QUASISIMILARITY OF RATIONALLY CYCLIC SUBNORMAL OPERATORS

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ABSTRACT. We give a new proof of Raphael's theorem that quasisimilar rationally cyclic subnormal operators have naturally isomorphic commutants. We also classify all subnormal operators quasisimilar to $R_{K,\mu}$, when $R(K)$ is a hypo-Dirichlet algebra, and μ is supported on the boundary of K.

INTRODUCTION

Two bounded linear operators on a Hilbert space, S and T, are said to be *quasisimilar* if there exist quasi-affinities X and Y (*i.e.* bounded operators with dense ranges and no kernels) such that $SX = XT$ and $YS = TY$. Quasisimilarity is an interesting equivalence relation for subnormal operators, because, on the one hand, it preserves such properties as the spectrum [4] and the essential spectrum [14], and on the other, it provides a large class of equivalent operators which, in certain cases, are amenable to a nice classification.

If K is a compact subset of $\mathbb{C}$, $Rat(K)$ will denote the rational functions with poles off K, and an operator S on $\mathcal{H}$ is called $Rat(K)$-cyclic if the spectrum of S is contained in K, and there is some vector ξ in $\mathcal{H}$, such that $\{r(S)\xi | r \in Rat(K)\}$ is dense in $\mathcal{H}$. Any $Rat(K)$-cyclic subnormal operator is unitarily equivalent to multiplication by the independent variable on $R^2(K,\mu)$, the closure of $Rat(K)$ in $L^2(\mu)$, for some finite, positive Borel measure supported on K [6,p.146]. We shall call this latter operator $R_{K,\mu}$, and μ will be called the *symbol* of $R_{K,\mu}$.

If f is in $R^2(K,\mu) \cap L^\infty(\mu)$, M_f, the operator of multiplication by f on $R^2(K,\mu)$, is clearly in the commutant of $R_{K,\mu}$, and, by a theorem of Yoshino [15], anything in the commutant is of this form. In [12], Raphael proved that the commutants of quasisimilar rationally cyclic (*i.e.* $Rat(K)$-cyclic for some K) subnormal operators are isometrically isomorphic and weak-star homeomorphic. In Section 1 we give a

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new proof of this theorem, which is shorter, and, we think, simpler. We would like to thank John Conway for bringing [12] to our attention.

In [3], Clary classified all subnormal operators quasisimilar to the unilateral shift. This was extended by Hastings [10] to isometries of finite cyclic multiplicity. In Section 2 we extend this in a different direction, namely to classifying all subnormal operators quasisimilar to $R_{K,\mu}$, when $R(K)$ is a hypo-Dirichlet algebra, and μ is supported on the boundary of K. $R(K)$ denotes the uniform closure of $Rat(K)$ in $C(K)$, and $R(K)$ is called a *hypo-Dirichlet algebra* if (i) the uniform closure of $\Re(R(K))$ is of finite codimension in $\Re(C(\partial K))$, and (ii) the linear span of $\{log|r| : r, r^{-1} \in R(K)\}$ is dense in $\Re(C(\partial K))$. If $\Re(R(K))$ is actually dense in $\Re(C(\partial K))$, $R(K)$ is called a Dirichlet algebra. If K is finitely connected then $R(K)$ is hypo-Dirichlet [2], but this is not necessary (although the interior of K must always be finitely connected).

By "measure" we shall always mean a compactly supported finite Borel measure on $\mathbb{C}$. Two measures will be called *equivalent* if they are mutually absolutely continuous. If K is polynomially convex, $R^2(K,\mu)$ is just the closure of the polynomials in $L^2(\mu)$, which we shall write as $P^2(\mu)$, and in this case we shall write S_μ for $R_{K,\mu}$. The (bounded linear) operators on a Hilbert space $\mathcal{H}$ will be written $B(\mathcal{H})$. The weak-star closure of $Rat(K)$ in $L^\infty(\mu)$ will be written $R^\infty(K,\mu)$. Note that $R^\infty(K,\mu)$ is contained in $R^2(K,\mu) \cap L^\infty(\mu)$.

Section 1

We first need a structure theorem to translate from operator theory to function theory. The result we need is from [5], where it is proved for the polynomial case, but the rational case works exactly the same way.

