COMPOSITION PRESERVES RIGIDITY

B. A. LOTTO AND J. E. M^C^CARTHY

30 June 1992

We work in the unit disk $\mathbf{D}$ in the complex plane or on its boundary $\partial\mathbf{D}$, the unit circle. For $1 \leq p \leq \infty$ we denote by L^p the Lebesgue space with respect to normalized Lebesgue measure on $\partial\mathbf{D}$. Any function in L^p has a harmonic extension into $\mathbf{D}$ given by the Poisson integral. The boundary value function on $\partial\mathbf{D}$ may be recovered from the harmonic function in $\mathbf{D}$, and we follow the custom of identifying the two functions. We are concerned primarily with the classical Hardy space H^p, which is the closed subspace of L^p consisting of those functions that are holomorphic in $\mathbf{D}$. Good references on these matters are [8], [3], and [5].

A function f in H^p is said to be *rigid* in H^p if, whenever g is in H^p and $\arg f(\zeta) = \arg g(\zeta)$ for almost every ζ in $\partial\mathbf{D}$, then $g = cf$ for some nonnegative constant c. (Here and below, the argument of a complex number is computed modulo 2π.) In the case $p = 1$, rigid functions come up in many problems and have been studied extensively. There are connections to extremal problems in H^∞, the geometry of the unit ball of H^1, kernels of Toeplitz operators, complete nondeterminacy of stationary stochastic processes, and the structure of de Branges-Rovnyak spaces. Some references are [1, 5, 6, 11, 12].

A basic problem is to find a good function theoretic description of rigid functions. Rigid functions are outer (Proof: if $f = uh$ where u is nonconstant and inner, then $\arg f = \arg(1+u)^2 h)$ but the converse is not true (Proof: if u is nonconstant and inner, then $(1+u)^2$ is outer and $\arg(1+u)^2 = \arg u$). The best characterization of rigid functions so far is due to Helson [7]. It states that an outer function in H^p is rigid if and only if $f/(u+v)^2 \notin H^p$ for all inner functions u and v that are not both constant. Unfortunately, a description of those functions that can be written as $u+v$ with u and v inner remains elusive. For more on this problem, see [10].

The goal of this paper is to prove the following theorem. The $p = 1$ case was proved by Younis [13]. His proof is indirect and relies on the operator theoretic characterization of rigidity [1]. Our proof is direct and function theoretic in nature.

Theorem. *Suppose that f is rigid in H^p and that u is nonconstant and inner. Then $f \circ u$ is rigid in H^p.*

Proof. We should start by clearing up a point of possible ambiguity. For the moment, let us distinguish between functions in L^p and their harmonic extensions by using lower case for the former and upper case for the latter. If u is an inner function, the inverse image of any m-null set is again m-null [4], so if f is defined m-a.e. on $\partial\mathbf{D}$, then $f \circ u$ is defined m-a.e. as well. A theorem of W. Rudin [9, Theorem 3.1] states that if f is in L^1, then the harmonic extension of $f \circ u$ equals $F \circ U$. Hence, composition with an inner function preserves the correspondence between boundary value functions and their harmonic extensions. We therefore now return to the convention of identifying the two.

Assume for the moment that the theorem holds under the additional hypothesis that $u(0) = 0$. Now suppose v is an arbitrary nonconstant inner function and that f is rigid in H^p. Let $u = \phi_{v(0)} \circ v$, where $\phi_\alpha(z) = (\alpha - z)/(1 - \bar{\alpha} z)$. Then u is inner and $u(0) = 0$. Since f is rigid and $\phi_{v(0)}$ is a homeomorphism of $\overline{\mathbf{D}}$ onto itself, it follows that $f \circ \phi_{v(0)}$ is rigid. Then $f \circ \phi_{v(0)} \circ u = f \circ v$ is rigid as well.

So we need only prove the theorem with the additional assumption $u(0) = 0$. Thinking of u as a function on $\partial\mathbf{D}$, we choose a Borel representative for u that equals 1 in modulus everywhere on $\partial\mathbf{D}$. Now u preserves normalized Lebesgue measure (which we denote by m); in particular, it preserves m-null sets. Hence, the composition operator $C_u\psi = \psi \circ u$ is well-defined and isometric on L^1 and hence for any L^p with $p < \infty$. It follows that C_u is isometric on L^∞ as well, since $\|\psi\|_\infty = \lim_{p\to\infty} \|\psi\|_p$. Following [2, p. 318] the adjoint operator C_u^* (which is called the expectation operator and denoted by E there) maps L^1 into itself; that is,

$$\int \psi \cdot (\rho \circ u)\, dm = \int C_u^* \psi \cdot \rho \, dm$$

for all ψ in L^1 and ρ in L^∞. It is easily seen that C_u^* maps L^p into itself for all p and that the above formula is valid for all ψ in L^p and ρ in $L^{p'}$ (where p', as usual, denotes the exponent conjugate to p). Furthermore, there exists a disintegration of m with respect to u. This is a collection $\{\mu_\zeta\colon \zeta \in \partial\mathbf{D}\}$ of probability measures on $\partial\mathbf{D}$ that possess the following three properties:

(1) μ_ζ is carried on $\{u = \zeta\}$ for m-a.e. ζ;
(2) For all ψ in L^1, $C_u^*\psi(\zeta) = \int \psi \, d\mu_\zeta$ for m-a.e. ζ; and
(3) For all ψ in L^1, $\int \psi \, dm = \iint \psi(\eta)\, d\mu_\zeta(\eta)\, dm(\zeta)$.

