(G(z)^* \otimes y)$ extends to an isometric operator $M_{\tilde{G}}^* : \mathcal{H}_k(\mathcal{E}) \rightarrow \mathcal{H}_s(\mathcal{C} \otimes \mathcal{E})$. Thus $M_{\tilde{G}}^{**} = M_{\tilde{G}} : \mathcal{H}_s(\mathcal{C} \otimes \mathcal{E}) \rightarrow \mathcal{H}_k(\mathcal{E})$ is a contraction which proves the claim.

Now let $F \in \mathcal{H}_k(\mathcal{E})$ with $\|F\|_{\mathcal{H}_k(\mathcal{E})} \leq 1$, and let $H = M_{\tilde{G}}^*F$. Then $H \in \mathcal{H}_s(\mathcal{C} \otimes \mathcal{E})$, $\|H\|_{\mathcal{H}_s(\mathcal{C} \otimes \mathcal{E})} \leq 1$, and $F = M_{\tilde{G}}H$, because $M_{\tilde{G}}^*$ is an isometry. By the previous discussion, there are $\psi \in \text{Mult}(\mathcal{H}_s)$ and $\Gamma \in \text{Mult}(\mathcal{H}_s, \mathcal{H}_s(\mathcal{C} \otimes \mathcal{E}))$ such that $\psi(z_0) = 0$, $\|\psi h\|^2 + \|\Gamma h\|^2 \leq \|h\|^2$ for every $h \in \mathcal{H}_s$ and $H(z) = \frac{1}{1-\psi(z)}\Gamma(z)$, which gives $F(z) = \frac{1}{1-\psi(z)}\Phi(z)$ with

$$(3.2) \quad \Phi(z) = \tilde{G}(z)\Gamma(z),$$

and the result follows from the fact that \tilde{G} is a contractive multiplier from $\mathcal{H}_s(\mathcal{C} \otimes \mathcal{E})$ into $\mathcal{H}_k(\mathcal{E})$.

As the following result shows, this reasoning leads to additional information about the factorization in Theorem 1.1. For simplicity, we shall consider only the scalar-valued case.

Corollary 3.1. *Let s be a normalized CNP kernel on Ω and let $k = sg$, with*

$$g_w(z) = \sum_{n=0}^{\infty} g_n(z)\bar{g}_n(w).$$

A function f belongs to the unit ball of \mathcal{H}_k if and only if there exist $\psi, \varphi_n \in \text{Mult}(\mathcal{H}_s)$, $n \geq 1$, with $\psi(z_0) = 0$ and

$$\|\psi h\|_{\mathcal{H}_s}^2 + \sum_n \|\varphi_n h\|_{\mathcal{H}_s}^2 \leq \|h\|_{\mathcal{H}_s}^2, \quad h \in \mathcal{H}_s,$$

such that

$$f = \frac{\sum_n g_n \varphi_n}{1 - \psi}.$$

Proof. This is a direct application of Theorem 1.1 (i) together with the equality (3.2). Here $\mathcal{E} = \mathbb{C}$ and Γ becomes an element of $\text{Mult}(\mathcal{H}_s, \mathcal{H}_s(l^2))$. \square

Some concrete examples of this type are discussed in Section 4.

3.2. The constructive approach. As already mentioned, our approach is different and it is based on an idea of Sarason [25, 26] which was further developed in [15]. Here is a short motivation for it. Recall from Lemma 2.2 that for each $z \in \Omega$, $s_z \in \text{Mult}(\mathcal{H}_k(\mathcal{E}))$. Therefore, for $F \in \mathcal{H}_k(\mathcal{E})$, the Sarason function of F

$$(3.3) \quad V_F(z) = 2\langle F, s_z F \rangle_{\mathcal{H}_k(\mathcal{E})} - \|F\|_{\mathcal{H}_k(\mathcal{E})}^2, \quad z \in \Omega,$$

is well defined. A straightforward computation shows that if s is the Szegő kernel and $k = s$ then $\text{Re } V_F$ is just the Poisson integral of $\|F\|_{\mathcal{E}}^2$, hence it satisfies

$$0 \leq \|F(z)\|_{\mathcal{E}}^2 \leq \text{Re } V_F(z).$$

It is the least harmonic majorant of $\|F\|_{\mathcal{E}}$. A remarkable fact proved in [15, Section 2] is that this inequality continues to hold for arbitrary normalized CNP kernels s , more precisely

$$(3.4) \quad 0 \leq \|F(z)\|_{\mathcal{E}}^2 \leq \frac{\|s_z F\|_{\mathcal{H}_s(\mathcal{E})}^2}{\|s_z\|^2} \leq \text{Re } V_F(z),$$

in particular, $\text{Re } V_F$ is a majorant of $\|F\|_{\mathcal{E}}$ as well. Moreover, if $\Omega \subset \mathbb{C}^d$, \mathcal{H}_s consists of analytic functions, V_F is harmonic when $d = 1$, or pluriharmonic when $d > 1$. Now assume that $\|F\|_{\mathcal{H}_s(\mathcal{E})} = 1$, set $\psi = \frac{V_F - 1}{V_F + 1}$, $\Phi = \frac{2}{V_F + 1} F$. Then $\psi(z_0) = 0$, $F = \frac{1}{1 - \psi} \Phi$ and a straightforward computation shows that

$$|\psi(z)|^2 + \|\Phi(z)\|_{\mathcal{E}}^2 = \frac{|V_F(z)|^2 - 2 \text{Re } V_F(z) + 1 + 4\|F(z)\|_{\mathcal{E}}^2}{|V_F + 1|^2} \leq 1.$$

For example, if $\mathcal{E} = \mathbb{C}$ and $M(\mathcal{H}_s) = H^\infty$ with equality of norms, then this proves part of our main theorem, and in fact for a single function $f \in H^2$ this proof was given by Sarason in [25, 26]. In the general case considered here, pointwise estimates as above cannot lead to a proof of the main theorem. However, the intuition behind our approach is the argument outlined here.

3.3. The proof of Theorem 1.1. For the remainder of this section, let k and s be reproducing kernels on Ω as in the statement of Theorem 1.1, i.e. s is a CNP kernel, normalized at z_0 , and $k = sg$ with $g \gg 0$. The key step is the following far-reaching generalization of the inequality (3.4).

Lemma 3.2. *Let $F \in \mathcal{H}_k(\mathcal{E})$. Then*

$$\langle s_w F, F \rangle_{\mathcal{H}_k(\mathcal{E})} + \langle F, s_z F \rangle_{\mathcal{H}_k(\mathcal{E})} - \|F\|_{\mathcal{H}_k(\mathcal{E})}^2 - \frac{\langle s_w F, s_z F \rangle_{\mathcal{H}_k(\mathcal{E})}}{s_w(z)} \gg 0.$$

Proof. Let $s_w(z) = \frac{1}{1-u_w(z)}$ with $u_w(z) = \sum_n \overline{u_n(w)}u_n(z)$, $u_n(z_0) = 0$ for all n . Then $\frac{1}{s_w(z)} = 1 - u_w(z)$ and hence

$$(3.5) \quad \begin{aligned} & \langle s_w F, F \rangle_{\mathcal{H}_k(\mathcal{E})} + \langle F, s_z F \rangle_{\mathcal{H}_k(\mathcal{E})} - \|F\|_{\mathcal{H}_k(\mathcal{E})}^2 - \frac{\langle s_w F, s_z F \rangle_{\mathcal{H}_k(\mathcal{E})}}{s_w(z)} \\ & = u_w(z) \langle s_w F, s_z F \rangle_{\mathcal{H}_k(\mathcal{E})} - \langle (s_w - 1)F, (s_z - 1)F \rangle_{\mathcal{H}_k(\mathcal{E})}. \end{aligned}$$

Let $z_1, \dots, z_k \in \Omega$, $a_1, \dots, a_k \in \mathbb{C}$, set $h = \sum_{i=1}^k a_i s_{z_i}$, and $h_n = \sum_i a_i \overline{u_n(z_i)} s_{z_i}$ for $n = 1, 2, \dots$. Then $h, h_n \in \text{Mult}(\mathcal{H}_s) \subset \text{Mult}(\mathcal{H}_k(\mathcal{E}))$,

$$\begin{aligned} \sum_n \|h_n F\|_{\mathcal{H}_k(\mathcal{E})}^2 &= \sum_n \left\| \sum_i a_i \overline{u_n(z_i)} s_{z_i} F \right\|_{\mathcal{H}_k(\mathcal{E})}^2 \\ &= \sum_{i,j} a_i \bar{a}_j u_{z_i}(z_j) \langle s_{z_i} F, s_{z_j} F \rangle_{\mathcal{H}_k(\mathcal{E})}. \end{aligned}$$

Moreover, since $h - h(z_0) = \sum_i a_i (s_{z_i} - 1)$,

$$\|(h - h(z_0))F\|_{\mathcal{H}_k(\mathcal{E})}^2 = \sum_{i,j} a_i \bar{a}_j \langle (s_{z_i} - 1)F, (s_{z_j} - 1)F \rangle_{\mathcal{H}_k(\mathcal{E})}.$$

By Lemma 2.1 (ii) we have $h - h(z_0) = \sum_n M_{u_n} M_{u_n}^* h = \sum_n u_n h_n$ and hence by Lemma 2.2

$$\|(h - h(z_0))F\|_{\mathcal{H}_k(\mathcal{E})}^2 = \left\| \sum_n u_n (h_n F) \right\|_{\mathcal{H}_k(\mathcal{E})}^2 \leq \sum_n \|h_n F\|_{\mathcal{H}_k(\mathcal{E})}^2.$$

Thus, by (3.5)

$$\begin{aligned} & \sum_{i,j} a_i \bar{a}_j \left(\langle s_{z_i} F, F \rangle_{\mathcal{H}_k(\mathcal{E})} + \langle F, s_{z_j} F \rangle_{\mathcal{H}_k(\mathcal{E})} - \|F\|_{\mathcal{H}_k(\mathcal{E})}^2 - \frac{\langle s_{z_i} F, s_{z_j} F \rangle_{\mathcal{H}_k(\mathcal{E})}}{s_{z_i}(z_j)} \right) \\ & = \sum_{i,j} a_i \bar{a}_j (u_{z_i}(z_j) \langle s_{z_i} F, s_{z_j} F \rangle_{\mathcal{H}_k(\mathcal{E})} - \langle (s_{z_i} - 1)F, (s_{z_j} - 1)F \rangle_{\mathcal{H}_k(\mathcal{E})}) \\ & = \sum_n \|h_n F\|_{\mathcal{H}_k(\mathcal{E})}^2 - \|(h - h(z_0))F\|_{\mathcal{H}_k(\mathcal{E})}^2 \geq 0. \end{aligned}$$

□

An immediate application of the lemma yields the general version of the inequality (3.4).

