The range of Toeplitz operators on the ball

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0 Introduction

Let B_d be the unit ball in $\mathbb{C}^d$, S_d be the boundary of B_d, and σ_d be normalized Lebesgue measure on S_d. The Hardy space $H^2(B_d)$ is the closure in $L^2(S_d, \sigma_d)$ of the analytic polynomials. The space $H^\infty(B_d)$ of bounded functions in $H^2(B_d)$ is precisely the space of functions that are radial limits (σ_d-almost everywhere) of bounded analytic functions on B_d. Let P denote the orthogonal projection from $L^2(S_d, \sigma_d)$ onto $H^2(B_d)$. If m is in $H^\infty(B_d)$, the co-analytic Toeplitz operator $T_{\bar{m}}^{H^2(B_d)}$ is defined by

$$T_{\bar{m}}^{H^2(B_d)} f = P\bar{m}f.$$

The purpose of this paper is to study the common range of all the co-anlytic Toeplitz operators $T_{\bar{m}}^{H^2(B_d)}$.

For the case $d = 1$, it was shown in [2] that a function f is in the range of every non-zero co-analytic Toeplitz operator $T_{\bar{m}}^{H^2(B_1)}$ if and only if the Taylor coefficients of f at zero satisfy

$$\hat{f}(n) = O(e^{-c\sqrt{n}})$$

for some $c > 0$. It was also shown that, for the case $d > 1$, if the Taylor coefficients of some f in $H^2(B_d)$ satisfy

$$\hat{f}(\alpha) = O(e^{-c|\alpha|^{d/(d+1)}})$$

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for some $c > 0$, then the function f will be in the range of every non-zero co-analytic Toeplitz operator on $H^2(B_d)$. It was asked if this sufficient condition were also necessary. Our main theorem answers this question in the negative:

Theorem 1 *Let $f(z_1, \ldots, z_d) = f_1(z_1) = \sum_{n=0}^{\infty} a_n z_1^n$, let $\varepsilon > 0$, and suppose that $a_n = O(e^{-cn^{\frac{1}{2}+\varepsilon}})$ for some $c > 0$. Then f is in the range of the Toeplitz operator $T_{\bar{m}}^{H^2(B_d)}$ for every non-zero m in $H^\infty(B_d)$.*

The exponent $n^{\frac{1}{2}+\varepsilon}$ is not optimal - using results of [3] it can be improved to $\sqrt{n}\log n$. We do not know what necessary and sufficient conditions are for a function to be in the range of every non-zero co-analytic Toeplitz operator.

In dimension $d = 1$, Szegö's theorem [9] states that a necessary and sufficient condition for a positive bounded function g on the circle to be the modulus of a non-zero function in $H^\infty(B_1)$ is

$$\int_{S_d} log(g) d\sigma_d > -\infty \tag{0.1}$$

For $d > 1$, condition (0.1) is necessary and sufficient for g to be the modulus of a function in the larger Nevanlinna class $N(B_d)$, consisting of those holomorphic functions f on the ball for which

$$T(f, 1) := \sup_{0<r<1} \int_{S_d} \log^+ |f(r\zeta)| d\sigma_d(\zeta) < \infty$$

[7, Theorem 10.11]. It is no longer sufficient, however, for g to be the modulus of a bounded analytic function, because the function

$$\zeta \mapsto \text{ess} \sup_{-\pi \leq \theta \leq \pi} |m(e^{i\theta}\zeta)|$$

must be lower semi-continuous on S_d if m is in $H^\infty(B_d)$ [7]. In [7, Theorem 12.5], Rudin proves that if g is log-integrable, and there exists some non-zero f in $H^\infty(B_d)$ with $g \geq |f|$ a.e. and $g/|f|$ lower semi-continuous, then there does exist m in $H^\infty(B_d)$ with $g = |m|$ a.e. We show

Theorem 2 *Let $d \geq 2$. There is a non-negative continuous function g on S_d, with $\int_{S_d} log(g) d\sigma_d > -\infty$, and which vanishes at only one point, but such that for no non-zero function m in $H^\infty(B_d)$ is $|m| \leq g$ almost everywhere $[\sigma_d]$.*

This answers question 15 of [7] in the negative.

