# REFLEXIVITY OF SUBNORMAL OPERATORS

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*Dedicated to Donald Sarason, in admiration of the range of his pioneering work*

Abstract. We give a new proof that subnormal operators are reflexive. We extend this to certain subnormal $n$-tuples. We give the first complete proof that a pair of doubly commuting isometries is reflexive.

## 0. Introduction

Let $A$ be a weakly closed algebra of bounded linear operators on a Hilbert space $\mathcal{H}$. Its lattice, $Lat(A)$, is the set of all closed subspaces of $\mathcal{H}$ that are left invariant by every element of $A$. The set of operators that leave invariant every space in $Lat(A)$ is denoted $AlgLat(A)$. The algebra $A$ is called *reflexive* if $A = AlgLat(A)$. An operator (or set of operators) $T$ is called reflexive if the weakly closed unital algebra it generates, $W(T)$, is reflexive.

D. Sarason proved that normal operators are reflexive, and that so are analytic Toeplitz operators [Sa1]. R. Olin and J. Thomson extended this result in 1979 to prove that all subnormal operators (*i.e.* restrictions of normal operators to invariant subspaces) are reflexive [OT]. Whilst their original proof has been somewhat simplified since then [Th1], [Th2], [Co1], to date all proofs have relied on an elaborate construction of "full analytic subspaces". We show that Thomson's work on bounded point evaluations [Th3] allows a much simpler proof (Theorem 1).

An $n$-tuple of operators $N = (N_1, \dots, N_n)$ is called *normal* if each $N_i$ is normal, and $N_i N_j = N_j N_i$ for all $i, j$. The $n$-tuple $S = (S_1, \dots, S_n)$ of operators on $\mathcal{H}$ is called *subnormal* if there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$, and a normal $n$-tuple $(N_1, \dots, N_n)$ on $\mathcal{K}$, such that each $N_i$ leaves $\mathcal{H}$ invariant, and $N_i|_{\mathcal{H}} = S_i$. Just as in the cyclic case,

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX a subnormal $n$-tuple is reflexive if its restriction to every cyclic subspace is. Moreover, any cyclic subnormal $n$-tuple is unitarily equivalent to $(M_{z_1}, \ldots, M_{z_n})$ on $P^2(\mu)$ for some compactly supported measure $\mu$ on $\mathbb{C}^n$ (see Section 2). So studying reflexivity reduces to studying the spaces $P^2(\mu)$ and the $n$-tuple $S_\mu$ of multiplication by the variables.

A point $\lambda$ in $\mathbb{C}$ is called a *bounded point evaluation* for $P^2(\mu)$ if the functional of evaluating at $\lambda$, defined on the polynomials, is bounded on the space $P^2(\mu)$. If $\lambda$ is a bounded point evaluation, then there is a function $k_\lambda$ such that $p(\lambda) = \int p\overline{k_\lambda}d\mu$. Integration against this kernel function gives a well-defined meaning for $f(\lambda)$ for any function $f$ in $P^2(\mu)$, and we shall use this without further comment (this is a direct generalization of the usual practice of thinking of functions in the Hardy space $H^2$ both as functions on the unit circle with vanishing negative Fourier coefficients, and as analytic functions on the unit disk). The point $\lambda$ in $\mathbb{C}$ is called an *analytic bounded point evaluation* for $P^2(\mu)$ if $\lambda$ is in the interior of the bounded point evaluations, and for each $f$ in $P^2(\mu)$, the function sending $\zeta$ to $\int f\overline{k_\zeta}d\mu$ is analytic in a neighbourhood of $\lambda$.

The most important result about single subnormal operators since S. Brown's proof of the existence of invariant subspaces [Br] was Thomson's proof of the existence of analytic bounded point evaluations [Th3]. He showed that if $S_\mu$ is pure, all the bounded point evaluations are analytic bounded point evaluations, and that the kernel functions $k_\lambda$ span $P^2(\mu)$. We shall use this as one of the pillars of our proof of reflexivity of single subnormal operators.