Corollary 3.3. *Let s, k be reproducing kernels on the nonempty set Ω such that s is a normalized CNP kernel and $k = sg$ with $g >> 0$. Let*

$F \in \mathcal{H}_k(\mathcal{E})$ and let V_F be the Sarason function of F given by (3.3). Then

$$\frac{s_z(z)}{k_z(z)} \|F(z)\|_{\mathcal{E}}^2 \leq \operatorname{Re} V_F(z).$$

Proof. Apply Lemma 3.2 to obtain

$$\frac{V_F(z) + \overline{V_F(w)}}{2} - \frac{\langle s_w F, s_z F \rangle_{\mathcal{H}_k(\mathcal{E})}}{s_w(z)} \gg 0.$$

For $z = w \in \Omega$ this gives

$$\frac{\|s_z F\|_{\mathcal{H}_k(\mathcal{E})}^2}{s_z(z)} \leq \operatorname{Re} V_F(z),$$

and the standard estimate

$$s_z(z)^2 \|F(z)\|_{\mathcal{E}}^2 \leq k_z(z) \|s_z F\|_{\mathcal{H}_k}^2,$$

gives the inequality in the statement. \square

Our next step relates the positivity result in Lemma 3.2 to the multiplier-norm estimates in Theorem 1.1

Lemma 3.4. *Let $f_1, \dots, f_N \in \mathcal{H}_k$ be finite linear combinations of reproducing kernels in \mathcal{H}_k , let $\{e_n\}$ be an orthonormal basis in \mathcal{E} and set*

$$F(z) = \sum_{n=1}^N f_n(z) e_n.$$

Then $F \in \operatorname{Mult}(\mathcal{H}_s, \mathcal{H}_k(\mathcal{E}))$ and its Sarason function V_F belongs to $\operatorname{Mult}(\mathcal{H}_s)$. Moreover, if $h \in \mathcal{H}_s$ then

$$(3.6) \quad \|hF\|_{\mathcal{H}_k(\mathcal{E})}^2 \leq \operatorname{Re} \langle V_F h, h \rangle_{\mathcal{H}_s},$$

and if $a \in \mathbb{C}$, $\operatorname{Re} a > 0$,

$$(3.7) \quad \|(V_F - a)h\|_{\mathcal{H}_s}^2 + 4 \operatorname{Re} a \|hF\|_{\mathcal{H}_k(\mathcal{E})}^2 \leq \|(V_F + \bar{a})h\|_{\mathcal{H}_s}^2.$$

Proof. When F has the special form given in the statement, its Sarason function can be written as

$$V_F(z) = 2 \sum_{i,j=1}^m c_{ij} k_{z_i}(z_j) s_{z_i}(z) - \|F\|_{\mathcal{H}_k(\mathcal{E})}^2$$

for a suitable choice of scalars c_{ij} , which implies that $V_F \in \text{Mult}(\mathcal{H}_s)$. If $h = \sum_i a_i s_{z_i}$, then $h \in \text{Mult}(\mathcal{H}_k)$ and (3.6) is equivalent to

$$\begin{aligned} \sum_{i,j} a_i \bar{a}_j \langle s_{z_i} F, s_{z_j} F \rangle_{\mathcal{H}_k(\mathcal{E})} &\leq \frac{1}{2} \sum_{i,j} a_i \bar{a}_j s_{z_i}(z_j) (V_F(z_j) + \overline{V_F(z_i)}) \\ &= \sum_{i,j} a_i \bar{a}_j s_{z_i}(z_j) (\langle F, s_{z_j} F \rangle_{\mathcal{H}_k(\mathcal{E})} + \langle s_{z_i} F, F \rangle_{\mathcal{H}_k(\mathcal{E})} - \|F\|_{\mathcal{H}_k(\mathcal{E})}^2) \end{aligned}$$

and the inequality follows by an application of Lemma 3.2 and the Schur product theorem. Moreover, (3.7) is just a reformulation of (3.6), since

$$\|(V_F + \bar{a})h\|_{\mathcal{H}_s}^2 - \|(V_F - a)h\|_{\mathcal{H}_s}^2 = 4 \operatorname{Re} a \operatorname{Re} \langle V_F h, h \rangle_{\mathcal{H}_s}.$$

Finally, (3.6) together with the fact that finite linear combinations of reproducing kernels in \mathcal{H}_s are dense in \mathcal{H}_s , shows that $F \in \text{Mult}(\mathcal{H}_s, \mathcal{H}_k(\mathcal{E}))$, and that both inequalities hold for arbitrary $h \in \mathcal{H}_s$. \square

We can now prove the factorization in Theorem 1.1 (ii) in a slightly more general form, which turns out to be useful in applications.

Proposition 3.5. *Let $F \in \mathcal{H}_k(\mathcal{E})$ and let $a \in \mathbb{C}$ with $\operatorname{Re} a > 0$. If V_F is the Sarason function of F ,*

$$\psi_a = \frac{V_F - a}{V_F + \bar{a}}, \quad \Phi_a = \frac{2}{V_F + \bar{a}} F,$$

then $F = \frac{\operatorname{Re} a}{1 - \psi_a} \Phi_a$, $\psi_a \in \text{Mult}(\mathcal{H}_s)$, $\Phi_a \in \text{Mult}(\mathcal{H}_s, \mathcal{H}_k(\mathcal{E}))$ with

$$\|\psi_a h\|_{\mathcal{H}_s}^2 + \operatorname{Re} a \|\Phi_a h\|_{\mathcal{H}_k(\mathcal{E})}^2 \leq \|h\|_{\mathcal{H}_s}^2, \quad h \in \mathcal{H}_s.$$

Proof. Assume first that F has the form in Lemma 3.4 and recall from the lemma that in this case $V_F \in \text{Mult}(\mathcal{H}_s)$. If $h = (V_F + \bar{a})u$ with $u \in \mathcal{H}_s$, the inequality (3.7) applied to u can be rewritten as

$$\left\| \frac{V_F - a}{V_F + \bar{a}} h \right\|_{\mathcal{H}_s}^2 + \operatorname{Re} a \left\| \frac{2}{V_F + \bar{a}} F h \right\|_{\mathcal{H}_k(\mathcal{E})}^2 \leq \|h\|_{\mathcal{H}_s}^2,$$

which is precisely the inequality in the statement. We claim that $M_{V_F + \bar{a}}$ has dense range. Indeed, if $h \in (M_{V_F + \bar{a}} \mathcal{H}_s)^\perp$, then

$$\langle (M_{V_F + \bar{a}} + M_{V_F + \bar{a}}^*) h, h \rangle_{\mathcal{H}_s} = 0,$$

and by (3.6) it follows that

$$0 = \operatorname{Re} \langle (V_F + \bar{a}) h, h \rangle_{\mathcal{H}_s} \geq \|h\|_{\mathcal{H}_s}^2 \operatorname{Re} a + \|F h\|_{\mathcal{H}_k(\mathcal{E})}^2,$$

which implies that $h = 0$ and proves the claim. Thus for F as above we obtain that $\psi_a \in \text{Mult}(\mathcal{H}_s)$, $\Phi_a \in \text{Mult}(\mathcal{H}_s, \mathcal{H}_k(\mathcal{E}))$, and the inequality in the statement holds for all $h \in \mathcal{H}_s$.

Clearly, every $F \in \mathcal{H}_k(\mathcal{E})$ can be approximated in $\mathcal{H}_k(\mathcal{E})$ by a sequence (F_N) as above and from the previous argument we have that $\psi_a^N = \frac{V_{F_N} - a}{V_{F_N} + \bar{a}} \in \text{Mult}(\mathcal{H}_s)$, $(\sqrt{\text{Re } a})\Phi_a^N = \frac{2\sqrt{\text{Re } a}}{V_{F_N} + \bar{a}}F_N \in \text{Mult}(\mathcal{H}_s, \mathcal{H}_k(\mathcal{E}))$ are contractive multipliers. Also note that (V_{F_N}) converges pointwise to V_F in Ω .

Then for $h \in \mathcal{H}_s$, $(\psi_a^N h)$ converges weakly in \mathcal{H}_s to $\psi_a h$. Similarly, $(\Phi_a^N h)$ converges weakly in $\mathcal{H}_k(\mathcal{E})$ to $\Phi_a h$, because the sequence is bounded and satisfies

$$\lim_{N \rightarrow \infty} \langle \Phi_a^N h, k_z e \rangle_{\mathcal{H}_k(\mathcal{E})} = \lim_{N \rightarrow \infty} \frac{2}{V_{F_N}(z) + \bar{a}} \langle F_N(z), e \rangle_{\mathcal{E}} = \langle \Phi_a h, k_z e \rangle_{\mathcal{H}_k(\mathcal{E})},$$

for all $z \in \Omega$, $e \in \mathcal{E}$. Thus $(\psi_a^N h, \Phi_a^N h)$ converges weakly in $\mathcal{H}_s \oplus \mathcal{H}_k(\mathcal{E})$ which implies

$$\|\psi_a h\|_{\mathcal{H}_s}^2 + \text{Re } a \|\Phi_a h\|_{\mathcal{H}_k(\mathcal{E})}^2 \leq \liminf_{N \rightarrow \infty} (\|\psi_a^N h\|_{\mathcal{H}_s}^2 + \text{Re } a \|\Phi_a^N h\|_{\mathcal{H}_k(\mathcal{E})}^2) \leq \|h\|_{\mathcal{H}_s}^2,$$

and completes the proof. \square

Note that the factorization holds without any assumption on $\|F\|_{\mathcal{H}_k(\mathcal{E})}$, but we do not control the value $\psi_a(z_0)$.

The factorization in Theorem 1.1 (ii) is a direct application. If $\|F\|_{\mathcal{H}_k(\mathcal{E})} = 1$, the result is obtained for $a = 1$, $\psi = \psi_1$, $\Phi = \Phi_1$. Note that since $V_F(z_0) = 1$, we have $\psi(z_0) = 0$. For $\|F\|_{\mathcal{H}_k(\mathcal{E})} < 1$, the factorization, and hence the implication (a) \Rightarrow (b) in Theorem 1.1 (i), is obtained as follows.