When the original version of this paper was circulated in pre-print form (see the announcement in [1]), H. Alexander [private communication] produced a very simple constructive example of a function g satisfying the conclusion of Theorem 2, obviating the complicated construction in our proof. However, as we think our construction may be of some use in solving the problem of characterising exactly which functions are moduli of $H^\infty(B_d)$ functions, we elected to retain the proof of Theorem 2 in this paper.

1 Preliminary Lemmata

We need to know explicitly the projection from $L^2(B_d)$ onto $H^2(B_d)$. Let $\alpha = (\alpha_1, \ldots, \alpha_d)$ be a multi-index and $\zeta = (z_1, \ldots, z_d)$ a point in $\mathbb{C}^d$. The function ζ^α then maps ζ to $z_1^{\alpha_1} \ldots z_d^{\alpha_d}$. The notation $|\alpha|$ stands for $\alpha_1 + \ldots \alpha_d$, and $\alpha! = \alpha_1! \ldots \alpha_d!$.

Lemma 1.1

$$\int_{S_d} \zeta^\alpha \overline{\zeta^\beta} d\sigma_d = \delta_{\alpha,\beta} \frac{(d-1)!\,\alpha!}{(d-1+|\alpha|)!}. \tag{1.2}$$

Moreover, if $P_{H^2(B_d)}$ denotes the projection from $L^2(\sigma_d)$ onto $H^2(B_d)$, then

$$P_{H^2(B_d)} |z_2^{\alpha_2}|^2 \ldots |z_d^{\alpha_d}|^2 \overline{z_1^j} z_1^i = \begin{cases} 0 & \text{if } i < j \\ \frac{(d-1+i-j)!\, i!\, \alpha_2! \ldots \alpha_d!}{(i-j)!\,(d-1+i+\alpha_2+\ldots+\alpha_d)!} z_1^{i-j} & \text{if } i \geq j. \end{cases} \tag{1.3}$$

PROOF: Formula (1.2) is proved in [6]. The expression on the left-hand side of (1.3) is orthogonal to every monomial except z_1^{i-j}; taking inner products gives the constant. □

We need to consider co-analytic Toeplitz operators on different spaces. If μ is a compactly supported measure on $\mathbb{C}^d$, let $P^2(\mu)$ denote the closure of the polynomials in $L^2(\mu)$, and let $P_{P^2(\mu)}$ denote the orthogonal projection from $L^2(\mu)$ onto $P^2(\mu)$. If m is a bounded analytic function on the support of μ, the co-analytic Toeplitz operator $T_{\bar{m}}^{P^2(\mu)}$ is defined by

$$T_{\bar{m}}^{P^2(\mu)} f \;=\; P_{P^2(\mu)}\, \bar{m} f.$$

When μ is σ_d, the space $P^2(\mu)$ is the Hardy space $H^2(B_d)$, and we recover our original definition.

In order to transfer information about co-analytic Toeplitz operators with the same symbol on different spaces, we use the following lemma, whose proof is immediate:

Lemma 1.4 *Let $\mathcal{H}$ be a Hilbert space of holomorphic functions on B_d in which the monomials are mutually orthogonal. Let $m(z_1, \ldots, z_d) = \sum_{\beta \in \mathbb{N}^d} b_\beta \zeta^\beta$. Then*

$$T_{\bar{m}}^{\mathcal{H}} \frac{\zeta^\alpha}{\|\zeta^\alpha\|_{\mathcal{H}}^2} = \sum_{\beta \leq \alpha} \bar{b}_{\alpha-\beta} \frac{\zeta^\beta}{\|\zeta^\beta\|_{\mathcal{H}}^2} \tag{1.5}$$

This lemma also allows us to define Toeplitz operators with an unbounded conjugate analytic symbol. The formal definition (1.5) defines an upper triangular operator, with respect to the orthonormal basis of normalized monomials. It therefore has a domain which contains all the polynomials; we extend its domain to include all functions on which $T_{\bar{m}}$, thought of as a formal operator on the power series, gives a power series whose coefficients are the Taylor coefficients of some function in $\mathcal{H}$.