The second pillar is the structure of $A(S)$, the weak-star (*i.e.* $\sigma$-weakly) closed algebra generated by the subnormal operator $S$. Conway and Olin [CO] showed that this algebra is isometrically isomorphic and weak-star homeomorphic to $P^\infty(\mu)$, the weak-star closure of the polynomials in $L^\infty(\mu)$, where $\mu$ is the scalar spectral measure for the minimal normal extension of $S$. Sarason has shown [Sa2] that $P^\infty(\mu)$ consists of an $L^\infty$ summand plus the space of bounded analytic functions on the interior of a certain set $\Sigma(\mu)$, called the Sarason hull.

The third pillar is that the algebra $A(S)$ is *elementary,* (also called $A_1$) *i.e.* if $L$ is a weak-star continuous linear functional on $A(S)$, then there are vectors $x$ and $y$ such that

$L(p(S)) =< p(S)x, y >$ for any polynomial $p$. This idea first surfaced in [Br], and was proved in general in [OT]. The proof was refined in [Th2], and the state-of-the-art proof is in [BC]. One of the consequences is that $A(S) = W(S)$. See [Co1] for an exposition of all these results.

These three results (the existence of analytic bounded point evaluations, the representation of $A(S)$ as $P^\infty(\mu)$, together with Sarason's description of $P^\infty(\mu)$, and the factoring of weak-star continuous linear functionals) are fundamental to the modern theory of single subnormal operators. However, for subnormal tuples the first two results are in general false, and the validity of the third (whether the $\sigma$-weakly closed algebra a subnormal tuple generates is elementary) is open. We can make some progress if we impose (fairly stringent) restrictions, for example:

**Theorem 4** *Let $\mu$ be a measure on $\mathbb{C}^n$, and let $\Omega$ be the set of analytic bounded point evaluations for $P^2(\mu)$. Suppose $\Omega$ has polynomially convex closure, and is either (i) strongly pseudoconvex or (ii) star-shaped. Suppose also that the span of the kernel functions corresponding to points of $\Omega$ is dense in $P^2(\mu)$ Then $S_\mu$ is reflexive.*

In section 3 we prove that if two isometries doubly commute then they are reflexive. An incomplete proof of this is given in [Pt1]. In section 4 we give an example to show that non-commuting isometries need not be reflexive.

## 1. Single Subnormal Operators

The key part of the proof of reflexivity for subnormal operators is proving it for pure cyclic subnormal operators. It is well-known that any cyclic subnormal operator is unitarily equivalent to multiplication by the independent variable on some space $P^2(\mu)$, where $\mu$ is a compactly supported measure on $\mathbb{C}$. We shall denote this operator $S_\mu$; it is pure (*i.e.* has no normal summand) if $P^2(\mu)$ does not have $L^2(\mu|_E)$ as a summand for any set $E$ of positive measure. For this, and other basic facts about subnormal operators, see J. Conway's book [Co1].

**Theorem 1** *Pure cyclic subnormal operators are reflexive.*

*Proof.* Assume the operator is $S_\mu$ for some $\mu$. Let $T$ be in $AlgLat(A(S_\mu))$.

(i) The kernel functions $k_\lambda$ are eigenvectors of $S_\mu^*$, so are left invariant by $T^*$. Therefore $T^*$ commutes with $S_\mu^*$ on each of the one-dimensional spaces spanned by some $k_\lambda$, and as their span is dense, $T^*$ commutes with $S_\mu^*$. Therefore $T$ commutes with $S_\mu$, and is given by multiplication by some function $\phi$ in $P^2(\mu) \cap L^\infty(\mu)$.
(ii) Because $A(S_\mu)$ is isometrically isomorphic and weak-star homeomorphic to $P^\infty(\mu)$, we must show $\phi$ is in $P^\infty(\mu)$; this requires showing that $\phi$ is analytic on the interior of $\Sigma(\mu)$.