Let $w \in \Omega$ and apply the previous argument to the function

$$F_w = \left(F, \sqrt{\frac{1 - \|F\|_{\mathcal{H}_k(\mathcal{E})}^2}{k_w(w)}} k_w \right) \in \mathcal{H}_k(\mathcal{E} \oplus \mathbb{C}),$$

which has unit norm in the space above. The Sarason function of F_w is

$$V_{F_w} = V_F + (1 - \|F\|_{\mathcal{H}_k(\mathcal{E})}^2)(2s_w - 1),$$

and if

$$\psi_w = \frac{V_{F_w} - 1}{V_{F_w} + 1}, \quad \Phi_w = \frac{2}{V_{F_w} + 1} F_w,$$

we obtain $F = \frac{1}{1 - \psi_w} P_{\mathcal{E}} \Phi_w$. Moreover, it is easy to verify that if $w, w' \in \Omega$ with $s_w \neq s_{w'}$, the corresponding factorizations are different.

In order to complete the proof of Theorem 1.1 we need to verify the implication (b) \Rightarrow (a) together with the uniqueness of the factorization in the case when $\|F\|_{\mathcal{H}_k(\mathcal{E})} = 1$. The argument is based on the following lemma which contains a useful approximation result.

Lemma 3.6. *Let $\psi \in \text{Mult}(\mathcal{H}_s)$, $\Phi \in \text{Mult}(\mathcal{H}_s, \mathcal{H}_k(\mathcal{E}))$, and assume that*

$$(3.8) \quad \|\psi h\|_{\mathcal{H}_s}^2 + \|\Phi h\|_{\mathcal{H}_k(\mathcal{E})}^2 \leq \|h\|_{\mathcal{H}_s}^2, \quad h \in \mathcal{H}_s.$$

(i) *For $0 < r < 1$, the function $F^r = \frac{\Phi}{1-r\psi}$ belongs to $\text{Mult}(\mathcal{H}_s, \mathcal{H}_k(\mathcal{E}))$ and satisfies*

$$(3.9) \quad \|F^r h\|_{\mathcal{H}_k(\mathcal{E})}^2 \leq \text{Re} \left\langle \frac{1+r\psi}{1-r\psi} h, h \right\rangle_{\mathcal{H}_s}, \quad h \in \mathcal{H}_s.$$

(ii) *If $|\psi(z_0)| < 1$, F^r converges weakly in $\mathcal{H}_k(\mathcal{E})$ to $F = \frac{\Phi}{1-\psi}$ when $r \rightarrow 1^-$, and we have*

$$s_w(z) \left(\frac{1+\psi(z)}{1-\psi(z)} + \overline{\frac{1+\psi(w)}{1-\psi(w)}} \right) - 2\langle s_w F, s_z F \rangle_{\mathcal{H}_k(\mathcal{E})} \gg 0, \quad z, w \in \Omega.$$

Proof. (i) Fix $0 < r < 1$. From (3.8) it follows that $\psi \in \text{Mult}(\mathcal{H}_s)$ is contractive, i.e. $\frac{1}{1-r\psi} \in \text{Mult}(\mathcal{H}_s)$. Since $\Phi \in \text{Mult}(\mathcal{H}_s, \mathcal{H}_k(\mathcal{E}))$ and $F^r = \frac{\Phi}{1-r\psi}$ we obtain that $F^r \in \text{Mult}(\mathcal{H}_s, \mathcal{H}_k(\mathcal{E}))$. Moreover, (3.8) also implies that

$$r^2 \|\psi h\|_{\mathcal{H}_s}^2 + \|\Phi h\|_{\mathcal{H}_k(\mathcal{E})}^2 \leq \|h\|_{\mathcal{H}_s}^2, \quad h \in \mathcal{H}_s,$$

and when applied to $(1-r\psi)^{-1}h$ it yields

$$\|F^r h\|_{\mathcal{H}_k(\mathcal{E})}^2 \leq \left\| \frac{1}{1-r\psi} h \right\|^2 - \left\| \frac{r\psi}{1-r\psi} h \right\|^2.$$

Then the result follows from $\frac{1+r\psi}{1-r\psi} - 1 = \frac{2r\psi}{1-r\psi}$, $\frac{1+r\psi}{1-r\psi} + 1 = \frac{2}{1-r\psi}$.

(ii) We have that $F^r(z) \rightarrow F(z)$ when $r \rightarrow 1^-$, and an application of (i) with $h = 1 = s_{z_0}$, gives $\|F^r\|_{\mathcal{H}_k(\mathcal{E})} \leq \frac{1+|\psi(z_0)|}{1-|\psi(z_0)|}$ which shows that $F^r \rightarrow F$ weakly in $\mathcal{H}_k(\mathcal{E})$ when $r \rightarrow 1^-$. To see the second assertion, let $h \in \mathcal{H}_s$ with

$$h = \sum_{i=1}^n a_i s_{z_i},$$

and apply again (i) to obtain

$$\begin{aligned}
& \sum_{i,j=1}^n a_i \bar{a}_j \langle s_{z_i} F, s_{z_j} F \rangle_{\mathcal{H}_k(\mathcal{E})} = \|hF\|_{\mathcal{H}_k(\mathcal{E})}^2 \leq \liminf_{r \rightarrow 1^-} \|hF^r\|_{\mathcal{H}_k(\mathcal{E})}^2 \\
& \leq \limsup_{r \rightarrow 1^-} \operatorname{Re} \left\langle \frac{1+r\psi}{1-r\psi} h, h \right\rangle_{\mathcal{H}_s} \\
& = \sum_{i,j=1}^n a_i \bar{a}_j \frac{1}{2} s_{z_i}(z_j) \left(\frac{1+\psi(z_j)}{1-\psi(z_j)} + \overline{\frac{1+\psi(z_i)}{1-\psi(z_i)}} \right),
\end{aligned}$$

which completes the proof. \square

An immediate consequence is the implication (b) \Rightarrow (a) in Theorem 1.1 (i). Indeed, if (b) holds, then (3.8) holds and in addition, $\psi(z_0) = 0$, hence (3.9) with $h = 1$ gives $\|F^r\|_{\mathcal{H}_k(\mathcal{E})}^2 \leq 1$. By part (ii) of the lemma we obtain $\|F\|_{\mathcal{H}_k(\mathcal{E})}^2 \leq 1$.

The uniqueness assertion in part (ii) of Theorem 1.1 is another direct application. Let

$$L(z, w) = s_w(z) \left(\frac{1+\psi(z)}{1-\psi(z)} + \overline{\frac{1+\psi(w)}{1-\psi(w)}} \right) - 2\langle s_w F, s_z F \rangle_{\mathcal{H}_k(\mathcal{E})}$$

denote the positive definite function from part (ii) of Lemma 3.6. If $\psi(z_0) = 0$ and $\|F\|_{\mathcal{H}_k(\mathcal{E})} = 1$, it follows that $L(z_0, z_0) = 0$. Then the standard inequality

$$|L(z, z_0)|^2 \leq L(z_0, z_0)L(z, z),$$

gives $L(z, z_0) = 0$, hence

$$\frac{1+\psi(z)}{1-\psi(z)} = 2\langle F, s_z F \rangle_{\mathcal{H}_k(\mathcal{E})} - 1 = V_F(z),$$

i.e., $\psi = \frac{V_F - 1}{V_F + 1}$, and the assertion follows.

This completes the proof of Theorem 1.1.

Finally, we record the following sharpening of the first part of Lemma 3.6 (ii). Let $F \in \mathcal{H}_k(\mathcal{E})$ with $\|F\|_{\mathcal{H}_k(\mathcal{E})} = 1$ and let $F = \frac{\Phi}{1-\psi}$ be the unique factorization of Theorem 1.1. For $0 < r < 1$, define as above $F^r = \frac{\Phi}{1-r\psi}$. Then F^r converges in norm to F as $r \rightarrow 1^-$. Indeed, we already saw that F^r converges weakly to F . Moreover, as remarked after the proof of Lemma 3.6, $\|F^r\|_{\mathcal{H}_k(\mathcal{E})} \leq 1$ for all $r < 1$. Since $\|F\|_{\mathcal{H}_k(\mathcal{E})} = 1$, it follows that the convergence is actually in norm in this case.

4. EXAMPLES AND APPLICATIONS

4.1. **Examples.** 1) *Weighted Bergman spaces.* Let μ be a finite positive measure on \mathbb{B}_d , $d \in \mathbb{N}$, such that for all $f \in \text{Hol}(\mathbb{B}_d)$ and for any compact subset $K \subset \mathbb{B}_d$, there exists $c_K > 0$ such that

$$|f(z)|^2 \leq c_K \int_{\mathbb{B}_d} |f|^2 d\mu,$$

for all $f \in \text{Hol}(\mathbb{B}_d)$ and all $z \in K$.

The corresponding Bergman space $L_a^2(\mu) = L^2(\mu) \cap \text{Hol}(\mathbb{B}_d)$ is a Hilbert space with reproducing kernel k^μ . Here we shall only consider the scalar case, but all considerations extend to the vector-valued version $L_a^2(\mu, \mathcal{E})$ defined correspondingly.

Since every analytic contractive $\mathcal{B}(l^2, \mathbb{C})$ -valued function induces a contractive multiplier on $L_a^2(\mu)$, it follows by Corollary 2.3 that $k^\mu/s \gg 0$ for any analytic normalized CNP kernel in \mathbb{B}_d . In what follows we shall focus on the Drury-Arveson kernel $s = \frac{1}{1-\langle z, w \rangle}$. In this case Theorem 1.1 shows that any $f \in L_a^2(\mu)$ of norm at most 1, can be written as $f = \frac{\varphi}{1-\psi}$, where $\psi \in \text{Mult}(H_d^2)$ with $\psi(0) = 0$, with multiplier norm at most 1, and $\varphi \in \text{Hol}(\mathbb{B}_d)$ satisfies that

$$\int_{\mathbb{B}_d} |g\varphi|^2 d\mu \leq \|g\|_{H_d^2}^2,$$

i.e. $|\varphi|^2 d\mu$ is a *Carleson measure* for H_d^2 . These objects are best understood when $d = 1$.