Proposition 1.1. *Suppose R_1 and R_2 are rationally cyclic subnormal operators. Then they are quasisimilar if and only if there exist measures μ_1 and μ_2, a compact set K, an R_{K,μ_1}-cyclic vector ϕ in $R^2(K,\mu_1) \cap L^\infty(\mu_1)$, and constants C_1 and C_2, such that R_i is unitarily equivalent to R_{K,μ_i}, $(i = 1,2)$, and*

$$\int |r\phi|^2 d\mu_1 \le C_1 \int |r|^2 d\mu_2 \le C_2 \int |r|^2 d\mu_1$$

for all r in $Rat(K)$.

We next prove two lemmata, which are just special cases of Theorem 1.4. The first says that if $R_{K,\mu}$ and $R_{K,\nu}$ are actually similar, and ν is absolutely continuous with respect to μ, then their commutants are naturally isomorphic.

Lemma 1.2. *Suppose μ and ν are measures supported on K, ν is absolutely continuous with respect to μ, and*

$$\int |r|^2 d\mu \leq C_1 \int |r|^2 d\nu \leq C_2 \int |r|^2 d\mu$$

for all r in $Rat(K)$. Then the identity map on $Rat(K)$ extends to an isometric isomorphism and weak-star homeomorphism from $R^2(K,\mu) \cap L^\infty(\mu)$ onto $R^2(K,\nu) \cap L^\infty(\nu)$.

Proof. By continuity, the map $X : r \mapsto r$, for r in $Rat(K)$, extends to a bounded invertible operator from $R^2(K,\mu)$ to $R^2(K,\nu)$, which intertwines $R_{K,\mu}$ and $R_{K,\nu}$. Let f be in $R^2(K,\mu) \cap L^\infty(\mu)$. Then there is a sequence r_n in $Rat(K)$ which converges to f in $L^2(\mu)$, and, by passing to a subsequence if necessary, we can assume r_n tends to f μ-almost everywhere. Xf is the $L^2(\nu)$ limit of $Xr_n = r_n$, and because ν is absolutely continuous with respect to μ, r_n converges ν-almost everywhere, so $Xf = f$, ν-almost everywhere. Therefore

$$\|Xf\|_{L^\infty(\nu)} \leq \|f\|_{L^\infty(\mu)}.$$

Moreover $XM_fX^{-1} = M_{Xf}$, so

$$\begin{aligned}
\|f\|_{L^\infty(\mu)} &= \|M_f\|_{B(R^2(K,\mu))} \\
&= \|X^{-1}M_{Xf}X\|_{B(R^2(K,\mu))} \\
&\leq \|X\|\|X^{-1}\|\|M_{Xf}\|_{B(R^2(K,\nu))} \\
&\leq \|X\|\|X^{-1}\|\|Xf\|_{L^\infty(\nu)}.
\end{aligned}$$

Therefore $\tilde{X} : R^2(K,\mu) \cap L^\infty(\mu) \to R^2(K,\nu) \cap L^\infty(\nu)$, the restriction of X, is bounded below in the $L^\infty(\mu) - L^\infty(\nu)$ norms, and is multiplicative (since it's just a restriction map), so it must be an isometry. $\tilde{X}$ is surjective because X is invertible. Finally, $\tilde{X}$ is weak-star continuous, since ν is absolutely continuous with respect to

μ, and so by the Krein-Smulian theorem and the Banach-Alaoglu theorem, it is a weak-star homeomorphism. ////

The second lemma requires only that $R_{K,\mu}$ and $R_{K,\nu}$ be quasisimilar, but adds the hypothesis that μ and ν be mutually absolutely continuous.

Lemma 1.3. *Suppose μ and ν are mutually absolutely continuous measures, ϕ is an essentially bounded $Rat(K)$-cyclic vector for $R_{K,\mu}$, and that for some constants C_1 and C_2,*

$$\int |r\phi|^2 d\mu \leq C_1 \int |r|^2 d\nu \leq C_2 \int |r|^2 d\mu$$

for all r in $Rat(K)$. Then the identity map on $Rat(K)$ extends to an isometric isomorphism and weak-star homeomorphism from $R^2(K,\mu) \cap L^\infty(\mu)$ onto $R^2(K,\nu) \cap L^\infty(\nu)$.

Proof. As in the previous lemma, the map X from $R^2(K,\mu)$ to $R^2(K,\nu)$, which leaves $Rat(K)$ fixed, restricts to a map $\tilde{X}$ from $R^2(K,\mu) \cap L^\infty(\mu)$ to $R^2(K,\nu) \cap L^\infty(\nu)$ which is the identity map (point-wise) on functions, and therefore is isometric, multiplicative and weak-star continuous. It only remains to show that it is surjective.