Let $f$ be in $P^2(\mu)$. The closure of $\{pf : p \textit{ a polynomial } \}$ is an invariant subspace for $S_\mu$, so must be $\phi$ invariant. Therefore one can approximate $\phi f$ by $p_n f$, where $p_n$ are polynomials; but this means one can approximate $\phi$ by $p_n$ in $L^2(|f|^2\mu)$, so $\phi$ is in $P^2(|f|^2\mu)$. Therefore $\phi$ is analytic on the set of analytic bounded point evaluations of $P^2(|f|^2\mu)$, for every $f$.
(iii) Let $\lambda$ be an arbitrary point in the interior of $\Sigma(\mu)$. Evaluation at $\lambda$ is weak-star continuous, so there are functions $x$ and $y$ in $P^2(\mu)$ such that $p(\lambda) = \int px\bar{y}d\mu$. Therefore $\lambda$ is a bounded point evaluation for $P^2(|x|^2\mu)$, hence an analytic bounded point evaluation (as $S_{|x|^2\mu}$ is pure if $S_\mu$ is); therefore $\phi$ is analytic at $\lambda$, as desired. □

To prove that all subnormal operators are reflexive is now fairly quick. For completeness, we include a proof based on that in Conway [Co1] Theorem VII.8.5; notice that the proof of the previous theorem allows one to skip completely sections VII.6 and VII.7 in [Co1].

**Theorem 2** *All subnormal operators are reflexive.*

*Proof (Conway).* (i) First, we show that reflexivity of cyclic subnormals implies it for all subnormals.

Let $S$ be subnormal on $\mathcal{H}$, with minimal normal extension $N$, and let $T$ be in $AlgLat(A(S))$. Let $x$ and $y$ be vectors in $\mathcal{H}$ such that $x$, $y$ and $x+y$ are all separating vectors for the von Neumann algebra generated by $N$. Such vectors are dense in $\mathcal{H}$.
By hypothesis (and the separating requirement), on the cyclic spaces generated by $x$, $y$ and $x+y$, $T$ agrees with operators in $A(S)$, which we shall call respectively $X$, $Y$ and $Z$. So

$$T(x+y) = Z(x+y) = Tx + Ty = Xx + Yy.$$

Therefore

$$(X - Z)x = (Z - Y)y.$$

Since this last vector is in both the space generated by $x$ and the space generated by $y$, both $X$ and $Y$ must agree on it. Therefore

$$0 = (X - Y)(X - Z)x = (X - Y)(Z - Y)y$$

Because both $x$ and $y$ are separating,

$$0 = (X - Y)(X - Z) = (X - Y)(Z - Y).$$

Adding then yields that $(X - Y)^2 = 0$, so $X = Y$.

(ii) For a normal operator $N$, the spectral theorem implies that any operator leaving $N$'s reducing subspaces invariant must commute with it. So parts (ii) and (iii) of the proof of Theorem 1 prove that cyclic normal operators are reflexive, and hence all normal operators are.

(iii) To get rid of purity, note that if $S = N_0 \oplus S_1$ is the decomposition of $S$ into a normal and pure part, and $T$ is in $AlgLat(A(S))$, then $T = T_0 \oplus T_1$ in $A(N_0) \oplus A(S_1)$. Because this last algebra is elementary, any weak-star continuous functional is of the form $(x_0 \oplus x_1) \otimes (y_0 \oplus y_1)$. If this annihilates $A(S)$, $(y_0 \oplus y_1)$ is orthogonal to the subspace generated by $A(S)(x_0 \oplus x_1)$, and hence to $T(x_0 \oplus x_1)$. Therefore $T$ is in $A(S)$, as desired. $\square$

## 2. Subnormal Tuples

Many of the properties of subnormal operators carry over to subnormal tuples, but as we know of no source where they are written down, we list some here. The proofs are fairly straightforward generalizations of the proofs for single subnormal operators, so we shall not include them here. For other properties of subnormal tuples, see [CS] and [Co2]. We assume in what follows that $S = (S_1, \dots, S_n)$ is a subnormal $n$-tuple on $\mathcal{H}$, with minimal normal extension $N$ on $\mathcal{K}$ (minimal means that $\mathcal{K}$ is the closed linear span of $\{N_1^{*i_1} \dots N_n^{*i_n}\xi : \xi \in \mathcal{H}, i_1, \dots, i_n \in \mathbb{N}\}$).