Lemma 1.6 *Let g be in the Nevanlinna class $N(B_1)$, with $g(0) \neq 0$, and $1 \leq \alpha < 2$. Then*

$$\int_{B_1} (\log^- |g|)^\alpha dA \leq K,$$

where K is some constant depending only on $T(g,1)$, $|g(0)|$ and α.

Proof: The proof is in two parts. First we prove it for g zero-free, then we prove it for g a Blaschke product.

(a) Suppose g has no zeroes in B_1, and without loss of generality assume $\|g\|_\infty < 1$. Then there is a singular measure μ_s such that, for any $0 < r < 1$,

$$\log^- |g(re^{i\theta})| = \frac{1}{2\pi} \int_0^{2\pi} P_{re^{i\theta}}(e^{i\phi})[\log^- |g(e^{i\phi})| d\phi + d\mu_s(\phi)]$$

where $P_{re^{i\theta}}(e^{i\phi})$ is the Poisson kernel. Therefore

$$\begin{aligned} &\int_{B_1} |\log^- |g(re^{i\theta})||^\alpha r dr d\theta \\ &= \int_0^1 r dr \int_0^{2\pi} d\theta \{ \int_0^{2\pi} P_{re^{i\theta}}(e^{i\phi})[\log^- |g(e^{i\phi})| d\phi + d\mu_s(\phi)] \}^\alpha \\ &\leq \{ \int_0^{2\pi} (\int_0^1 r dr \int_0^{2\pi} d\theta [P_{re^{i\theta}}(e^{i\phi})]^\alpha)^{\frac{1}{\alpha}} [\log^- |g(e^{i\phi})| d\phi + d\mu_s(\phi)] \}^\alpha \end{aligned}$$

by Minkowski's inequality. As the $L^\alpha(A)$ norm of the Poisson kernel for a fixed boundary point is at most $(\frac{8}{2-\alpha})^{\frac{1}{\alpha}}$, we get

$$\int_{B_1} (\log^- |g(re^{i\theta})|)^\alpha r dr d\theta \leq (\frac{8}{2-\alpha})(\frac{1}{\log |g(0)|})^\alpha.$$

(b) Suppose now g is a Blaschke product with zero-set $\{w_n\}$. Then

$$\begin{aligned}[\int_{B_1} |\log|g(z)||^\alpha dA(z)]^{\frac{1}{\alpha}} &= [\int_{B_1}(\sum_{n=0}^\infty \log|\tfrac{1-\bar{w}_n z}{z-w_n}|)^\alpha dA(z)]^{\frac{1}{\alpha}} \\ &\leq \sum_{n=0}^\infty [\int_{B_1}(\log|\tfrac{1-\bar{w}_n z}{z-w_n}|)^\alpha dA(z)]^{\frac{1}{\alpha}} \end{aligned} \tag{1.7}$$

Now let us estimate $\int_{B_1}(\log|\frac{1-\bar{w}_n z}{z-w_n}|)^\alpha dA(z)$.

The terms for $|w_n| \leq \frac{1}{2}$ are dominated by $T(g,1) + \log^- |g(0)|$, by Jensen's formula. For convenience, assume w is positive, and make the change of variables $\zeta = re^{i\theta} = \frac{z-w}{1-wz}$. Then

$$\begin{aligned}\int_{B_1}(\log|\frac{1-wz}{z-w}|)^\alpha dA(z) &= \frac{1}{\pi}\int_{B_1}(\log\frac{1}{r})^\alpha \frac{(1-w^2)^2}{|1-wre^{i\theta}|^4} r dr d\theta \\ &= 2(1-w^2)^2 \int_0^1 (\log\frac{1}{r})^\alpha \frac{1+(rw)^2}{(1-(rw)^2)^3} r dr \end{aligned} \tag{1.8}$$