To this end, define two auxiliary maps $W : R^2(K,\nu) \to R^2(K,|\phi|^2\mu)$, and $U : R^2(K,|\phi|^2\mu) \to R^2(K,\mu)$, by $W(r) = r$ and $U(r) = \phi r$ for all r in $Rat(K)$, and extending them by continuity. Again, $\tilde{W}\tilde{X}$ maps $R^2(K,\mu) \cap L^\infty(\mu)$ into $R^2(K,|\phi|^2\mu) \cap L^\infty(|\phi|^2\mu)$ by the identity map on functions, and if $\tilde{W}\tilde{X}$ is surjective, so is $\tilde{X}$. But U is invertible, and $UR_{K,|\phi|^2\mu}U^{-1} = R_{K,\mu}$, so if g is in $R^2(K,|\phi|^2\mu) \cap L^\infty(|\phi|^2\mu)$, then $UM_gU^{-1} = M_g$ is in the commutant of $R_{K,\mu}$, so g is in $R^2(K,\mu) \cap L^\infty(\mu)$, and hence in the range of $\tilde{W}\tilde{X}$. Therefore $\tilde{X}$ is surjective, as desired. ////

The general case of the theorem now follows easily.

Theorem 1.4. *Suppose R_1 and R_2 are quasisimilar rationally cyclic subnormal operators. Then their commutants are naturally isometrically isomorphic and weak-star homeomorphic.*

Proof. By Proposition 1.1, we can find measures μ_1, $\tilde{\mu}_1$, μ_2, and constants C_1, C_2, such that R_1 is unitarily equivalent to both R_{K,μ_1} and $R_{K,\tilde{\mu}_1}$, R_2 is unitarily equivalent to R_{K,μ_2}, and

$$\int |r|^2 d\tilde{\mu}_1 \leq C_1 \int |r|^2 d\mu_2 \leq C_2 \int |r|^2 d\mu_1$$

for all r in $Rat(K)$.

Let $\sigma = (\mu_1 + \mu_2)$, and $\tau = (\tilde{\mu}_1 + \mu_2)$. Then

$$\int |r|^2 d\sigma \leq (1 + \frac{C_2}{C_1}) \int |r|^2 d\mu_1 \leq (1 + \frac{C_2}{C_1}) \int |r|^2 d\sigma,$$

so $R_{K,\sigma}$ is similar to R_{K,μ_1}, and, by Lemma 1.2, the identity map on $Rat(K)$ extends to an isometric isomorphism and weak-star homeomorphism from $R^2(K,\sigma) \cap L^\infty(\sigma)$ onto $R^2(K,\mu_1) \cap L^\infty(\mu_1)$. Likewise, $R_{K,\tau}$ is similar to R_{K,μ_2}, and so $R^2(K,\tau) \cap L^\infty(\tau)$ is mapped isometrically and weak-star homeomorphically onto $R^2(K,\mu_2) \cap L^\infty(\mu_2)$.

R_{K,μ_1} is unitarily equivalent to $R_{K,\tilde{\mu}_1}$, so μ_1 and $\tilde{\mu}_1$ are equivalent [6, p.217], so σ and τ are equivalent. Because quasisimilarity is clearly preserved under similarity transformations, $R_{K,\sigma}$ is quasisimilar to $R_{K,\tau}$. Applying Proposition 1.1 to $R_{K,\sigma}$ and $R_{K,\tau}$, we get unitarily equivalent $R_{K,\tilde{\sigma}}$ and $R_{K,\tilde{\tau}}$ satisfying the hypotheses of Lemma 1.3, and so we can finally conclude that all of the six operators $R_{K,\mu_2}, R_{K,\tau}, R_{K,\tilde{\tau}}, R_{K,\tilde{\sigma}}, R_{K,\sigma}, R_{K,\mu_1}$ have isometrically isomorphic and weak-star homeomorphic commutants, under maps that leave R^∞ fixed. ////

To show both steps are needed in the above proof, here is a simple example of two similar, cyclic subnormal operators with mutually singular symbols: Let λ_n be Lebesgue measure on the circle of centre 0 and radius $1 - 2^{-n}$, normalized to have total mass 2^{-n}. Let μ_1 be $\sum_{m=0}^{\infty} \lambda_{2m+1}$, and μ_2 be $\sum_{m=0}^{\infty} \lambda_{2m+2}$. Then for any polynomial p,

$$\int |p|^2 d\mu_1 \leq 2 \int |p|^2 d\mu_2 \leq 4 \int |p|^2 d\mu_1.$$

So S_{μ_1} is similar to S_{μ_2}, and μ_1 and μ_2 are mutually singular.