There is a measure $\mu$ supported on a compact subset $K$ of $\mathbb{C}^n$, and an isometric isomorphism and weak-star homeomorphism $\rho$ from $L^\infty(\mu)$ onto $W^*(N)$, the von Neumann algebra generated by $N$, taking $N_i$ onto the $i^{th}$ coordinate function $z_i$. Spatially, $N$ is unitarily equivalent to the $n$-tuple of multiplication operators $M_z := (M_{z_1}, \ldots, M_{z_n})$ on $L^2(\mu; D)$ of square-summable functions from $K$ to $D$, a Hilbert space of possibly varying dimension. If $S$ is cyclic, it is unitarily equivalent to $S_\mu$, $M_z$ on $P^2(\mu)$ (this was first observed in [Ha]). Separating vectors for $W^*(N)$ are dense in both $\mathcal{K}$ and $\mathcal{H}$; and if $x$ and $y$ are separating vectors, and $\epsilon > 0$, there is some $\alpha$, $0 < \alpha < \epsilon$, such that $x + \alpha y$ is also separating.

If $\phi$ in $L^\infty(\mu)$ satisfies $\rho(\phi(N))\mathcal{H} \subseteq \mathcal{H}$, we will write $\phi(S)$ for $\rho(\phi(N))|_{\mathcal{H}}$; it is then true that $\|\phi(S)\| = \|\phi(N)\| = \|\phi\|_\infty$. From this it follows that both $A(S)$ and $A(N)$ are naturally isometrically isomorphic and weak-star homeomorphic to $P^\infty(\mu)$. If $S$ is cyclic and $T$ is an operator that commutes with each $S_i$, then $T$ is multiplication by some bounded function in $P^2(\mu)$.

All normal $n$-tuples are reflexive. A subnormal $n$-tuple is reflexive if its restriction to every cyclic subspace is reflexive.

It is not known in general whether subnormal tuples are reflexive (even when a pair of commuting isometries), or whether weak-star continuous linear functionals on $A(S)$ can be factored as $x \otimes y$. The methods of the previous section, relying as they do on analyticity, can certainly not be applied to general subnormal tuples. To see how bad these can be, consider the following example.

*Example.* Let $X$ be a compact subset of $\mathbb{C}$. The Banach algebra $R(X)$, the closure in $C(X)$ of the rational functions with poles off $X$, is doubly generated, as is well-known; choose one generator to be the coordinate function $z$, call the other generator $f$. Let $\iota : z \mapsto (z, f(z))$ map $X$ onto $K$, a subset of $\mathbb{C}^2$. Then $\iota^*$ gives an isomorphism from $P(K)$ to $R(X)$, and $\iota^{**}$ gives an isomorphism from the measures on $X$ to the measures on $K$. Moreover, $K$ is polynomially convex, because the maximal ideal space of $P(K)$ is $\iota(X) = K$. Thus, $P^2(\mu)$, for $\mu$ a measure on $K$, is just as bad as $R^2(X, \nu)$ can be. In particular, there exists a measure $\mu$ on a polynomially convex set $K$ such that $P^2(\mu)$ has no $L^2$ summands, but

(i) the weak-star continuous point evaluations of $P^\infty(\mu)$ have no interior;

(ii) $P^2(\mu)$ has no bounded point evaluations at all [Bre].

However, the natural $P^2(\mu)$ spaces to study are those that do possess analytic structure. In particular, the obvious measures are volume and surface-area, and these clearly have analytic bounded point evaluations (via the Bergman and Szegö kernels). The difficulty lies in identifying $P^\infty(\mu)$. Notice that the proof of Theorem 1 yields the following for cyclic subnormal tuples:

**Theorem 3** *Let $\mu$ be a measure on $\mathbb{C}^n$, and let $\Omega$ be the set of analytic bounded point evaluations for $P^2(\mu)$. Suppose*

*(a) the span of the kernel functions corresponding to points of $\Omega$ is dense in $P^2(\mu)$;*

*(b) $P^2(\mu) \cap H^\infty(\Omega) = P^\infty(\mu)$.*

*Then the subnormal $n$-tuple $S_\mu$ is reflexive.*

As we said, the difficulty lies in checking (b). Here are two cases where it holds.