Break the integral (1.8) into two pieces: from 0 to $\frac{1}{e}$, where the integrand is bounded by some constant C_1 independent of w, and from $\frac{1}{e}$ to 1. For the latter integral, use the inequality $\log\frac{1}{r} \leq 2(1-wr)$. One gets that (1.8) is bounded by $C_2 \frac{1}{2-\alpha}(1-w)^\alpha$, where C_2 depends on neither w nor α. So (1.7) is dominated by $(\frac{C_2}{2-\alpha})^{\frac{1}{\alpha}} \sum_{n=0}^\infty (1-|w_n|)$, and Jensen's formula again means we can dominate everything by a constant depending on α, $\log|g(0)|$ and $T(g,1)$. $\square$

Let A^{-n} consist of all holomorphic functions m in the unit disk that satisfy $|m(z)| = O((1-|z|)^{-n})$. The space A^0 is $H^\infty(B_1)$.

Lemma 1.9 *Let f be in A^{-n} for some n, and $0 < \alpha < 2$. Then*

$$\int_{B_1} (\log^- |f|)^\alpha dA < \infty.$$

Proof: We can assume that $f(0) \neq 0$. As f need not be in $N(B_1)$ we cannot apply Lemma 1.6 directly; but f is in the Nevanlinna class of certain smaller domains that touch the boundary of B_1 at only one point, and we shall average over these.

Fix p strictly between 1 and $\frac{2}{\alpha}$, let $a = \alpha p < 2$, let $q = \frac{p}{p-1}$ and let $N > q$. Let D_1 be a smoothly bounded convex domain inside the disk, containing $\{z : |z| < \frac{1}{2}\}$, whose closure touches the unit circle only at 1, and which has a high degree of tangency at 1: let the boundary of D_1 be $\{\rho(\theta)e^{i\theta} : -\pi \leq \theta \leq \pi\}$, and assume $1 - \rho(\theta) \sim |\theta|^N$. For any other point $\zeta = e^{i\theta_0}$ on the boundary of the unit disk, let $D_\zeta = e^{i\theta_0} D_1$.

Let ψ_ζ be the Riemann map of D_ζ onto B_1 that takes 0 to 0 and ζ to ζ. As the boundary of D_ζ is smooth, it follows from the Kellog-Warschawski theorem (see e.g. [5]) that ψ_ζ and its derivatives extend continuously to the closure of D_ζ, so distances before and after the conformal mapping are comparable.

If $r < \frac{1}{N}$, then f is in $H^r(D_\zeta)$, and $\sup_{\zeta \in S_1} \|f \circ \psi_\zeta^{-1}\|_{H^r} < \infty$. Thus

$$\int_{D_{e^{i\theta}}} (\log^- |f|)^a dA \leq C, \quad for\ all\ e^{i\theta}.$$

Integrating with respect to θ and changing the order of integration yields

$$\int_{B_1} (\log^- |f(re^{i\phi})|)^a (1-r)^{\frac{1}{N}} r dr d\phi < \infty.$$

Now

$$\begin{aligned} \int_{B_1} (\log^- |f|)^\alpha dA &\leq \left[\int_{B_1} (\log^- |f|)^{\alpha p} (1-r)^{\frac{p}{N}} dA\right]^{\frac{1}{p}} \left[\int_{B_1} (1-r)^{-\frac{q}{N}} dA\right]^{\frac{1}{q}} \\ &< \infty. \end{aligned}$$

□

Let μ_n be the measure on the unit disk given by $d\mu_n(z) = \frac{1}{\pi}(1-|z|^2)^n dA(z)$, and let $\mathcal{H}_n$ be $P^2(\mu_n)$. It is routine to verify that in $\mathcal{H}_n$ the monomials are mutually orthogonal, and

$$\|z^k\|^2_{\mathcal{H}_n} = \frac{n!}{(k+1)\dots(k+n+1)}.$$

The space $\mathcal{H}_0$ is the usual Bergman space for the disk. The following lemma is proved in [3] (in fact a slightly sharper form is proved). We include the following proof, which is sufficient for our purposes, for completeness:

Lemma 1.10 *Let $n \geq 0$, and m be a function in A^{-n}, not identically zero. Suppose $f(z) = \sum_{k=0}^\infty a_k z^k$ where $a_k = O(e^{-ck^{\frac{1}{2}+\varepsilon}})$ for some ε and c greater than 0. Then for any $s \geq 2n$ there exists g in $\mathcal{H}_s$ such that $T_{\bar{m}}^{\mathcal{H}_s} g = f$.*

Proof: First, observe that $f = T_{\bar{m}}^{\mathcal{H}_s} g$ for some g if and only if there is a constant C such that for all polynomials p

$$|\langle p, f\rangle_{\mathcal{H}_s}| \leq C\sqrt{\int |p|^2 |m|^2 d\mu_s}.$$

So it is sufficient to prove that

$$|\sum_{k=0}^{\infty} \bar{a}_k \hat{p}(k) \frac{1}{(k+1)\dots(k+s+1)}| \leq C\sqrt{\int |p|^2|m|^2 d\mu_s}.$$

This in turn will follow from the Banach-Steinhaus theorem if we can show that for any function h in $P^2(|m|^2\mu_s)$,

$$\hat{h}(k) = O(e^{ck^{\frac{1}{2}+\varepsilon}}). \tag{1.11}$$

Now Stoll showed in [8] that if h satisfies

$$\int_{B_1} (\log^+ |h|)^\alpha dA < \infty$$

for some $\alpha > 0$ then $\hat{h}(k) = O(e^{o(\frac{2}{2+\alpha})})$. We can assume ε is small, and take $\alpha = \frac{2-4\varepsilon}{1+2\varepsilon}$. As h is in $P^2(|m|^2\mu_s)$, $h(z)m(z)(1-|z|^2)^{s/2} = k(z)$ is in $L^2(dA)$, and

$$\log^+ |h| \leq \log^+ |k| + \log^- |(1-|z|^2)^{s/2}| + \log^- |m|.$$

The first two terms on the right are clearly integrable to the α^{th} power, and so is the third by Lemma 1.9; therefore h satisfies (1.11) as desired. □

We want to be able to restrict functions in the ball to planes and factor out zeros without losing control of the size of the function; the next lemma allows us to do this.

Lemma 1.12 *Let m be holomorphic on B_d and satisfy*

$$|m(z_1,\dots,z_d)| \leq C(1-\sqrt{|z_1|^2+\dots|z_d|^2})^{-s}.$$

Suppose also that

$$m(z_1,\dots,z_d) = z_d^t m_2(z_1,\dots,z_d) + z_d^{t+1} m_3(z_1,\dots,z_d),$$

where m_2 and m_3 are analytic. Let $m_1(z_1,\dots,z_{d-1}) = m_2(z_1,\dots,z_{d-1},0)$. Then

$$|m_1(z_1,\dots,z_{d-1})| \leq (3d)^{s+t} C(1-\sqrt{|z_1|^2+\dots|z_{d-1}|^2})^{-(s+t)}.$$

PROOF: Let $(z_1,\dots,z_{d-1})$ be in B_{d-1}, and let $\varepsilon = \frac{1}{3d}(1-\sqrt{|z_1|^2+\dots+|z_{d-1}|^2})$. Then the polydisk centered at $(z_1,\dots,z_{d-1},0)$ with multi-radius $(\varepsilon,\dots,\varepsilon)$ is contained in $(1-\varepsilon)B_d$.

Integrating on the distinguished boundary of the polydisk we get

$$\begin{aligned} |m_1(z_1,\ldots,z_{d-1})| = & \quad |m_2(z_1,\ldots,z_{d-1},0)| \\ = & \left| \int_{(z_1,\ldots,z_{d-1},0)+\varepsilon T^d} \frac{m(\zeta_1,\ldots,\zeta_d)}{\zeta_3^t} \right| \\ \leq & \quad \frac{C}{\varepsilon^{s+t}}. \end{aligned}$$

□

2 Common Range of $T_{\bar{m}}$

We can now prove that a function that depends on only one variable is in the range of every $T_{\bar{m}}^{H^2(B_d)}$ if its Taylor coefficients decay like $e^{-ck^{\frac{1}{2}+\varepsilon}}$.