Section 2

Harmonic measure for a connected open set U is defined in the following way: pick a point a in U. For any continuous real-valued function f on the boundary of

U, define a function $\hat{f}$ on U by

$$\hat{f}(z) = sup\{g(z) : g \text{ is subharmonic on } U, \text{ and } \limsup_{z\to\zeta} g(z) \leq f(\zeta), \forall \zeta \in \partial U\}.$$

The functional $f \mapsto \hat{f}(a)$ is continuous on $C_{\Re}(\partial U)$, and therefore comes from a measure, ω_a, which is called harmonic measure for U at a. Whilst a different choice of a will yield a different measure, the two measures will be boundedly mutually absolutely continuous, so, by an abuse of language, we refer to harmonic measure for U without specifying a point a. Harmonic measure for a compact set K, with interior components U_n, is $\omega = \sum 2^{-n}\omega_n$, where ω_n is harmonic measure for U_n. For a fixed compact set K, μ_a and μ_s will mean, respectively, the absolutely continuous and singular parts of μ with respect to ω, and μ_{an} and μ_{sn} will mean the absolutely continuous and singular parts with respect to ω_n.

A point ζ is called a *bounded point evaluation* for $R^2(K,\mu)$ if there exists a constant C_ζ satisfying

$$|r(\zeta)| \leq C_\zeta \|r\|_{2,\mu}$$

for every r in $Rat(K)$. If ζ is a bounded point evaluation, it makes sense to talk about the value of $f(\zeta)$ for any function f in $R^2(K,\mu)$, because evaluation at ζ, defined a priori only for rational functions, extends by continuity to all of $R^2(K,\mu)$. The point ζ is called an *analytic bounded point evaluation* if it is in the interior of the bounded point evaluations, and if, for every $f \in R^2(K,\mu)$, the function $z \mapsto f(z)$ is analytic in a neighbourhood of ζ.

A *(Gleason) part* of $R(K)$ is a maximal set P in $\mathbb{C}$ such that for x and y in P, if e_x and e_y denote the functionals of evaluation at x and y respectively, then $\|e_x - e_y\|_{R(K)^*} < 2$. By [**7**, p.146] representing measures for points in different parts are mutually singular.

Let us recall some facts about hypo-Dirichlet algebras. These facts, and their proofs, all come from [1].

If the codimension of $\Re(R(K))$ in $\Re(C(\partial K))$ is t, then there are t measures $\eta_1, \ldots, \eta_t$ which annihilate $\Re(R(K))$ and are absolutely continuous with respect to harmonic measure for K. There are t invertible functions $F_1, \ldots, F_t$ in $R(K)$ such that $\int log|F_j| d\eta_k = \delta_{jk}$. A function h in $R^2(K,\omega)$ is called *inner* if, for some real numbers $\alpha_1, \ldots, \alpha_t$, $|h| = |F_1|^{\alpha_1} \ldots |F_t|^{\alpha_t}$ ω-almost everywhere. If v in $L^2(\mu)$ satisfies $\int log|v|d\omega > -\infty$, then there exists an inner function h and a function g that is a $Rat(K)$-cyclic vector for $R_{K,\omega}$, with $|v| = |g||h|$ ω-almost everywhere.

We are now able to prove our main theorem.

Theorem 2.1. *Let K be a compact planar set, such that $R(K)$ is a hypo-Dirichlet algebra. Let μ be a measure supported on the boundary of K, and let ν be a measure supported on K. Then $R_{K,\nu}$ is quasisimilar to $R_{K,\mu}$ if and only if*

(a) $(\nu|_{\partial K})_s$ *is mutually absolutely continuous with respect to* μ_s*;*

(b) If

$$\int_{\partial U_n} log \frac{d\mu}{d\omega_n} = -\infty$$

then

$$\int_{\partial U_n} log \frac{d\nu}{d\omega_n} = -\infty,$$

ν_{an} *is equivalent to* μ_{an}*, and* $\nu(U_n) = 0$*;*

(c) If

$$\int_{\partial U_n} log \frac{d\mu}{d\omega_n} > -\infty \tag{1}$$

then

$$\int_{\partial U_n} log \frac{d\nu}{d\omega_n} > -\infty. \tag{2}$$

Proof. (i) Necessity.