Theorem 1.1 and Corollary 3.3 lead to interesting inequalities: If $f \in L_a^2(\mu)$ with $\|f\| = 1$, then the Sarason function of f is given by

$$V_f(z) = \int_{\mathbb{B}_d} |f(w)|^2 \frac{1 + \langle z, w \rangle}{1 - \langle z, w \rangle} d\mu$$

and we conclude

$$\left\| \frac{V_f - 1}{V_f + 1} h \right\|_{H_d^2}^2 + \left\| \frac{2fh}{V_f + 1} \right\|_{L_a^2(\mu)}^2 \leq \|h\|_{H_d^2}^2$$

for all $h \in H_d^2$. Moreover, Corollary 3.3 says that

$$|f(z)|^2 \leq \frac{k_z^\mu(z)}{(1 - |z|^2)} \int_{\mathbb{B}_d} |f(w)|^2 \frac{1 - |\langle z, w \rangle|^2}{|1 - \langle z, w \rangle|^2} d\mu(w).$$

Let us apply Corollary 3.1 when $\mu = A$, the normalized area measure on the unit disc $\mathbb{B}_1 = \mathbb{D}$. In this case $L_a^2(A)$ is denoted simply by L_a^2

and its reproducing kernel is

$$k_w^A(z) = \frac{1}{(1 - \bar{w}z)^2} = s_w^2(z),$$

where $s_w(z) = \frac{1}{1 - z\bar{w}}$ is the Szegő kernel. Here we have that $k^A/s = G(z)G(w)^*$ with $G(z) = (1, z, z^2, \dots)$.

Then by Corollary 3.1 and the fact that $\text{Mult}(\mathcal{H}_s) = H^\infty$, it follows that $f \in L_a^2$ with $\|f\|_{L_a^2} \leq 1$, if and only if there are $\psi, \varphi_n \in H^\infty$, $n \geq 1$ with $\psi(0) = 0$, $|\psi(z)|^2 + \sum_{n=1}^\infty |\varphi_n(z)|^2 \leq 1$, such that

$$f(z) = \frac{\sum_{n=1}^\infty z^n \varphi_n(z)}{1 - \psi(z)}.$$

Similar calculations can be carried out for standard weighted Bergman on the unit ball \mathbb{B}_d , or the polydisc \mathbb{D}^d .

2) *Hardy spaces.* The Hardy space $H^2(\mathbb{B}_d)$ is defined as the closure of analytic polynomials in $L^2(\sigma_d)$, where σ_d is the normalized Lebesgue measure on the unit sphere. Its reproducing kernel k is given by $k = s^d$, where s is the Drury-Arveson kernel. All considerations above apply, the corresponding calculations go through and one obtains similar results. For example, for $d = 2$ Corollary 3.1 implies that f is in the unit ball of $H^2(\mathbb{B}_2)$ if and only if

$$(4.1) \quad f(z) = \frac{\sum_\alpha c_\alpha z^\alpha \varphi_\alpha(z)}{1 - \psi(z)},$$

where $\alpha = (\alpha_1, \alpha_2)$ is a multi-index, $\|c_\alpha z^\alpha\|_{H_2^2} = 1$, $\psi(0) = 0$ and

$$\begin{pmatrix} \psi \\ \varphi_{(1,0)} \\ \varphi_{(0,1)} \\ \vdots \end{pmatrix}$$

is a contraction from H_2^2 to $H_2^2(l^2)$.

If $f \in H^2(\mathbb{B}_d)$, $d \in \mathbb{N}$ has unit norm, its Sarason function is

$$V_f(z) = \int_{|w|=1} |f(w)|^2 \frac{1 + \langle z, w \rangle}{1 - \langle z, w \rangle} d\sigma_d(w),$$

and Theorem 1.1 gives

$$\left\| \frac{V_f - 1}{V_f + 1} h \right\|_{H_d^2}^2 + \left\| \frac{2fh}{V_f + 1} \right\|_{H^2(\mathbb{B}_d)}^2 \leq \|h\|_{H_d^2}^2$$

for all $h \in H_d^2$. Also, by Corollary 3.3 we have

$$(1 - |z|^2)^{d-1} |f(z)|^2 \leq \int_{|w|=1} |f(w)|^2 \frac{1 - |\langle z, w \rangle|^2}{|1 - \langle z, w \rangle|^2} d\sigma_d(w).$$

In product domains, for example \mathbb{D}^d , $d \in \mathbb{N} \cup \{\infty\}$, the Hardy space $H^2(\mathbb{D}^d)$ is defined as the closure of analytic polynomials in $L^2(\sigma_1^d)$, where σ_1^d is the product of d copies of σ_1 . In the case when $d = \infty$, we consider analytic polynomials in a finite number of variables. It is well known (see for example [16]), that $H^2(\mathbb{D}^\infty)$ can be identified with the space of Dirichlet series with square summable coefficients. It can be viewed as a reproducing kernel Hilbert space on the set Ω consisting of points in \mathbb{D}^∞ whose coordinates form an l^2 -sequence. In all cases the reproducing kernel is given by

$$k_w(z) = \prod_{j=1}^d s_{w_j}^0(z_j), \quad z = (z_j), w = (w_j)$$

where s^0 is the Szegő kernel. Clearly, each factor of this product is a normalized CNP factor of k . Similar calculations can be performed using these kernels. Also, according to Corollary 2.3 (i), if $u \in H^\infty(\mathbb{D}^d)$ of norm at most 1 with $u(0) = 0$, and $s_w(z) = \frac{1}{1-u(z)u(w)}$, then $k/s \gg 0$.

4.2. Invariant subspaces. A direct application of Theorem 1.1 gives (see [4] for the case $k = s$) that if $k = sg$ with s a normalized CNP kernel and g positive definite, then the zero-sets of \mathcal{H}_k -functions coincide with the zero-sets of functions in $\text{Mult}(\mathcal{H}_s, \mathcal{H}_k)$. This is a somewhat surprising result since the second space of functions might be considerably smaller. The idea extends to multiplier-invariant subspaces in a natural way. If \mathcal{E} is a separable Hilbert space, a closed subspace \mathcal{M} of $\mathcal{H}_k(\mathcal{E})$ is called multiplier-invariant if $\varphi\mathcal{M} \subset \mathcal{M}$ whenever $\varphi \in \text{Mult}(\mathcal{H}_k)$. The multiplier-invariant subspace generated by $\mathcal{S} \subset \mathcal{H}_k(\mathcal{E})$ is the closure of in \mathcal{H}_k of $\{\varphi F : \varphi \in \text{Mult}(\mathcal{H}_k), F \in \mathcal{S}\}$ and will be denoted by $[S]$. We also write $[\{F\}] = [F]$.

Corollary 4.1. *If $F \in \mathcal{H}_k(\mathcal{E})$, $\|F\|_{\mathcal{H}_k(\mathcal{E})} = 1$, and let $F = \frac{1}{1-\psi}\Phi$ be the factorization given by Theorem 1.1. Then $[F] = [\Phi]$. In particular, every multiplier-invariant subspace of $\mathcal{H}_k(\mathcal{E})$ is generated by elements of $\text{Mult}(\mathcal{H}_s, \mathcal{H}_k(\mathcal{E}))$.*

Proof. Let $F \in \mathcal{H}_k(\mathcal{E})$, $\|F\|_{\mathcal{H}_k(\mathcal{E})} = 1$ with $F = \frac{1}{1-\psi}\Phi$ as in Theorem 1.1. Then by definition and Lemma 2.2, $\Phi = (1-\psi)F \in [F]$. Conversely, the same argument shows that for $0 < r < 1$, we have $F^r = \frac{1}{1-r\psi}\Phi \in [\Phi]$, hence by Lemma 3.6 (ii) it follows that $F \in [\Phi]$. The second part of the statement is an obvious consequence of the first. \square

4.3. Extremal functions. Let k, s be reproducing kernels on Ω such that s is a normalized CNP kernel, $s_{z_0} = 1$, and $k/s \gg 0$. We say that $F \in \mathcal{H}_k(\mathcal{E})$ is extremal if

$$\langle \varphi F, F \rangle_{\mathcal{H}_k(\mathcal{E})} = \varphi(z_0)$$

for all $\varphi \in \text{Mult}(\mathcal{H}_k)$. These functions generate wandering subspaces for shift-invariant subspaces and in certain cases they play an essential role in that theory (see for example, [5], [29], [8]). If F is extremal in \mathcal{H}_k then it has unit norm and

$$V_F(z) = 2\langle F, s_z F \rangle_{\mathcal{H}_k(\mathcal{E})} - 1 = 1,$$

hence in the notation in Theorem 1.1 we have $\psi = 0$, and $\Phi = F$.

Corollary 4.2. *Let k, s be as above.*

(i) *Every extremal function in $F \in \mathcal{H}_k(\mathcal{E})$ is a contractive multiplier from \mathcal{H}_s into $\mathcal{H}_k(\mathcal{E})$. In particular, if F is extremal in $\mathcal{H}_k(\mathcal{E})$, then*

$$\|F(z)\|_{\mathcal{E}}^2 \leq \frac{k_z(z)}{s_z(z)}.$$

(ii) *If the linear span of the kernels s_z , $z \in \Omega$, is dense in \mathcal{H}_k , then a function $F \in \mathcal{H}_k(\mathcal{E})$ of norm 1 is an extremal function in $\mathcal{H}_k(\mathcal{E})$ if and only if it is a contractive multiplier from \mathcal{H}_s into $\mathcal{H}_k(\mathcal{E})$.*

Proof. The first part is from Theorem 1.1, while the second can be proved either directly, or by an application of Corollary 3.3. (ii) Suppose that F is a contractive multiplier from \mathcal{H}_s into $\mathcal{H}_k(\mathcal{E})$. By Theorem 1.1 (ii) there is a unique representation $F = \Phi/(1 - \psi)$, where $\|\psi h\|_{\mathcal{H}_s}^2 + \|\Phi h\|_{\mathcal{H}_k(\mathcal{E})}^2 \leq \|h\|_{\mathcal{H}_s}^2$ for all $h \in \mathcal{H}_s$. Since F is a contractive multiplier from \mathcal{H}_s into $\mathcal{H}_k(\mathcal{E})$, it follows from uniqueness that $\Phi = F$ and $\psi = 0$. In particular, the Sarason function of F satisfies $V_F = 1$, and hence $\langle F, s_z F \rangle_{\mathcal{H}_k(\mathcal{E})} = 1$ for all $z \in \Omega$. The density assumption now implies that F is an extremal function in $\mathcal{H}_k(\mathcal{E})$. \square