**Theorem 4** *Suppose $\Omega$ has polynomially convex closure, and is either (i) strongly pseudoconvex or (ii) star-shaped. Then, in the previous theorem, condition (a) implies condition (b).*

*Proof.* (i) Let $f$ be in $P^2(\mu) \cap H^\infty(\Omega)$. Because $\Omega$ is strongly pseudoconvex, there are functions $f_j$, analytic on a neighborhood of $\overline{\Omega}$, converging to $f$ uniformly on compacta, and satisfying $\|f_j\|_\infty \leq \|f\|_\infty$ [CR] - see also [BF]. As $\overline{\Omega}$ is polynomially convex, the $f_j$'s in turn can be uniformly approximated by polynomials $p_{j,k}$. As the ball of $P^\infty(\mu)$ is weak-star compact, some subsequence of $p_{j,j}$ will converge weak-star to some function $g$ in $P^\infty(\mu)$. But

$$\int g\overline{k_\lambda}d\mu = g(\lambda) = \lim p_{jj}(\lambda) = \int f\overline{k_\lambda}d\mu$$

for all $\lambda$ in $\Omega$, so $f$ is in $P^\infty(\mu)$.

The reverse inclusion is obvious.

(ii) Assume for simplicity that $\Omega$ is star-shaped with respect to zero. Then the above argument works, with the obvious choice for $f_j$ of $f(r_j z)$, where $r_j$ increases to 1. $\square$

Case (ii) above generalizes the results of M. Ptak in [Pt2].

## 3. Doubly Commuting Isometries

An $n$-tuple of operators $(T_1, \dots, T_n)$ is said to *doubly commute* if, for any distinct indices $i$ and $j$, $T_i T_j = T_j T_i$ and $T_i^* T_j = T_j T_i^*$. In this section we prove that any pair of doubly commuting isometries is reflexive. That single isometries are reflexive was first proved by J. Deddens [De]. In [Pt1], Ptak claims to prove that a pair of doubly commuting isometries is reflexive; however, his proof contains a gap.

Specifically, let $S$ be the shift on $H^2(\mathcal{H})$, the space of square-summable analytic functions with values in the Hilbert space $\mathcal{H}$ (for information on shifts of multiplicity greater than one, which can be represented as multiplication by $z$ on some $H^2(\mathcal{H})$, see the book [Sz-NF]). Let $T_0$ be an operator on $\mathcal{H}$, and let $T$ be the operator in $\{S\}' = H^\infty(B(\mathcal{H}))$ defined by $T(z) = T_0$ for all $z$ in the circle. Then in [Pt1], Proposition 4 Ptak asserts that if $T_0$ has property $C_\infty$, then $\{S, T\}$ has property $C$ (see definitions below), and says the proof is a small modification of the proof of [Wo], Lemma 2. But Wogen's proof, which he gives for the case that $T_0$ is positive, can only be generalized to reductive operators $T_0$ (see Lemma 5 below). We note that if $T_0$ is the unilateral shift, Ptak's assertion is equivalent to the following question in function theory:
Let $f_1$, $f_2$ be two functions in $H^2(\mathbb{T}^2)$. Does there necessarily exist $f_0$ in $H^2(\mathbb{T}^2)$ such that $|f_0|^2 = |f_1|^2 + |f_2|^2$ almost everywhere on the torus?