By hypothesis, there exists a $Rat(K)$-cyclic vector ϕ for $R_{K,\mu}$, and a $Rat(K)$-cyclic vector ψ for $R_{K,\nu}$, such that for all r in $Rat(K)$

$$\begin{aligned} \int |r\phi|^2 d\mu \leq \int |r|^2 d\nu \\ \int |r\psi|^2 d\nu \leq \int |r|^2 d\mu. \end{aligned} \tag{3}$$

Fix some interior component U_n. Every representing measure for a point of U_n that is supported on ∂K is absolutely continuous with respect to ω_n [1,3.1], so we can apply Forelli's lemma [7,p.42], which guarantees the existence of a sequence r_k in $Rat(K)$, with $\|r_k\|_\infty \leq 1$, and with r_k tending to 1 ω_n-almost everywhere, and to zero $(\mu + \nu|_{\partial K} + \omega)_{sn}$-almost everywhere. Therefore r_k converges ω weak-star to the characteristic function of $cl(U_n)$, so on the interior of K r_k converges pointwise to this characteristic function. Thus we can conclude that r_k actually converges $(\mu+\nu+\omega)$ weak-star to the characteristic function χ_{E_n} of a set E_n, where $\chi_{E_n} = 1 \quad (\omega_n + \nu|_{U_n})$ a.e., and $\chi_{E_n} = 0 \quad (\mu + \nu|_{\mathbb{C}\setminus U_n} + \omega)_{sn}$ a.e. Multiplying r by χ_{E_n} in inequalities (3), they become

$$\begin{aligned} \int_{cl(U_n)} |r\phi|^2 d\mu_n &\leq \int_{cl(U_n)} |r|^2 d\nu_n \\ \int_{cl(U_n)} |r\psi|^2 d\nu_n &\leq \int_{cl(U_n)} |r|^2 d\mu_n, \end{aligned} \tag{3'}$$

where $\mu_n = \mu_{an}$ and $\nu_n = \nu_{an} + \nu|_{U_n}$.

Let $\nu|_{\partial K}$ be called σ. As $1 - \chi_{(\cup E_n)}$ is also in the $(\mu+\nu+\omega)$ weak-star closure of the rationals, inequalities (3) also restrict to the singular parts of the measures on the boundary:

$$\begin{aligned} \int_{\partial K} |r\phi|^2 d\mu_s &\leq \int_{\partial K} |r|^2 d\sigma \\ \int_{\partial K} |r\psi|^2 d\sigma &\leq \int_{\partial K} |r|^2 d\mu_s. \end{aligned} \tag{3''}$$

By the Kolmogoroff-Krein theorem [7, p.135], no point of the interior of K is a bounded point evaluation for $R^2(K, \mu_s)$. But bounded point evaluations are preserved under quasisimilarity [11], so no point in the interior of K is a bounded point evaluation for $R^2(K, \sigma)$ either. But if a point ζ is a positive distance from the support of a measure μ, it is a bounded point evaluation for $R^2(K, \mu)$ if and only if $\frac{1}{z-\zeta}$ is not in $R^2(K, \mu)$. Therefore $R^2(K, \mu_s) = R^2(\partial K, \mu_s)$, and $R^2(K, \sigma) = R^2(\partial K, \sigma)$.

But by [1,3.1] every point in ∂K has a unique representing measure for $R(K)$, and hence for $R(\partial K)$, so every point in ∂K is a peak point for $R(\partial K)$, and thus by Bishop's peak point criterion [7, p.54], $R(\partial K) = C(\partial K)$. Therefore $R^2(K, \mu_s) = L^2(\mu_s)$, $R^2(K, \sigma) = L^2(\sigma)$, and inequalities $(3'')$ imply that the normal operators N_{μ_s} (multiplication by the independent variable on $L^2(\mu_s)$) and N_σ are quasisimilar. This forces them to be unitarily equivalent [6, p.82], and so μ_s and σ are mutually absolutely continuous. Thus (a) holds.