Part (ii) extends to the general context a very recent result obtained by D. Seco [27] for the Dirichlet space. Part (i) of the corollary recovers some known results, especially when $\mathcal{E} = \mathbb{C}$. For example, when $k = s$ part (i) can be found in [19]. For weighted Bergman spaces on the unit disc a slightly stronger result holds, since (see subsection 4.1 above) the condition

$$\text{Re } V_F(z) = \int_{\mathbb{D}} \frac{1 - |wz|^2}{|1 - \bar{w}z|^2} \|F(z)\|_{\mathcal{E}}^2 d\mu(z) = 1,$$

is in general more restrictive than the conclusion of the corollary. Recently, Corollary 4.2 (i) has been established in [8] in the case when

$\Omega = \mathbb{B}_d$, $z_0 = 0$, multiplication by the coordinates form a d -contraction on \mathcal{H}_k , and s is the Drury-Arveson kernel. Recall that $k/s \gg 0$ by Corollary 2.3 (ii). In this case extremal functions F satisfy $\|F(z)\|_{\mathcal{E}}^2 \leq (1 - |z|^2)k_z(z)$. By Corollary 4.2, these results continue to hold when $d = \infty$. Another situation when the corollary applies is when $\Omega \subset \mathbb{D}^d$, $d \in \mathbb{N} \cup \{\infty\}$, and multiplication by each coordinate is a contraction on \mathcal{H}_k . If $z_0 = 0$, and $s_{jw}(z) = \frac{1}{1 - z_j \bar{w}_j}$ then $k/s_j \gg 0$, $j \geq 1$ and clearly,

$$\mathcal{H}_{s_j} = H^{2,j} = \{h : h(z) = v(z_j), v \in H^2\}.$$

Corollary 4.2 implies that extremal functions $F \in \mathcal{H}_k(\mathcal{E})$ are contractive multipliers from each $H^{2,j}$ into \mathcal{H}_k and satisfy $\|F(z)\|_{\mathcal{E}}^2 \leq (1 - \max_j |z_j|^2)k_z(z)$.

The pointwise estimates for extremal functions can also be obtained from Corollary 2.3 (i).

Corollary 4.3. *Let k be a reproducing kernel on Ω and set*

$$\alpha_k(z) = \sup\{u(z)u^*(z) : u \in \text{Mult}(\mathcal{H}_k(l^2), \mathcal{H}_k), u(z_0) = 0, \|M_u\| \leq 1\}.$$

If F is extremal in $\mathcal{H}_k(\mathcal{E})$, then

$$\|F(z)\|_{\mathcal{E}}^2 \leq (1 - \alpha_k(z))k_z(z).$$

Proof. If $\text{Mult}(\mathcal{H}_k(l^2), \mathcal{H}_k)$ contains only constant functions there is nothing to prove. If there exists a non-zero $u \in \text{Mult}(\mathcal{H}_k(l^2), \mathcal{H}_k)$ with $u(z_0) = 0$, $\|M_u\| \leq 1$, then $s_w(z) = \frac{1}{1 - u(z)u^*(w)}$ is a normalized CNP kernel and Corollary 2.3 (i) shows that $k/s \gg 0$. Then by Corollary 4.2 every extremal function $F \in \mathcal{H}_k(\mathcal{E})$ satisfies

$$\|F(z)\|_{\mathcal{E}}^2 \leq (1 - u(z)u^*(z))k_z(z),$$

and the result follows. \square

In some cases, the function $\alpha_k(z)$ can be easily estimated. For example, if $\Omega = \mathbb{B}_d$, $z_0 = 0$, and \mathcal{H}_k consists of analytic functions, it follows that $\text{Mult}(\mathcal{H}_k(l^2), \mathcal{H}_k)$ is contractively contained in the space of bounded analytic $\mathcal{B}(l^2, \mathbb{C})$ -valued functions. Let $u \in \text{Mult}(\mathcal{H}_k(l^2), \mathcal{H}_k)$, $u(0) = 0$, with supremum norm at most 1, and let $z \in \mathbb{B}_d$ be fixed. Apply the maximum principle to the subharmonic function

$$\lambda \mapsto \frac{uu^*(\lambda z)}{|\lambda|^2}, \quad |\lambda z| < 1,$$

to obtain that $u(z)u^*(z) \leq |z|^2$, i.e. $\alpha_k(z) \leq |z|^2$. Similarly, if $\Omega = \mathbb{D}^d$, $z_0 = 0$, and \mathcal{H}_k consists of analytic functions, it follows with the above argument that $\alpha_k(z) \leq \max_j |z_j|$, where $z = (z_j)$. These estimates continue to hold in the case when $d = \infty$, and become equalities

when the identity function belongs to the unit ball of $\text{Mult}(\mathcal{H}_k(l^2), \mathcal{H}_k)$ (i.e. when the multiplication operators by the coordinate functions form a row contraction on \mathcal{H}_k), respectively when multiplication by each coordinate is contractive.

4.4. Multipliers. Let s be a normalized CNP kernel on the nonvoid set Ω , and let k be a reproducing kernel on Ω with $k/s \gg 0$. We are interested in the space $\text{Mult}(\mathcal{H}_s, \mathcal{H}_k(\mathcal{E}))$. In most cases we lack a complete characterization of such multipliers, and our aim is to discuss some sufficient conditions for a function to belong to this space. Our conditions are expressed in terms of the Sarason functions V_F , $F \in \mathcal{H}_k(\mathcal{E})$.

Proposition 3.5 turns out to be useful in this context. A direct application shows that for $F \in \mathcal{H}_k(\mathcal{E})$, $x > 0$, we have

$$\Phi_y = \frac{1}{V_F + x - iy} F \in \text{Mult}(\mathcal{H}_s, \mathcal{H}_k(\mathcal{E})),$$

for all $y \in \mathbb{R}$, and the functions are uniformly bounded in this multiplier space. Using this observation we can construct other functions in $\text{Mult}(\mathcal{H}_s, \mathcal{H}_k(\mathcal{E}))$ in the following way. Given a finite Borel measure μ supported on the imaginary axis, we let $\hat{\mu}$ be its Cauchy transform,

$$\hat{\mu}(z) = \int_{i\mathbb{R}} \frac{d\mu(iy)}{iy - z}, \quad z \in \mathbb{C} \setminus i\mathbb{R}.$$

For $x > 0$, let $\hat{\mu}_x(z) = \hat{\mu}(x + z)$ be defined in the right half-plane $\{\text{Re } z > 0\}$. From above we obtain that

$$(4.2) \quad \hat{\mu}_x(V_F)F \in \text{Mult}(\mathcal{H}_s, \mathcal{H}_k(\mathcal{E})),$$

for any $F \in \mathcal{H}_k(\mathcal{E})$, any $x > 0$ and any finite Borel measure μ on the imaginary axis.

For our next application we need to recall the following notion. We say that the *one-function corona theorem* holds for $\text{Mult}(\mathcal{H}_s)$ if $\varphi^{-1} \in \text{Mult}(\mathcal{H}_s)$ whenever $\varphi \in \text{Mult}(\mathcal{H}_s)$ and φ is bounded below on Ω . The condition is certainly fulfilled if the full corona theorem holds for $\text{Mult}(\mathcal{H}_s)$, in the sense that evaluations at points of Ω are dense in the maximal ideal space of this algebra. For example, from results in [10], [13] and [23, Theorem 5.4] the one-function corona theorem holds for $\text{Mult}(\mathcal{H}_s)$ if $\Omega = \mathbb{B}_d$, $z_0 = 0$, and $s_w(z) = (1 - \langle z, w \rangle)^{-\gamma}$, $0 < \gamma \leq 1$, or $s_w(z) = \frac{1}{\langle z, w \rangle} \log \frac{1}{1 - \langle z, w \rangle}$. On the other hand, [4, Theorem 1.5] shows that the one-function corona theorem may fail even for (radial) normalized CNP kernels on the unit disc.

Corollary 4.4. *Assume that the one-function corona theorem holds for $\text{Mult}(\mathcal{H}_s)$. If $F \in \mathcal{H}_k(\mathcal{E})$ and its Sarason function V_F is bounded in Ω , then $F \in \text{Mult}(\mathcal{H}_s, \mathcal{H}_k(\mathcal{E}))$, and $V_F \in \text{Mult}(\mathcal{H}_s)$.*

Proof. Apply Proposition 3.5 with $a = 1$, or Theorem 1.1 for $\|F\|_{\mathcal{H}_k(\mathcal{E})} = 1$, to conclude that $\psi = \frac{V_F - 1}{V_F + 1} \in \text{Mult}(\mathcal{H}_s)$, hence $\frac{2}{V_F + 1} = 1 - \psi \in \text{Mult}(\mathcal{H}_s)$. If V_F is bounded, $\frac{2}{V_F + 1}$ is bounded below in Ω , hence by assumption it is invertible in $\text{Mult}(\mathcal{H}_s)$, i.e., $V_F \in \text{Mult}(\mathcal{H}_s)$. Moreover, since $\frac{1}{V_F + 1}F \in \text{Mult}(\mathcal{H}_s, \mathcal{H}_k(\mathcal{E}))$ and $V_F \in \text{Mult}(\mathcal{H}_s)$, we obtain $F \in \text{Mult}(\mathcal{H}_s, \mathcal{H}_k(\mathcal{E}))$. \square

We remark that without the assumption that the one-function corona theorem holds, the preceding corollary may fail. Indeed, by the results of [4, Section 5], there exists a space \mathcal{H}_s of continuous functions on \mathbb{D} with a normalized CNP kernel s such that $\text{Mult}(\mathcal{H}_s) \subsetneq \mathcal{H}_s$ (called a Salas space there). For every $f \in \mathcal{H}_s \setminus \text{Mult}(\mathcal{H}_s)$, the Sarason function V_f is bounded since the multiplier norm of s_z is uniformly bounded over $z \in \mathbb{D}$, even though $f \notin \text{Mult}(\mathcal{H}_s)$.

In several cases the condition that the Sarason function is bounded in Ω , can be replaced by the weaker assumption that its real part is bounded. This is certainly not sufficient to make V_F a multiplier of \mathcal{H}_s , but it sometimes implies that $F \in \text{Mult}(\mathcal{H}_s, \mathcal{H}_k(\mathcal{E}))$.