*Definitions.* If $T$ is an operator on the Hilbert space $\mathcal{H}$, then $T^{(n)}$ acting on $\mathcal{H}^{(n)}$ is the direct sum of $n$ copies of $T$ with itself. If $\mathcal{T}$ is a subset of $B(\mathcal{H})$, and $x$ is a vector in $\mathcal{H}$, then $C(\mathcal{T}, x)$ is the smallest closed subspace containing $x$ and invariant under every element of $\mathcal{T}$. A subset $\mathcal{T}$ of $B(\mathcal{H})$ has *property* $C$ (introduced in [Wo]) if, for every positive integer $n$ and vector $\eta$ in $\mathcal{H}^{(n)}$, there is a vector $\xi$ in $\mathcal{H}$ and a unitary operator $U : C(\mathcal{T}^{(n)}, \eta) \to C(\mathcal{T}, \xi)$ such that, for every $T$ in $\mathcal{T}$,

$$UT^{(n)}|_{C(\mathcal{T}^{(n)}, \eta)} U^* = T|_{C(\mathcal{T}, \xi)}.$$

If this also holds for $n = \infty$, say that $\mathcal{T}$ has *property* $C_\infty$.

The proof of the following lemma is based on [Wo], Lemma 2, but we include it for completeness. A normal operator is *reductive* if every invariant subspace is also invariant for its adjoint. In terms of the scalar spectral measure $\mu$, this is equivalent to saying that $P^\infty(\mu) = L^\infty(\mu)$.

**Lemma 5** *Let $S$ be the unilateral shift on $H^2(\mathcal{H})$, and let $\mathcal{T}_0$ be a finite set of commuting normal operators on $\mathcal{H}$ that are all reductive. Extend each operator $T_0$ in $\mathcal{T}_0$ to an operator $T$ on $H^2(\mathcal{H})$, by defining $T(z) = T_0$ for all $z$ in the circle; call the new collection $\mathcal{T}$. Then $A(S, \mathcal{T})$ has property $C_\infty$.*

*Proof.* Let $\eta$ be in $H^2(\mathcal{H})^{(n)}$, for some $1 \leq n \leq \infty$. Let $\mathcal{M} = C(\{S^{(n)}, \mathcal{T}^{(n)}\}, \eta)$ and $\mathcal{M}_0 = \mathcal{M} \ominus S^{(n)}\mathcal{M}$. For each $T$ in $\mathcal{T}$, $S^{(n)}\mathcal{M}$ is invariant under $T^{(n)}$, so $H^2(\mathcal{H})^{(n)} \ominus S^{(n)}\mathcal{M}$ is invariant for ${T^{(n)}}^*$. Because $T^{(n)}$ is reductive, $H^2(\mathcal{H})^{(n)} \ominus S^{(n)}\mathcal{M}$ is also invariant for $T^{(n)}$, so $T^{(n)}\mathcal{M}_0$ is contained in $\mathcal{M}_0$. Therefore $\mathcal{M}_0$ is a reducing subspace for each $T^{(n)}$, and $T^{(n)}|_{\mathcal{M}_0}$ is always normal.

The space $\mathcal{M}_0$ is precisely $C(\mathcal{T}^{(n)}, P_{\mathcal{M}_0}\eta)$, where $P_{\mathcal{M}_0}$ is the orthogonal projection onto $\mathcal{M}_0$. By the spectral theorem for normal tuples, and as each $T$ is just a direct sum of countably many copies of $T_0$, there is a $\mathcal{T}_0$-cyclic subspace $\mathcal{N}_0$ of $\mathcal{H}$ and a unitary map $U_0$ from $\mathcal{M}_0$ onto $\mathcal{N}_0$ such that $U_0 T^{(n)}|_{\mathcal{M}_0} U_0^* = T_0|_{\mathcal{N}_0}$ for each $T$ in $\mathcal{T}$. Now extend this to a map from $\mathcal{M}$ into $H^2(\mathcal{H})$ by embedding $\mathcal{N}_0$ into the constant functions of $H^2(\mathcal{H})$ (*i.e.* the kernel of $S^*$), and defining $U({S^{(n)}}^k m_0) = z^k U_0(m_0)$ for $m_0$ in $\mathcal{M}_0$. As the range of $U$ is $A(S, \mathcal{T})$-cyclic, $A(S, \mathcal{T})$ has property $C_\infty$, as required. □

We also need the following lemma. It is also proved in [Pt1].