Theorem 1 *Let $f(z_1,\ldots,z_d) = f_1(z_1) = \sum_{n=0}^{\infty} a_n z_1^n$, let $\varepsilon > 0$, and suppose that $a_n = O(e^{-cn^{\frac{1}{2}+\varepsilon}})$ for some $c > 0$. Then f is in the range of the Toeplitz operator $T_{\bar{m}}^{H^2(B_d)}$ for every non-zero m in $H^\infty(B_d)$.*

PROOF: For $d = 1$, this is proved (without the ε) in [2], so assume $d \geq 2$. Fix m in $H^\infty(B_d)$;

$$m(z_1,\ldots,z_d) = \sum_{i_1,\ldots,i_d=0}^{\infty} b_{i_1,\ldots,i_d} z_1^{i_1} \ldots z_d^{i_d}.$$

Let

$$S = \{(i_2,\ldots,i_d) : \text{for some } i_1,\ b_{i_1,\ldots,i_d} \neq 0\}.$$

Define

$$t_d = \inf\{i_d : \text{for some } i_2,\ldots,i_{d-1},\ (i_2,\ldots,i_{d-1},i_d) \in S\},$$

and define t_k inductively by

$$t_k = \inf\{i_k : \text{for some } i_2,\ldots,i_{k-1},\ (i_2,\ldots,i_{k-1},i_k,t_{k+1},\ldots,t_d) \in S\}.$$

Let $n = t_2 + \ldots + t_d$.

Case (a): $n = 0$.

Then the function

$$m_1(z_1) = m(z_1,0,\ldots,0)$$

is not identically zero, and is in $H^\infty(B_1)$. By Lemma 1.1,

$$T_{\bar{m}}^{H^2(B_d)} z_1^i = \sum_j \bar{b}_{j,0,\ldots,0} \frac{(i-j+1)\ldots(i-j+d-1)}{(i+1)\ldots(i+d-1)} z_1^{i-l}.$$

So by Lemma 1.4, if one can solve the equation

$$T_{\bar{m}_1}^{\mathcal{H}_{d-2}} g_1 = f_1 \tag{2.1}$$

for some g_1 in $\mathcal{H}_{d-2}$, then $g(z_1,\ldots,z_d) = g_1(z_1)$ solves

$$T_{\bar{m}}^{H^2(B_d)} g = f,$$

and by Equation (1.2) $\|g\|_{H^2(B_d)} = \sqrt{(d-1)!}\|g_1\|_{\mathcal{H}_{d-1}} < \infty$. By Lemma 1.10, equation (2.1) has a solution.

Case (b): $n > 0$.

One can decompose m as

$$m(z_1,\ldots,z_d) = z_2^{t_2}\ldots z_d^{t_d} m_2(z_1,\ldots,z_d) + m_3(z_1,\ldots,z_d),$$

where each term in the expansion of m_3 is divisible by some $z_k^{t_k+1}$. Applying Lemma 1.12 inductively, $m_1(z) = m_2(z,0,\ldots,0)$ is in A^{-n}, and by the choice of $t_2,\ldots,t_d$, it is not identically zero. Consider the function

$$f_2(z) = \sum_{k=0}^{\infty} a_k (k+d)(k+d+1)\ldots(k+dn+1) z^k.$$

As $d \geq 2$, we can apply Lemma 1.10 with $s = dn$, so there is

$$g_2(z) = \sum_{k=0}^{\infty} \gamma_k (k+1)(k+2)\ldots(k+dn+1) z^k$$

in $\mathcal{H}_{dn}$ with

$$T_{\bar{m}_1}^{\mathcal{H}_{dn}} g_2 = f_2. \tag{2.2}$$

Define g by

$$g(z_1,\ldots,z_d) = \frac{1}{t_2!\ldots t_d!} z_2^{t_2}\ldots z_d^{t_d} \sum_{k=0}^{\infty} \gamma_k (k+1)(k+2)\ldots(k+n+d-1) z_1^k.$$