The simplest example of this type is when s is the Szegő kernel on \mathbb{D} , and $k = s$, $\mathcal{E} = \mathbb{C}$, hence $\mathcal{H}_s = \mathcal{H}_k = H^2$. If g is an unbounded analytic function in the unit disc with $0 < a \leq \text{Re } g(z) \leq b < \infty$, then there is a bounded outer function f with $|f|^2 = \text{Re } g$ a.e. on the unit circle. Therefore, f is a multiplier of H^2 , but V_f , which agrees with g up to an additive constant, is not.

Another example is provided by a recent result in [12] which asserts that if s is the unweighted Dirichlet kernel, k is the reproducing kernel in some weighted Bergman space $L_a^2(\mu)$ on \mathbb{D} , and $\text{Re } V_1$ is bounded in \mathbb{D} , then $1 \in \text{Mult}(\mathcal{H}_s, \mathcal{H}_k)$, i.e. \mathcal{H}_s is continuously contained in \mathcal{H}_k . In fact the argument used in [12] is based on the very general Lemma 24 in [7] and can be extended to arbitrary normalized CNP kernels. For kernels on \mathbb{B}_d of the form $s_w(z) = \frac{1}{\langle z, w \rangle} \log \frac{1}{1 - \langle z, w \rangle}$, or $s_w(z) = (1 - \langle z, w \rangle)^{-\gamma}$, $0 < \gamma < 1$, or more generally, if $s = s_1^\gamma$, $0 < \gamma < 1$, for some analytic normalized CNP kernel s_1 , and k a weighted Bergman kernel, then $k/s \gg 0$ by Corollary 2.3 (iii), and the boundedness of $\text{Re } V_1$ obviously implies that V_1 is bounded, as $\text{Re } s$ and s are comparable for such kernels. On the other hand, this is no longer the case when s is the Drury-Arveson kernel.

We are going to prove that for a significant class of kernels k and all normalized CNP kernels s with $k/s \gg 0$, the condition $\operatorname{Re} V_F$ bounded in Ω , implies that $F \in \operatorname{Mult}(\mathcal{H}_s, \mathcal{H}_k(\mathcal{E}))$.

Our basic assumption is that the norm on \mathcal{H}_k can be expressed with help of L^2 -norms of linear differential operators.

To be more precise, assume that $\Omega \subset \mathbb{R}^d$, let μ_1, \dots, μ_m be finite positive Borel measures on Ω , and for $1 \leq i \leq m$ let \mathcal{L}_i be a linear differential operator of the form

$$\mathcal{L}_i = \sum_{|\alpha| \leq N} a_{i,\alpha} \partial^\alpha,$$

where $N \in \mathbb{N}$ is fixed, the coefficients $a_{i,\alpha}$ are μ_i -measurable functions, and, as usual, $\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1 x_1} \dots \partial^{\alpha_d x_d}}$.

Now assume that \mathcal{H}_k contains a dense set \mathcal{D} , such that the functions in \mathcal{D} , together with all multipliers in $\operatorname{Mult}(\mathcal{H}_s)$ are continuous on Ω and have partial derivatives of order $\leq N$ μ_i -a.e., for $1 \leq i \leq m$. Moreover, assume that there are absolute constants $c_1, c_2 > 0$ with

$$(4.3) \quad c_1 \|f\|_{\mathcal{H}_k}^2 \leq \sum_{i=1}^m \int_{\Omega} |\mathcal{L}_i f|^2 d\mu_i \leq c_2 \|f\|_{\mathcal{H}_k}^2, \quad f \in \mathcal{D}.$$

Note that in this case each \mathcal{L}_i extends to a bounded linear operator from \mathcal{H}_k into $L^2(\mu_i)$. If we denote these extensions by $\tilde{\mathcal{L}}_i$, it follows that

$$(4.4) \quad \|f\|^2 = \sum_{i=1}^m \int_{\Omega} |\tilde{\mathcal{L}}_i f|^2 d\mu_i,$$

defines an equivalent norm on \mathcal{H}_k .

Sobolev spaces provide standard examples of such spaces. Also, most of the examples considered in this paper satisfy these assumptions; weighted Bergman and Hardy spaces on \mathbb{B}_d , or \mathbb{D}^d , weighted Dirichlet spaces on the unit disc, or more generally weighted Besov spaces on \mathbb{B}_d , and in particular the Drury-Arveson spaces H_d^2 , $d \in \mathbb{N}$ (see [31]).

Theorem 4.5. *Let $\Omega, \mu_i, \mathcal{L}_i$, $1 \leq i \leq m$, \mathcal{H}_k and \mathcal{D} be as above and assume that (4.3) holds. If $F \in \mathcal{H}_k(\mathcal{E})$ and $\operatorname{Re} V_F$ is bounded in Ω , then $F \in \operatorname{Mult}(\mathcal{H}_s, \mathcal{H}_k(\mathcal{E}))$, and there exists a constant $c_N > 0$ depending only on N , such that*

$$\|F\|_{\operatorname{Mult}(\mathcal{H}_s, \mathcal{H}_k(\mathcal{E}))} \leq c_N (\|\operatorname{Re} V_F\|_{\infty} + 3)^{N+\frac{1}{2}}.$$

Proof. For $0 \leq r \leq m$, let $\tilde{\mathcal{L}}_r$ be the extended differential operator from (4.4). We shall prove that there exists a constant $C_N > 0$, depending

only on N , such that whenever $F \in \mathcal{H}_k(\mathcal{E})$ and $h \in \mathcal{H}_s$ is a finite linear combination of reproducing kernels in \mathcal{H}_s , we have

$$(4.5) \quad \int_{\Omega} \frac{\|(\tilde{\mathcal{L}}_r \otimes 1_{\mathcal{E}})hF\|_{\mathcal{E}}^2}{(\operatorname{Re} V_F + 3)^{2N+1}} d\mu_r \leq C_N \|h\|_{\mathcal{H}_s}^2.$$

Clearly, the theorem follows directly from this inequality.

Note first that it will be sufficient to prove (4.5) for F in dense subset of $\mathcal{H}_k(\mathcal{E})$, since $h \in \operatorname{Mult}(\mathcal{H}_s) \subset \operatorname{Mult}(\mathcal{H}_k(\mathcal{E}))$, and if $F_n \rightarrow F$ in $\mathcal{H}_k(\mathcal{E})$, then $V_{F_n}(z) \rightarrow V_F(z)$, $z \in \Omega$. Then (4.4) together with Fatou's lemma show that the estimate holds for arbitrary $F \in \mathcal{H}_k(\mathcal{E})$ with the same constant C_N . Consequently, for a fixed orthonormal basis $\{e_n\}$ of \mathcal{E} we can consider \mathcal{E} -valued functions F of the form

$$F = \sum_{n \leq M} f_n e_n,$$

with $f_n \in \mathcal{D}$, $1 \leq n \leq M$, the dense subset of \mathcal{H}_k from our assumption. In this case F has \mathcal{E} -valued partial derivatives of order $\leq N$ μ_r -a.e., and these are μ_r -measurable.

Start with the inequality in Proposition 3.5 to obtain for $\operatorname{Re} a > 0$,

$$\begin{aligned} 4\operatorname{Re} a \left\| \frac{1}{V_F + \bar{a}} hF \right\|_{\mathcal{H}_k(\mathcal{E})}^2 &\leq \|h\|_{\mathcal{H}_s}^2 - \left\| \frac{V_F - a}{V_F + \bar{a}} h \right\|_{\mathcal{H}_s}^2 \\ &\leq 4\operatorname{Re} a \operatorname{Re} \left\langle \frac{1}{V_F + \bar{a}} h, h \right\rangle_{\mathcal{H}_s}, \end{aligned}$$

and from (4.3)

$$\int_{\Omega} \left\| \mathcal{L}_r \frac{1}{V_F + \bar{a}} hF \right\|_{\mathcal{E}}^2 d\mu_r \lesssim \operatorname{Re} \left\langle \frac{1}{V_F + \bar{a}} h, h \right\rangle_{\mathcal{H}_s}.$$

Now let $a = x + iy$, with $x > 0$ fixed, but arbitrary. Integration in y yields

$$(4.6) \quad \begin{aligned} \int_{\Omega} \int_{-\infty}^{\infty} \left\| \mathcal{L}_r \frac{1}{V_F + x - iy} hF(z) \right\|_{\mathcal{E}}^2 dy \mu_r(z) \\ \lesssim \int_{-\infty}^{\infty} \operatorname{Re} \left\langle \frac{1}{V_F + x - iy} h, h \right\rangle_{\mathcal{H}_s} dy. \end{aligned}$$

We claim that

$$(4.7) \quad \int_{-\infty}^{\infty} \operatorname{Re} \left\langle \frac{1}{V_F + x - iy} h, h \right\rangle_{\mathcal{H}_s} dy = \pi \|h\|_{\mathcal{H}_s}^2.$$

Indeed, if $h = \sum_j c_j s_{z_j}$,

$$\begin{aligned}
& \int_{-T}^T \operatorname{Re} \left\langle \frac{1}{V_F + x - iy} h, h \right\rangle_{\mathcal{H}_s} dy \\
&= \frac{1}{2} \sum_{j,l} c_j \bar{c}_l s_{z_j}(z_l) \int_{-T}^T \left(\frac{1}{V_F(z_l) + x - iy} + \frac{1}{\overline{V_F(z_l) + x + iy}} \right) dy \\
&= \sum_{j,l} c_j \bar{c}_l s_{z_j}(z_l) \frac{1}{2i} \left(\log \frac{V_F(z_l) + x + iT}{V_F(z_l) + x - iT} + \log \frac{\overline{V_F(z_j) + x + iT}}{\overline{V_F(z_j) + x - iT}} \right) \\
&\rightarrow \pi \sum_{j,l} c_j \bar{c}_l s_{z_l}(z_j), \quad T \rightarrow \infty,
\end{aligned}$$

and the claim follows by the monotone convergence theorem.

In order to estimate the left hand side of (4.6) from below, we note first that by assumption, V_F has partial derivatives of order $\leq N$ μ_r -a.e., as $\frac{2}{V_F+1} \in \operatorname{Mult}(\mathcal{H}_s)$, hence by the product rule we have μ_r -a.e.

$$\begin{aligned}
\mathcal{L}_r \frac{1}{V_F + x - iy} hF &= \frac{1}{V_F + x - iy} \mathcal{L}_r hF \\
&+ \sum_{j=1}^N \frac{1}{(V_F + x - iy)^{j+1}} \mathcal{L}_{rj} hF,
\end{aligned}$$

where \mathcal{L}_{rj} are linear differential operators of order $\leq N$ with μ_r -measurable coefficients.