**Lemma 6** *Let $S$ be the unilateral shift on $H^2(\mathcal{H})$, and let $\mathcal{T}_0$ be a finite commuting set of operators on $\mathcal{H}$ such that $AlgLat(\mathcal{T}_0) = A(\mathcal{T}_0)$. Extend each operator $T_0$ in $\mathcal{T}_0$ to an operator $T$ on $H^2(\mathcal{H})$, by defining $T(z) = T_0$ for all $z$ in $\mathbb{T}$; call the new collection $\mathcal{T}$. Then $AlgLat(S, \mathcal{T}) = W(S, \mathcal{T}) = A(S, \mathcal{T})$.*

*Proof.* Let $R$ be in $AlgLat(S, \mathcal{T})$. If $k_\lambda(z)$ is the kernel function $1/(1 - \bar{\lambda} z)$, then $\{k_\lambda.\mathcal{H}\}$ is the kernel of $(S - \lambda)^*$, and is in $Lat(S^*, \mathcal{T}^*)$. Therefore $R^*$ leaves it invariant, and so commutes with $S^*$ on it. As these spaces span $H^2(\mathcal{H})$, $R$ commutes with $S$. Therefore $R$

can be written as $R = \sum_{k=0}^{\infty} z^k R_k$, where each $R_k$ maps $\mathcal{H}$ to $\mathcal{H}$, and the series converges almost uniformly on the disk, and radially it converges strongly almost everywhere (see [Sz-NF]).

Let $\mathcal{M}$ be an arbitrary $\mathcal{T}_0$-invariant subspace. Then $H^2(\mathcal{M})$ is $A(S, \mathcal{T})$-invariant, so $R$-invariant. Therefore for almost every $w$ in $\mathbb{T}$, $R(w) = \lim_{r \uparrow 1} \sum_{k=0}^{\infty} (rw)^k R_k$ is in $AlgLat(\mathcal{T}_0)$. So if $\xi$ is in $\mathcal{M}$ and $\eta$ is in $\mathcal{M}^{\perp}$,

$$(\sum_{k=0}^{\infty} w^k R_k \xi, \eta) = 0 \quad a.e.$$

The left-hand side is a bounded analytic function of $w$, so it vanishes for all $w$ in the unit disk. In particular, for $w = 0$, this gives that $R_0$ is in $AlgLat(\mathcal{T}_0)$. Hence $\sum_{k=0}^{\infty} w^k R_{k+1}$ is in $AlgLat(\mathcal{T}_0)$ for almost every $w$, and by induction each $R_k$ is.

Let $R_r = \sum_{k=0}^{\infty} r^k z^k R_k$. For $0 < r < 1$, the partial sums $\sum_{k=0}^{N} r^k z^k R_k$ are all in $A(S, \mathcal{T})$, and converge to $R_r$ in norm. Moreover, as $\{R_r : 0 < r < 1\}$ is uniformly bounded, $R_r$ converges to $R$ in both the weak and $\sigma$-weak operator topologies. $\square$

**Theorem 7** *Let $T = (T_1, T_2)$ be a pair of doubly commuting isometries on the Hilbert space $\mathcal{H}$. Then $AlgLat(T) = W(T) = A(T)$.*

*Proof.* For each isometry $T_i$, $\mathcal{H}$ has a Wold decomposition, *i.e.* it decomposes into two reducing subspaces, on the first of which the operator is a pure shift, and on the second of which it is unitary. As the first space is the span of the kernels of $(T_i^*)^n$, it is left invariant by everything that commutes with $T_i^*$; so the double commutativity hypothesis yields that each of these spaces is also reducing for the other isometry. Therefore $\mathcal{H}$ decomposes into 4 spaces, on which each $T_i$ is either unitary or a pure shift.

Moreover, the spaces where we have a shift $S$ and a unitary $U$ decompose further. By multiplying by the spectral projection for $U$ corresponding to, say, the right semi-circle, $U$ decomposes into the direct sum of two reductive operators, and because $S$ commutes with $U$, it also commutes with this decomposition.