The function g is in $H^2(B_d)$ because

$$\begin{aligned}
\|g\|^2_{H^2(B_d)} &= \frac{(d-1)!}{t_2! \ldots t_d!} \sum_{k=0}^{\infty} |\gamma_k|^2 (k+1)\ldots(k+n+d-1) \\
&\leq \frac{(d-1)!}{t_2! \ldots t_d!} \sum_{k=0}^{\infty} |\gamma_k|^2 (k+1)\ldots(k+dn+1) \\
&= \frac{(d-1)!}{t_2! \ldots t_d!} \|g_2\|^2_{\mathcal{H}_{(d-1)n}} \\
&< \infty.
\end{aligned}$$

Moreover

$$T^{H^2(B_d)}_{\bar{m}} g = T^{H^2(B_d)}_{\bar{z}_2^{t_d} \ldots \bar{z}_d^{t_d} \overline{m_1(z_1)}} g$$

is a function of z_1 only; it is, in fact, f. For if $T^{H^2(B_d)}_{\bar{m}} g = \sum_{k=0}^{\infty} e_k z_1^k$, and $m_1(z) = \sum_{k=0}^{\infty} c_k z^k$, then taking the inner product with z_1^j we get

$$\begin{aligned}
\frac{(d-1)!}{(j+1)\ldots(j+d-1)} e_j &= \langle T^{H^2(B_d)}_{\bar{m}} g, z_1^j \rangle_{H^2(B_d)} \\
&= \langle g, z_2^{t_2} \ldots z_d^{t_d} m_1 z_1^j \rangle_{H^2(B_d)} \\
&= (d-1)! \textstyle\sum_{k=j}^{\infty} \gamma_k \bar{c}_{k-j} \qquad (2.3)
\end{aligned}$$

Taking the inner product with z^j in Equation(2.2), we get

$$\begin{aligned}
\frac{1}{(j+1)\ldots(j+d-1)} a_j &= \langle T^{\mathcal{H}_{dn}}_{\bar{m}_1} g_2, z^j \rangle_{\mathcal{H}_{dn}} \\
&= \langle g_2, m_1 z^j \rangle_{\mathcal{H}_{dn}} \\
&= \textstyle\sum_{k=j}^{\infty} \gamma_k \bar{c}_{k-j} \qquad (2.4)
\end{aligned}$$

Comparing Equations(2.3) and (2.4), we see that $T^{H^2(B_d)}_{\bar{m}} g = f$, as desired. □

3 Boundary moduli

Define $F_{c,w}$ by

$$F_{c,w}(z) = \exp\left(c \frac{1-|w|^2}{(1-\langle z, w\rangle)^{d+1}}\right) \qquad (3.1)$$

We need the following two results. The first was proved by Drewnowski; a proof is given in [4, Lemma 3.2].

Lemma 3.2 (Drewnowski)

$$\lim_{c\to 0} \sup_{w\in B_d} \int_{S_d} \log(1+|cF_{c,w}|)d\sigma_d = 0.$$

The second result, due to Nawrocki, estimates the growth of the Taylor coefficients of $F_{c,w}$. We are interested in $w = re_1 = (r, 0, \ldots, 0)$; in this case all the Taylor coefficients of F_{c,re_1} are positive, and the following follows easily from the proof of [4, Lemma 3.3]:

Lemma 3.3 (Nawrocki) *For each $c > 0$ there exists $\varepsilon > 0$ such that*

$$\inf_{i\in\mathbb{N}} \sup_{0<r<1} \sqrt{\frac{(d-1+i)!}{(d-1)!\,i!}} \hat{F}_{c,re_1}(i, 0, \ldots, 0) e^{-\varepsilon i^{\frac{d}{d+1}}} > 0.$$

We can now use our knowledge of the common range of co-analytic Toeplitz operators to prove:

Theorem 2 *Let $d \geq 2$. There is a continuous non-negative function g on S_d, vanishing only at the point e_1, and satisfying $\int_{S_d} log(g)d\sigma_d > -\infty$, with the property that the only function m in $H^\infty(B_d)$ with $|m| \leq g$ almost everywhere $[\sigma_d]$ is the zero function.*