Now recall that $y \mapsto \frac{\operatorname{Re} V_F(z) + x}{\pi |V_F(z) + x + it - iy|^2}$, $z \in \Omega, t \in \mathbb{R}$ is the Poisson kernel for the right half-plane at the point $w = V_F(z) + x + it$, and note that for $2|t| \leq x$

$$\frac{\operatorname{Re} V_F(z) + x}{\pi |V_F(z) + x + it - iy|^2} \leq 2 \frac{\operatorname{Re} V_F(z) + x}{\pi |V_F(z) + x - iy|^2}.$$

Moreover, for μ_r -almost every $z \in \Omega$ the function

$$U_z(w) = \mathcal{L}_r hF(z) + \sum_{j=1}^N \frac{1}{(V_F + x + \bar{w})^j} \mathcal{L}_{rj} hF(z), \quad \operatorname{Re} w > 0,$$

is anti-analytic in the right half-plane, continuous and bounded in its closure. Consequently, $\|U_z\|_{\mathcal{E}}^2$ is subharmonic, bounded and continuous in the closure of the right half-plane, hence its Poisson integral is a (the least) harmonic majorant in the right half-plane. In particular,

$$\|U_z(V_F(z) + x + it)\|_{\mathcal{E}}^2 \leq \int_{-\infty}^{\infty} \frac{\operatorname{Re} V_F(z) + x}{\pi |V_F(z) + x + it - iy|^2} \|U_z(iy)\|_{\mathcal{E}}^2 dy.$$

From the last two inequalities we obtain for all $x \in [1, 2]$, $t \in [-\frac{1}{2}, \frac{1}{2}]$, and μ_r -a.e. on Ω

$$\begin{aligned}
 & \int_{-\infty}^{\infty} \left\| \mathcal{L}_r \frac{1}{V_F + x - iy} hF(z) \right\|_{\mathcal{E}}^2 dy \\
 &= \frac{\pi}{\operatorname{Re} V_F(z) + x} \int_{-\infty}^{\infty} \frac{\operatorname{Re} V_F(z) + x}{\pi |V_F(z) + x - iy|^2} \|U_z(iy)\|_{\mathcal{E}}^2 dy \\
 &\geq \frac{\pi}{2(\operatorname{Re} V_F(z) + x)} \int_{-\infty}^{\infty} \frac{\operatorname{Re} V_F(z) + x}{\pi |V_F(z) + x + it - iy|^2} \|U_z(iy)\|_{\mathcal{E}}^2 dy \\
 &\geq \frac{\pi}{2(\operatorname{Re} V_F(z) + x)} \|U_z(V_F(z) + x + it)\|_{\mathcal{E}}^2 \\
 &\geq \frac{\pi \left\| \sum_{j=0}^N (2\operatorname{Re} V_F + 2x - it)^{N-j} \mathcal{L}_{r_j} hF(z) \right\|_{\mathcal{E}}^2}{2^{2N+1} (\operatorname{Re} V_F(z) + 3)^{2N+1}},
 \end{aligned}$$

where $\mathcal{L}_{r_0} = \mathcal{L}_r$. Thus, if $d\nu = \frac{d\mu_r}{(\operatorname{Re} V_F + 3)^{2N+1}}$, this estimate together with (4.7) and (4.6) gives

$$\left\| \sum_{j=0}^N (2\operatorname{Re} V_F + 2x - it)^{N-j} \mathcal{L}_{r_j} hF \right\|_{L^2(\nu, \mathcal{E})}^2 \leq A_N \|h\|_{\mathcal{H}_s}^2,$$

for some constant $A_N > 0$ depending only on N , and for all $2x - it \in [2, 4] \times [-\frac{1}{2}, \frac{1}{2}]$. In particular, for every $u \in L^2(\nu, \mathcal{E})$, the polynomial

$$p_u(w) = \left\langle \sum_{j=0}^N (2\operatorname{Re} V_F + w)^{N-j} \mathcal{L}_{r_j} hF, u \right\rangle_{L^2(\nu, \mathcal{E})},$$

satisfies

$$|p_u(2x - it)| \leq A_N^{1/2} \|u\|_{L^2(\nu, \mathcal{E})} \|h\|_{\mathcal{H}_s}, \quad 2x - it \in [2, 4] \times [-\frac{1}{2}, \frac{1}{2}].$$

Then

$$|p_u^{(N)}(3)| \leq B_N \|u\|_{L^2(\nu, \mathcal{E})} \|h\|_{\mathcal{H}_s},$$

with $B_N > 0$ depending only on N , which implies (4.5) and completes the proof. \square

The result is of interest even in the special case when $k = s$ and $\mathcal{E} = \mathbb{C}$. Actually, the main motivation for this theorem was the Drury-Arveson kernel, and we record the corresponding result as an immediate application.

Corollary 4.6. *If $s_w(z) = \frac{1}{1-\langle z, w \rangle}$, $z, w \in \mathbb{B}_d$, and $f \in H_d^2$ satisfies*

$$\sup_{z \in \mathbb{B}_d} \operatorname{Re} \langle f, s_z f \rangle_{H_d^2} < \infty,$$

then $f \in \text{Mult}(H_d^2)$.

Recall from Equation (3.4) that $\frac{\|s_z f\|^2}{\|s_z\|^2} \leq \text{Re } V_f(z)$ for all $f \in \mathcal{H}_s$. While Corollary 4.6 shows that boundedness of $\text{Re } V_f$ implies that $f \in \text{Mult}(H_d^2)$, the main result of [14] shows that boundedness of $\frac{\|s_z f\|^2}{\|s_z\|^2}$ is not sufficient for f to belong to $\text{Mult}(H_d^2)$.

The natural question which arises is whether $\text{Re } V_F$ is bounded for all $F \in \text{Mult}(\mathcal{H}_s, \mathcal{H}_k(\mathcal{E}))$? As we shall see below, the answer is negative.

Another direct consequence of Theorem 4.5 concerns the embedding of \mathcal{H}_s into \mathcal{H}_k .

Corollary 4.7. *Let $\Omega, \mu_i, \mathcal{L}_i$, $1 \leq i \leq m$ and \mathcal{H}_k be as in Theorem 4.5. Let s be a normalized CNP kernel such that $k/s \gg 0$. If $1 \in \mathcal{H}_k$ and $\text{Re } V_1$ is bounded in Ω , then \mathcal{H}_s is continuously contained in \mathcal{H}_k .*

As pointed out in the Introduction, if \mathcal{H}_k is a weighted Bergman space, as defined in subsection 4.1, this extends the result in [12]. The general framework for Carleson embeddings involves L^2 -spaces rather than Bergman spaces, but the additional step is trivial in many cases. For example, if s is an analytic normalized CNP kernel on \mathbb{B}_d and μ is an arbitrary finite positive Borel measure on \mathbb{B}_d satisfying

$$(4.8) \quad \sup_{z \in \mathbb{B}_d} \text{Re} \int_{\mathbb{B}_d} s_z d\mu < \infty,$$

then Corollary 4.7 easily implies that μ is a Carleson measure for \mathcal{H}_s . Indeed, note that if v denotes the Lebesgue measure on \mathbb{B}_d , then \mathcal{H}_s is continuously embedded in $L_a^2(\mu_0)$, where

$$d\mu_0(z) = \max_{|w|=|z|} s_w(w)^{-1/2} dv(z).$$

Then $\mathcal{H}_k = L_a^2(\mu_0 + \mu)$ is a weighted Bergman space with $1 \in \mathcal{H}_k$ and since μ_0 is radial,

$$\int_{\mathbb{B}_d} s_z d\mu_0 = \mu_0(\mathbb{B}_d),$$

hence by (4.8) it follows that $\text{Re } V_1$ bounded in \mathbb{B}_d . Thus \mathcal{H}_s is continuously contained in \mathcal{H}_k and consequently, it is also continuously contained in $L^2(\mu)$. In concrete cases, (4.8) together with the method in [12] can be used to derive one-box conditions for Carleson measures for \mathcal{H}_s .

4.5. Sarason functions in $D(\mu)$ -spaces. In the general context considered above it is difficult to compute the Sarason function, or even to estimate its real part. There is an important class of spaces where the second problem appears more tractable due to the work of Shimorin ([28],[30]).

The local Dirichlet integral of $f \in H^2$ at $\zeta \in \overline{\mathbb{D}}$ is defined by

$$D_\zeta(f) = \int_{\mathbb{T}} \left| \frac{f(z) - f(\zeta)}{z - \zeta} \right|^2 dm(z).$$

Here m denotes the normalized arclength measure on the unit circle \mathbb{T} , and if $\zeta \in \mathbb{T}$, the value $f(\zeta)$ is the nontangential limit of f at ζ which exists whenever $D_\zeta(f)$ is finite (see [22]). Given a finite positive Borel measure on $\overline{\mathbb{D}}$, $D(\mu)$ is the Hilbert space consisting of all H^2 -functions f with

$$\|f\|_{D(\mu)}^2 = \|f\|_{H^2}^2 + \int_{\overline{\mathbb{D}}} D_\zeta(f) d\mu(\zeta) < \infty.$$

These norms can be expressed with help of the first derivative. We have (see [3], [21])

$$\|f\|_{D(\mu)}^2 = \|f\|_{H^2}^2 + \int_{\mathbb{D}} |f'(z)|^2 U_\mu(z) dA(z),$$

where

$$U_\mu(z) = \int_{\mathbb{D}} \log \left| \frac{1 - \bar{\zeta}z}{z - \zeta} \right| d\mu(\zeta) + \int_{\mathbb{T}} \frac{1 - |z|^2}{|1 - \bar{\zeta}z|^2} d\mu(\zeta).$$

$D(\mu)$ -spaces appeared first in [21] for measures supported on \mathbb{T} in connection with functional models for two-isometries. The general case was considered in [3]. The most common examples are the standard weighted Dirichlet spaces D_α , $0 \leq \alpha < 1$, consisting of analytic functions in \mathbb{D} with

$$(4.9) \quad \|f\|_\alpha^2 = \|f\|_{H^2}^2 + \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^\alpha dA(z) < \infty.$$

If $\mu_0 = m$, and for $0 < \alpha < 1$, $d\mu_\alpha = -(1 - |z|^2)\Delta(1 - |z|^2)^\alpha dA$, where Δ denotes the Laplacian, we have $D_\alpha = D(\mu_\alpha)$, with equality of norms.