The Szegö theorem for hypo-Dirichlet algebras [1, 10.1] proves that in case (b), no point of U_n is a bounded point evaluation for $R^2(K, \mu_n)$, so by the above argument,

μ_n and ν_n are equivalent. Moreover, since points of U_n are not bounded point evaluations for $R^2(K, \nu_n)$,

$$\int_{\partial U_n} log \frac{d\nu}{d\omega_n} d\omega_n = -\infty,$$

and so (b) holds.

Part (c) would follow from the Szegö theorem and the preservation under quasisimilarity of bounded point evaluations if ν had no mass on U_n. Since it may, however, have mass on the interior, we use Riemann maps to localise it to the boundary of each component of the complement of $cl(U_n)$. We must be somewhat careful, though, as these boundaries need not be Jordan curves, so there are technical difficulties in extending the map to the boundary.

Assume U_n satisfies (1). Let V be a component of $\mathbb{C} \setminus cl(U_n)$. Let W be the complement of $cl(V)$. Then W is simply connected (in the Riemann sphere), contains U_n, and $\partial W = \partial V \subseteq \partial U_n$. By inverting, if necessary, we can assume without loss of generality that V is the unbounded component of $\mathbb{C} \setminus cl(U_n)$, and hence that $cl(W)$ is compact in $\mathbb{C}$. (This assumption is purely for convenience, allowing us to quote results directly. The same conclusions will hold for bounded V.)

Let g be the Riemann map from W onto the unit disk, and let m be harmonic measure for W. Since $cl(W)$ is simply connected, $R(cl(W))$ is a Dirichlet algebra, so by [6, p.356] there is a function G in $R^2(cl(W), m)$ with $|G| = 1$ m-almost everywhere, and $\int G dm_b = g(b)$ for all b in W (*i.e.* G is the extension of g to the boundary).

Since G is bounded, by [1,4.1] there exist rational functions ρ_k in $Rat(cl(W))$, with $\|\rho_k\|_\infty \leq 1$, which tend to G m-almost everywhere, and so, by the dominated convergence theorem, $\rho_k(b)$ tends to $g(b)$ for all b in W. So if $\hat{G}$ is the function that equals G on ∂W and g on W, then, because $\omega_n|_{\partial W}$ is absolutely continuous with respect to m [9], ρ_k tends to $\hat{G}$ in $L^1(\omega_n)$. Therefore $\hat{G}$ is in $R^1(cl(W), \omega_n) \cap L^\infty(\omega_n)$, which, by [1,4.1], is just $R^\infty(cl(W), \omega_n)$. Since μ_n is equivalent to ω_n, $\hat{G}$ is in $R^\infty(cl(W), \mu_n)$, and hence, by Theorem 1.4, in $R^\infty(cl(W), \nu_n)$.

Therefore, inequalities $(3')$ remain true if r is replaced by $r(\hat{G})^k$. Letting k tend to infinity kills off the interior contribution, because $|\hat{G}|$ is less than one on the interior, so

$$\int_{\partial V} |r\phi|^2 d\mu_n \leq \int_{\partial V} |r|^2 d\nu_n$$

and

$$\int_{\partial V} |r\psi|^2 d\nu_n \leq \int_{\partial V} |r|^2 d\mu_n. \tag{4}$$

Summing inequalities (4) over all V in the complement of U_n, we finally get that R_{K,μ_n} is quasisimilar to $R_{K,\nu_n|\partial U_n}$, so preservation of bounded point evaluations implies that (2) holds.

(ii) Sufficiency

We must show that the normal parts of $R_{K,\mu}$ and $R_{K,\nu}$ are unitarily equivalent, and that the pure parts are quasisimilar [6, p.223]. Note first that for different components U_n and U_m, the harmonic measures ω_n and ω_m are singular. This is because, by [**8**, 8.3] (applicable because of [**1**]), the parts of $R(K)$ are connected, and, as remarked above, any point in ∂K has a unique representing measure, so lies in a part of its own. Therefore each U_n constitutes a part, and so ω_n and ω_m are singular.

Now the normal parts correspond to the measures in cases (a) and (b), and by hypothesis these are mutually absolutely continuous, so the normal parts are unitarily equivalent. By applying Forelli's lemma again, it is therefore sufficient to prove R_{K,ν_n} is quasisimilar to R_{K,μ_n} in case (c). We shall do this by showing that if ν is a measure on $cl(U_n)$, and $\nu|_{\partial U_n}$ is absolutely continuous with respect to ω_n and satisfies (2), then $R_{K,\nu}$ is quasisimilar to R_{K,ω_n}.