Therefore $\mathcal{H}$ decomposes into (up to) 6 reducing subspaces, on which $T_1, T_2$ are either
(i) both unitary, or

(ii) one a pure shift, the other a reductive unitary, or

(iii) both pure shifts.

We claim that in each case the pair is reflexive, the weakly and $\sigma$-weakly closed algebras they generate coincide, and that this algebra is elementary. This implies that on the direct sum of the subspaces the pair is reflexive and the two algebras coincide.

(i) We have already remarked that normal tuples are reflexive. That the algebra $A(T)$ is elementary follows from the spectral theorem.

(ii) This follows from Lemmata 5 and 6, because property $C_\infty$ implies $A(T)$ is elementary.

(iii) Because $T_1$ and $T_2$ doubly commute, the space decomposes into a direct sum of copies of $H^2(\mathbb{T}^2)$, where the operators are multiplication by the coordinate functions. The pair is reflexive (and $A(T) = W(T)$) by Lemma 6. That $A(T)$ is elementary follows from the theorem of H. Bercovici and D. Westwood that any function $h$ in $L^1(\mathbb{T}^n, \sigma)$, where $\sigma$ is Lebesgue measure, can be factored as $h = fg$, for some $f$ in $H^2(\mathbb{T}^n)$ and $g$ in $L^2(\mathbb{T}^n, \sigma)$ [BW]. $\square$

## 4. Non-commuting Isometries

We finish with an example to show that non-commuting isometries need not be reflexive.

**Example 8** Let $S$ be the unilateral shift, and $U$ the diagonal operator with first two entries $-1$, other entries 1.

$$S = \begin{pmatrix} 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad U = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix}.$$

Let $A$ be the weakly closed algebra generated by $S$ and $U$. We claim that $A$ is not reflexive.

By Beurling's theorem, the invariant subspaces of $S$, thought of as an operator on the Hardy space $H^2$, are the spaces $uH^2$, where $u$ is an inner function. So $Lat(A)$ consists of those $uH^2$ which are $U$-invariant. Let $u(z) = \sum_{n=0}^{\infty} a_n z^n$. If $a_0 = a_1 = 0$, then $uH^2$ is clearly $U$-invariant.

Suppose $a_0 \neq 0$, and $uH^2$ is in $Lat(A)$. Then applying the operator $(I-U) - \frac{a_1}{a_0}(I-U)S$ to $u$, one gets that the constant function $2a_0$ is in $uH^2$, so $uH^2$ must be all of $H^2$. Similarly, if $a_0 = 0$ and $a_1 \neq 0$, then the operator $\frac{1}{2a_1}(I-U)$ applied to $u$ gives the function $z$, so $uH^2 = zH^2$. Therefore $Lat(A) = \{z^2uH^2 : u \textit{ inner }\} \cup \{zH^2\} \cup \{H^2\} \cup \{0\}$.

Let $R$ be the rank-one diagonal operator

$$R = \begin{pmatrix} 1 & & \\ & 0 & \\ & & \ddots \end{pmatrix}.$$

Clearly $R$ is in $AlgLat(A)$. We shall show it is not in $A$.

Let $p(U,S)$ be a non-commuting polynomial in $U$ and $S$. Because of the identities $U^2 = I$, $US^2 = S^2$ and $USUS = SUS$, $p(U,S)$ must have the reduced form

$$p(U,S) = q_1(S) + q_2(S)U + q_3(S)US + q_4(S)USU.$$

So if $\{e_n\}_{n=0}^{\infty}$ is the usual orthonormal basis for $H^2$, $< p(U,S)e_0, e_0 >=$ $< p(U,S)e_1, e_1 >$. As $A$ is the weak closure of polynomials in $U$ and $S$, anything in $A$ must have its first two diagonal entries equal. Therefore $R$ is not in $A$, as claimed. $\square$

Note that on a finite-dimensional space, all subnormal operators are normal and reductive. Therefore the weakly-closed algebra generated by any set of them is a von Neumann algebra, and hence reflexive.

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