Shimorin proved in [30] that for every finite positive Borel measure on $\overline{\mathbb{D}}$, the reproducing kernel s^μ in $D(\mu)$ is a normalized CNP kernel. Moreover, in [28], Proposition 3 and Corollary 4, he showed that for

every $f \in D(\mu)$ we have

$$(4.10) \quad \begin{aligned} \operatorname{Re} V_f(z) &= \int_{\mathbb{T}} \frac{(1 - |z|^2) |f(\zeta)|^2}{|1 - \bar{\zeta}z|^2} dm(\zeta) \\ &+ \int_{\mathbb{D}} (2 \operatorname{Re} s_z^\mu(\zeta) - 1) D_\zeta(f) d\mu(\zeta). \end{aligned}$$

We shall use this remarkable identity to show that the converse of Theorem 4.5 fails in D_α , $0 < \alpha < 1$.

Proposition 4.8. *For $0 < \alpha < 1$, there exists $u \in \operatorname{Mult}(D_\alpha)$ such that $\operatorname{Re} V_u$ is unbounded in \mathbb{D} .*

The result holds for $\alpha = 0$ as well, but the proof is more involved and will be omitted.

The proof of the proposition requires some preliminary observations. Throughout in what follows we shall assume that $0 < \alpha < 1$, and denote by s^α the reproducing kernel in D_α . Note that this is a radial kernel, i.e.

$$(4.11) \quad s_w^\alpha(z) = 1 + \sum_{n=1}^{\infty} c_n^\alpha (\bar{w}z)^n, \quad c_n^\alpha \geq 0, \quad c_n^\alpha \sim (n+1)^{\alpha-1}.$$

Therefore we can consider the analytic function

$$s_1^\alpha(z) = 1 + \sum_{n=1}^{\infty} c_n^\alpha z^n, \quad z \in \mathbb{D}.$$

Some properties of this function are listed below.

Lemma 4.9. (i) *We have $\operatorname{Re} s_1^\alpha(z) > \frac{1}{2}$ for all $z \in \mathbb{D}$, and $1 - \frac{1}{s_1^\alpha}$ is a contractive multiplier of D_α .*

(ii) *s_1^α and $(s_1^\alpha)'$ are positive on $[0, 1)$ and*

$$s_1^\alpha(r) \sim (1-r)^{-\alpha}, \quad (s_1^\alpha)'(r) \sim (1-r)^{-\alpha-1},$$

where the constants involved depend only on α .

(iii) *There exist $\varepsilon_\alpha \in (0, 1)$ and $\delta_\alpha > 0$ such that*

$$\operatorname{Re} s_1^\alpha(z) \geq \delta_\alpha (1 - |z|)^{-\alpha},$$

whenever z belongs to the set $S = \{z \in \mathbb{D} : |z - |z|| < \varepsilon_\alpha(1 - |z|)\}$.

Proof. (i) Since s^α is a normalized CNP kernel, by definition and Lemma 2.1 (i) it follows that both statements are true with s_w^α in place of s_1^α . Then the assertion follows by letting $w \rightarrow 1$ and using Fatou's lemma in (4.9).

(ii) is a straightforward application of (4.11) combined with the standard fact that for $s < 1$,

$$\sum_{n=1}^{\infty} n^{-s} r^n \sim (1-r)^{s-1}$$

as $r \rightarrow 1$.

(iii) The same straightforward estimate in (4.11) gives

$$(s_1^\alpha)'(z) \leq M_\alpha (1-|z|)^{-\alpha-1}, \quad z \in \mathbb{D}$$

for some constant $M_\alpha > 0$. Now if $0 < \varepsilon < 1$ and $|z - |z|| < \varepsilon(1 - |z|)$, we have

$$\begin{aligned} |\operatorname{Re} s_1^\alpha(z) - \operatorname{Re} s_1^\alpha(|z|)| &\leq |z - |z|| \sup_{t \in [0,1]} |(s_1^\alpha)'(t|z| + (1-t)z)| \\ &\leq M_\alpha \varepsilon (1-|z|)(1-|z|)^{-\alpha-1} \\ &= M_\alpha \varepsilon (1-|z|)^{-\alpha}. \end{aligned}$$

Using (ii), we choose $\varepsilon_\alpha \in (0, 1)$ such that

$$\varepsilon_\alpha M_\alpha (1-|z|)^{-\alpha} \leq \frac{1}{2} \operatorname{Re} s_1^\alpha(|z|)$$

for all $z \in \mathbb{D}$, and the result follows. \square

The key step for our construction is the following estimate derived from Shimorin's identity.

Lemma 4.10. *Let S be the set in Lemma 4.9 (iii). There exists $c_\alpha > 0$ such that for all $f \in D_\alpha$,*

$$\|f\|_\alpha^2 + \sup_{z \in \mathbb{D}} \operatorname{Re} V_f(z) \geq c_\alpha \int_S |f'(\zeta)|^2 dA(\zeta).$$

Proof. Note that (4.10) implies for all $f \in D_\alpha$,

$$\|f\|_\alpha^2 + \operatorname{Re} V_f(z) \geq \int_{\mathbb{D}} 2 \operatorname{Re} s_z^\alpha(\zeta) D_\zeta(f) d\mu_\alpha(\zeta),$$

hence by Fatou's lemma

$$\begin{aligned} \|f\|_\alpha^2 + \sup_{z \in \mathbb{D}} \operatorname{Re} V_f(z) &\geq \liminf_{z \rightarrow 1} \int_{\mathbb{D}} 2 \operatorname{Re} s_z^\alpha(\zeta) D_\zeta(f) d\mu_\alpha(\zeta) \\ &\geq \int_{\mathbb{D}} 2 \operatorname{Re} s_1^\alpha(\zeta) D_\zeta(f) d\mu_\alpha(\zeta). \end{aligned}$$

The standard estimate $(1 - |\zeta|^2)|h(\zeta)|^2 \leq \|h\|_{H^2}^2$, yields

$$D_\zeta(f) \geq (1 - |\zeta|^2)|f'(\zeta)|^2.$$

Thus

$$\|f\|_\alpha^2 + \sup_{z \in \mathbb{D}} \operatorname{Re} V_f(z) \geq \int_{\mathbb{D}} 2 \operatorname{Re} s_1^\alpha(\zeta) |f'(\zeta)|^2 (1 - |\zeta|^2) d\mu_\alpha(\zeta),$$

and since $d\mu_\alpha = -(1 - |z|^2)\Delta(1 - |z|^2)^\alpha dA$, the result follows by Lemma 4.9 (iii). \square

Proof of Propostion 4.8. Let $f = 1 - \frac{1}{s_1^\alpha}$ and recall from Lemma 4.9 (i) that $f \in \operatorname{Mult}(D_\alpha)$ with $\|M_f\| \leq 1$. Since $|f(z)| < 1$, $z \in \mathbb{D}$, it follows that $(M_f^*)^n s_z^\alpha = \overline{(f(z))^n} s_z^\alpha$ converges to zero in D_α , hence $(M_f^*)^n$ converges to zero in the strong operator topology. Thus M_f admits a contractive weak-* weak-* continuous H^∞ -functional calculus. Since this functional calculus extends the polynomial functional calculus, it is easy to check that $g(M_f) = M_{g \circ f}$ for all $g \in H^\infty$, i.e. $g \circ f \in \operatorname{Mult}(D_\alpha)$ for all $g \in H^\infty$ with $\|g \circ f\|_{\operatorname{Mult}(D_\alpha)} \leq \|g\|_\infty$ (a similar construction appears in [11, Lemma 12]). We claim that if B is an infinite interpolating Blaschke product with zeros in $[0, 1)$ then $\operatorname{Re} V_{B \circ f}$ is unbounded in \mathbb{D} .

Let $\{z_n : n \geq 1\}$ be the zero-set of B and use Lemma 4.9 (ii) to conclude that $z_n = f(w_n)$, $n \geq 1$, with $w_n \in [0, 1)$, $\lim_{n \rightarrow \infty} w_n = 1$. Then, with the notations in Lemma 4.9 (iii), there exists an infinite set $J \subset \mathbb{N}$ such that the discs $\Delta_n = \{|z - w_n| < \frac{\varepsilon_\alpha}{3}(1 - w_n)\}$, $n \in J$, are disjoint. Note that by the triangle inequality we have for $z \in \Delta_n$

$$1 - |z| > (1 - \frac{\varepsilon_\alpha}{3})(1 - w_n), \quad |z - |z|| < 2|z - w_n| < \frac{2\varepsilon_\alpha}{3}(1 - |w_n|) < \varepsilon_\alpha(1 - |z|),$$

so that $\Delta_n \subset S$. Since $\|B\|_\infty = 1$, Lemma 4.10 gives

$$1 + \sup_{z \in \mathbb{D}} V_{B \circ f}(z) \geq c_\alpha \sum_J \int_{\Delta_n} |(B \circ f)'|^2 dA,$$

and since $|(B \circ f)'|^2$ is subharmonic in \mathbb{D} , we obtain

$$1 + \sup_{z \in \mathbb{D}} V_{B \circ f}(z) \geq \frac{\pi \varepsilon_\alpha^2 c_\alpha}{9} \sum_J (1 - w_n)^2 |(B \circ f)'(w_n)|^2.$$

It suffices to show that the values $(1 - w_n)|(B \circ f)'(w_n)|$, $n \in \mathbb{N}$, are bounded below.

Since B is interpolating, there exists $\delta > 0$ such that

$$|B'(z_n)| \geq \delta(1 - z_n), \quad n \in \mathbb{N}.$$

From $z_n = f(w_n)$ it follows that $1 - z_n = \frac{1}{s_1^\alpha(w_n)}$, hence

$$(1 - w_n)|(B \circ f)'(w_n)| = (1 - w_n)|B'(z_n)| \frac{(s_1^\alpha)'(w_n)}{(s_1^\alpha(w_n))^2} \geq \delta(1 - w_n) \frac{(s_1^\alpha)'(w_n)}{s_1^\alpha(w_n)},$$

and the result follows by Lemma 4.9 (ii). \square

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