By the remarks preceding the theorem, there is an inner function h and a generator g so that $\frac{d\nu}{d\omega_n} = |gh|$ ω_n-almost everywhere. Let $A : R^2(K,\omega_n) \to hR^2(K,\omega_n)$ be given by $Ar = hr$. Because h is inner, its modulus on ∂K is essentially bounded away from zero, and is also bounded above. Therefore A is invertible.

Define $B : R^2(K,\nu) \to hR^2(K,\omega_n)$ by $Br = ghr$. B has dense range, and because

$$\int |ghr|^2 d\omega_n = \int_{\partial U_n} |r|^2 d\nu \leq \int_{cl(U_n)} |r|^2 d\nu,$$

B is bounded. Let $X = A^{-1}B$. Then X is a quasi-affinity satisfying $R_{K,\omega_n}X = XR_{K,\nu}$.

To get the other intertwining operator, let $\hat{\nu}$ be the sweep of ν, *i.e.* that measure on ∂U_n satisfying

$$\int_{cl(U_n)} \hat{f} d\nu = \int_{\partial U_n} f d\hat{\nu}$$

(where $\hat{f}$ is as defined at the beginning of Section 2). Then $\hat{\nu}$ is absolutely continuous with respect to ω_n [6, p.335], and

$$\int_{\partial U_n} log \frac{d\hat{\nu}}{d\omega_n} d\omega_n \geq \int_{\partial U_n} log \frac{d\nu}{d\omega_n} d\omega_n > -\infty,$$

so there exists an inner function $\tilde{h}$ and a generator $\tilde{g}$ satisfying $\frac{d\hat{\nu}}{d\omega_n} = |\tilde{h}\tilde{g}|$ ω_n-almost everywhere.

Define $C : R^2(K, |\tilde{g}|^2 d\omega_n) \to R^2(K, \omega_n)$ by $Cr = \tilde{g}r$. C is unitary.
Define $D : R^2(K, |\tilde{g}|^2 d\omega_n) \to \tilde{h}R^2(K, |\tilde{g}|^2 d\omega_n)$ by $Dr = \tilde{h}r$. D is invertible.
Define $E : R^2(K, \hat{\nu}) \to \tilde{h}R^2(K, |\tilde{g}|^2 d\omega_n)$ by $Er = \tilde{h}r$. E is unitary.
Define $F : R^2(K, \hat{\nu}) \to R^2(K, \nu)$ by $Fr = r$. F is a quasi-affinity.

Now, put $Y = FE^{-1}DC^{-1}$. Y is a quasi-affinity, and $YR_{K,\omega_n} = R_{K,\nu}Y$. Therefore $R_{K,\nu}$ is quasisimilar to R_{K,ω_n}, as required. ////

It is a result of Trent [13] that the set of analytic bounded point evaluations of $R^2(K, \mu)$ is precisely the spectrum of $R_{K,\mu}$ less the essential spectrum. Thus, Theorem 2.1 could be restated:

Suppose $R(K)$ is a hypo-Dirichlet algebra, and μ is supported on ∂K. Then $R_{K,\nu}$ is quasisimilar to $R_{K,\mu}$ if and only if the two operators have the same spectrum and essential spectrum, and their normal parts are unitarily equivalent.

This is not true in general, however. Here is an example of a family of pure, cyclic subnormal operators, which have the same spectral pictures, but are not quasisimilar.

Example 2.2. For α a positive real number, let μ_α be the weighted area measure $(1-r)^\alpha r dr d\theta$ on the unit disk. Suppose that for two numbers $\alpha < \gamma$, S_{μ_α} were quasisimilar to S_{μ_γ}, so that there were a quasi-affinity X intertwining them. If $X1$ were ϕ, then we would have

$$\int |p|^2 |\phi|^2 (1-r)^\alpha r dr d\theta \leq C \int |p|^2 (1-r)^\gamma r dr d\theta. \qquad (5)$$

Suppose $\phi(z) = \sum a_k z^k$. Then letting $p(z) = z^n$ in (5), we would get, in terms of the beta function,

$$\sum_{k=0}^{\infty} |a_k|^2 \beta(2n + 2k + 2, \alpha + 1) \leq C \beta(2n + 2, \gamma + 1).$$

But this is impossible, since the first non-zero element on the left-hand side behaves asymptotically like $n^{-(\alpha+1)}$, and the right-hand side is $O(n^{-(\gamma+1)})$.

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