Now suppose that g is in H^p and $\arg g = \arg f \circ u$ almost everywhere on $\partial\mathbf{D}$. We need to show that $g = c(f \circ u)$ for some nonnegative constant c.

Let $h = C_u^* g$. Then h is in L^p and

$$\int h \cdot \zeta^n \, dm = \int C_u^* g \cdot \zeta^n \, dm = \int g \cdot u^n \, dm = g(0)u(0)^n = 0$$

for $n > 0$. It follows that h is in H^p. Now

$$h(\zeta) = \int g \, d\mu_\zeta$$

for m-a.e. ζ in $\mathbf{D}$. Fix any such ζ. Since $f \circ u$ identically equals $f(\zeta)$ on $\{u = \zeta\}$, we have that $\arg g = \arg f \circ u$ equals $\arg f(\zeta)$ on $\{u = \zeta\}$. Since μ_ζ is a probability measure carried on $\{u = \zeta\}$, it follows that $\arg h(\zeta) = \arg f(\zeta)$. The rigidity of f now implies that $h = cf$ for some nonnegative constant c.

Consider the function $g/(f \circ u)$. We have

$$\int \left| \frac{g}{f \circ u} \right| dm = \int \int \frac{|g(\eta)|}{|f \circ u(\eta)|} \, d\mu_\zeta(\eta)\, dm(\zeta).$$

Again, μ_ζ is carried on $\{u = \zeta\}$, so $f \circ u(\eta) = f(\zeta)$ μ_ζ-a.e. Hence the integral on the right equals

$$\int \frac{1}{|f(\zeta)|} \int |g(\eta)|\, d\mu_\zeta(\eta)\, dm(\zeta). \tag{1}$$

As above, for m-a.e. ζ, g has constant argument on $\{u = \zeta\}$, so

$$\int |g(\eta)|\, d\mu_\zeta(\eta) = \left| \int g(\eta)\, d\mu_\zeta(\eta) \right| = |h(\zeta)| = c|f(\zeta)|$$

m-a.e. Hence the integral (1) becomes $\int c\,dm = c$. In particular, $g/(f \circ u)$ is in L^1. But f is outer, so

$$\int \log |f \circ u|\,dm = \int \log |f|\,dm = \log |f(0)| = \log |f \circ u(0)|$$

and $f \circ u$ is outer as well. Hence $g/(f \circ u)$ belongs to the Smirnov class N^+, and, as it is integrable, it must belong to H^1. But $g/(f \circ u)$ is nonnegative, from which it follows that it must be a constant; in fact, it must identically equal c. Hence $g = c(f \circ u)$, as desired. □

Remark 1. The above proof shows that if g is in H^p, then $C_u^* g$ is in H^p. In other words, the restriction of the adjoint of the composition operator C_u on L^p is the adjoint of the composition operator C_u on H^p. Hence, disintegration gives a means of computing the adjoint of C_u on H^p when u is inner with $u(0) = 0$.

Remark 2. In the above situation, we can actually "compute" the disintegration measures μ_ζ. Suppose that ψ is in L^1. Then the harmonic extension of $C_u^* \psi$ into $\mathbf{D}$ is given by

$$C_u^* \psi(z) = \int C_u^* \psi \cdot P_z\,dm = \int \psi \cdot (P_z \circ u)\,dm$$

where $P_z(\eta) = (1 - |z|^2)/|\eta - z|^2$ is the Poisson kernel at z in $\mathbf{D}$.

Now $C_u^* \psi(r\zeta) \to C_u^* \psi(\zeta)$ as $r \uparrow 1$ for m-a.e. ζ in $\partial \mathbf{D}$, and $C_u^* \psi(\zeta) = \int \psi\,d\mu_\zeta$ for m-a.e. ζ in $\partial \mathbf{D}$. Hence $\int \psi \cdot (P_{r\zeta} \circ u)\,dm \to \int \psi\,d\mu_\zeta$ as $r \uparrow 1$ for m-a.e. ζ in $\partial \mathbf{D}$, where the exceptional set depends on ψ. In particular, this happens for all ψ in C, the space of continuous functions on $\partial \mathbf{D}$. Now choose a sequence $\psi_1, \psi_2, \ldots$ that is dense in C and let E_j be the exceptional set corresponding to ψ_j. Then $E = \cup_{j=1}^{\infty} E_j$ is m-null. We claim that $\int \psi \cdot (P_{r\zeta} \circ u)\,dm \to \int \psi\,d\mu_\zeta$ for all ψ in C and all ζ in $\partial \mathbf{D} \setminus E$. To see this, let $\epsilon > 0$ and choose ψ_j so that $\|\psi - \psi_j\|_C < \epsilon$. Then

$$\left| \int \psi\,d\mu_\zeta - \int \psi \cdot (P_{r\zeta} \circ u)\,dm \right| \leq \\ \left| \int (\psi - \psi_j)(\,d\mu_\zeta - (P_{r\zeta} \circ u)\,dm) \right| + \left| \int \psi_j(\,d\mu_\zeta - P_{r\zeta} \circ u\,dm) \right|.$$

The first term on the right is bounded by 2ϵ and the second tends to 0 as $r \uparrow 1$, so the desired convergence is established. Hence

$$d\mu_\zeta = w^*\text{-}\lim_{r\uparrow 1}(P_{r\zeta} \circ u)\, dm$$

for m-a.e. ζ in $\partial\mathbf{D}$. Once can deduce the properties of disintegration mentioned above directly from this formula.

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Department of Mathematics, University of California, Berkeley CA 94720
E-mail address: lotto@math.berkeley.edu

Department of Mathematics, Washington University, St. Louis MO 63130
E-mail address: mccarthy@jezebel.wustl.edu