Proof: Let $V_n = \{\zeta \in S_d : |\zeta - e_1| \geq \frac{1}{n}\}$. By Lemma 3.3, for any sequence c_n tending to zero, one can choose i_n and r_n such that

$$\hat{F}_{c_n,r_ne_1}(i_n, 0, \ldots, 0) > \frac{n}{c_n} e^{(i_n)^{\frac{4}{7}}} \tag{3.4}$$

(because $\frac{4}{7} < \frac{d}{d+1}$). Moreover, by passing to a subsequence, one can assume that

$$\sup_{\zeta\in V_n} c_n|F_{c_n,r_ne_1}(\zeta)| \leq \frac{1}{2^n}, \tag{3.5}$$

because $\zeta \in V_n$ implies that $|1 - \langle\zeta, r_ne_1\rangle| \geq \frac{1}{2n^2}$, and that

$$\int_{S_d} \log(1 + |c_nF_{c_n,r_ne_1}|)d\sigma_d \leq \frac{1}{2^n},$$

by Lemma 3.2. Define g by

$$g(\zeta) = \sqrt{\frac{1}{1 + \sum_{n=1}^\infty |c_nF_{c_n,r_ne_1}(\zeta)|^2}}.$$

It follows from (3.5) that g is continuous and vanishes only at e_1. Moreover

$$\begin{aligned}\int_{S_d} \log g d\sigma_d = & -\tfrac{1}{2}\textstyle\int_{S_d} \log(1+\sum_{n=1}^{\infty}|c_n F_{c_n,r_n e_1}|^2) d\sigma_d \\ > & -\textstyle\int_{S_d} \log\prod_{n=1}^{\infty}(1+|c_n F_{c_n,r_n e_1}|)^2 d\sigma_d \\ = & -2\textstyle\sum_{n=1}^{\infty}\int_{S_d}\log(1+|c_n F_{c_n,r_n e_1}|) d\sigma_d \\ \geq & \qquad\qquad -2.\end{aligned}$$

Now suppose there is a non-zero m in $H^\infty(B_d)$ with $|m| \leq g$ a.e. Then each of the functions $c_n F_{c_n,r_n e_1}$, being analytic in the ball of radius $\frac{1}{r_n}$, is in $P^2(|m|^2\sigma)$; moreover they are all of norm less than one in this space, because

$$\int_{S_d} |c_n F_{c_n,r_n e_1}|^2 |m|^2 d\sigma \leq \int_{S_d} |c_n F_{c_n,r_n e_1}|^2 g^2 d\sigma < 1.$$

Let

$$f(z_1,\ldots,z_d) = \sum_{k=0}^{\infty} e^{-k^{\frac{4}{7}}} \frac{(k+d-1)!}{(d-1)!k!} z_1^k.$$

By Theorem 1, there is a function h in $H^2(B_d)$ with

$$T_{\bar{m}}^{H^2(B_d)} h = f.$$

It follows that the linear map

$$\Gamma : p \mapsto \langle p, f\rangle_{H^2(B_d)},$$

defined a priori on the polynomials, extends by continuity to a bounded linear map on $P^2(|m|^2\sigma)$, as

$$|\Gamma(p)| = |\langle p, P(\bar{m}h)\rangle| = |\int pm\bar{h} d\sigma_d| \leq \|h\|_{H^2(B_d)} \|p\|_{P^2(|m|^2\sigma)}.$$

Moreover, each function $c_n F_{c_n,r_n e_1}$ is uniformly approximated on S_d by the partial sums of its Taylor series; hence

$$\Gamma(c_n F_{c_n,r_n e_1}) = \sum_{k=0}^{\infty} c_n \hat{F}_{c_n,r_n e_1}(k) e^{-k^{\frac{4}{7}}}. \tag{3.6}$$

But all the terms on the right-hand side of (3.6) are positive, and the i_n^{th} term is at least n by Equation (3.4). This contradicts the boundedness of Γ. □

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