TOPOLOGIES ON THE SMIRNOV CLASS

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Abstract. The Smirnov Class N^+ of analytic functions on the disk has traditionally been studied with the complete metric topology given by

$$\rho(f,g) = \sup_{0<r<1} \frac{1}{2\pi} \int_{\mathbb{T}} log(1 + |f(re^{i\theta}) - g(re^{i\theta})|)d\theta.$$

However, the space can be represented as the union of certain weighted Hardy spaces $H^2(w)$ (the closure of the polynomials in $L^2(wd\theta)$), and hence given the inductive limit topology H.

We give a new proof of Yanagihara's characterisation of the dual of (N^+, ρ). We prove that (N^+, H) is an (incomplete) metrizable topological algebra in which Fourier series converge. We then study the individual spaces $H^2(w)$, prove asymptotic versions of Szegö's theorem and the Helson-Szegö theorem on these spaces, and characterise the universal multipliers of their duals.

0. Introduction

The *Smirnov class* N^+ consists of those functions f analytic on the unit disk $\mathbb{D}$ for which

$$\lim_{r\to 1} \frac{1}{2\pi} \int_{\mathbb{T}} log(1 + |f(re^{i\theta})|)d\theta = \frac{1}{2\pi} \int_{\mathbb{T}} log(1 + |f(e^{i\theta})|)d\theta < \infty.$$

(The boundedness of the integrals on the left implies that f is in the Nevanlinna class, and so has non-tangential limits almost everywhere on the circle. It is this boundary function that we mean in the second integral.) We shall define $\|f\|$ by

$$\|f\| = \sup_{0<r<1} \frac{1}{2\pi} \int_{\mathbb{T}} log(1 + |f(re^{i\theta})|)d\theta.$$

This is not a norm, but it does define a metric ($\rho(f, g) := \|f - g\|$), that makes N^+ into a topological vector space. For basic facts about N^+, see *e.g* [4], [7] or [18].

In [22], N. Yanagihara found asymptotic bounds on the growth of the Taylor coefficients of a function in N^+, and in [23], he used these to characterise the multipliers of N^+ into H^∞, and hence to find the dual of (N^+, ρ). The first two results have recently been generalised by M. Nawrocki [16] to Banach space valued functions. We give simple proofs of Yanagihara's theorems in Section 1.

The Nevanlinna brothers proved that a function is in N^+ if and only if it can be expressed as the ratio g/h, where g and h are in H^2, the classical Hardy space, and h is outer ([17], or see [4]). But

$$\begin{aligned} f = \frac{g}{h} &\iff hf \in H^2 \\ &\iff \inf\{\int_{\mathbb{T}} |hf - p|^2 d\theta : p \quad a \quad polynomial\} = 0 \\ &\iff \inf\{\int_{\mathbb{T}} |hf - hp|^2 d\theta : p \quad a \quad polynomial\} = 0 \end{aligned}$$

(since h is outer, so $\{hp\}$ is dense in H^2)

$$\begin{aligned} &\iff \inf\{\int_{\mathbb{T}} |f - p|^2 |h|^2 d\theta : p \quad a \quad polynomial\} = 0 \\ &\iff f \in H^2(|h|^2), \end{aligned}$$

where $H^2(|h|^2)$ denotes the closure of the polynomials in $L^2(|h|^2 d\theta)$. Thus

$$N^+ = \bigcup_{h \quad outer} H^2(|h|^2).$$

There are two topologies associated with this representation of N^+: the usual locally convex inductive limit topology, which we shall call the Helson topology and denote $\mathcal{H}$, in which a neighbourhood base for 0 is given by those balanced convex sets whose intersection with each $H^2(|h|^2)$ is a neighbourhood of zero in $H^2(|h|^2)$; and a not locally convex topology, which we shall denote $\mathcal{I}$, in which a neighbourhood base for zero is given by all sets whose intersection with each $H^2(|h|^2)$ is a neighbourhood of zero. It is not immediately obvious that $\mathcal{I}$ is strictly stronger than $\mathcal{H}$; but in [15], it was proved that $\mathcal{I}$ coincides with the metric topology ρ, which is not locally convex.

This sheds a lot of light on $\mathcal{H}$; for a start, as $\mathcal{H}$ is the finest locally convex topology coarser than $\mathcal{I}$, $(N^+, \mathcal{H})$ and $(N^+, \mathcal{I})$ must have the same duals. It is a theorem of Helson that the dual of $(N^+, \mathcal{H})$ can be identified with those functions that are in the range of every Toeplitz operator with co-analytic symbol [8]; so, combined with Yanagihara's characterisation of $(N^+, \rho)^*$ (Theorem 1.7 below), one gets that the common range of these Toeplitz operators are those functions whose n^{th} Fourier coefficients are $O(e^{-c\sqrt{n}})$, for some positive c; see [15].

As $\mathcal{H}$ is Hausdorff, and $\mathcal{I}$ is metrizable, $\mathcal{H}$ too must be metrizable; in fact the metric can be given explicitly, as Yanagihara did in [24]:

$$d(f(z) \sim \sum_{n=0}^{\infty} a_n z^n, g(z) \sim \sum_{n=0}^{\infty} b_n z^n) = \sum_{k=1}^{\infty} 2^{-k} \frac{\sum_{n=0}^{\infty} |a_n - b_n| e^{-\sqrt{n/k}}}{1 + \sum_{n=0}^{\infty} |a_n - b_n| e^{-\sqrt{n/k}}}. \tag{0.1}$$

We discuss this in Section 2. Multiplication with respect to this metric is continuous [25], so it also follows that $(N^+, \mathcal{H})$ is a topological algebra. In [9] Helson showed that multiplication is separately continuous, directly from the definition of $\mathcal{H}$, but proving joint continuity seems to require passing to the more tractable metric topology d in (0.1).

In Section 2 we also show that Fourier series converge in $\mathcal{H}$ (strictly speaking, we should talk about power series, because the functions are not necessarily summable on the unit circle; but as the series for an H^2 function is always called a Fourier series, we shall use this terminology). Observe that for any weight which is not a Helson-Szegö weight, for example $|e^{i\theta} - 1|^2 d\theta$, Fourier series do not always converge in $H^2(w)$ [**10**]. Indeed the Fourier series of $1/(1-z)$ does not converge in any $H^2(w)$, nor in $(N^+, \mathcal{I})$. That the imposition of local convexity on the inductive limit makes the divergences disappear seems to reflect, in some vague sense, the symmetry of divergent Fourier series.

In Section 3 we consider the individual spaces $H^2(w)$, the closure of the (analytic) polynomials in $L^2(wd\theta)$, where w is a positive integrable function. Letting $\mathcal{W}$ be the set of weights w for which $\int_{\mathbb{T}} log(w(e^{i\theta}))d\theta > -\infty$, it is a famous result of Szegö that evaluating a polynomial at zero extends to be a continuous linear functional on $H^2(w)$ if and only if w is in $\mathcal{W}$. If this holds, it is easy to show that evaluating the n^{th} Taylor coefficient at zero is also a continuous functional, and that the norms of these functionals grow more slowly than e^{cn}, for any positive c. We prove that the bound can actually be replaced by $e^{c\sqrt{n}}$, and that this cannot be improved for general w (Theorem 3.1). We also prove an asymptotic Helson-Szegö theorem, on the angle between the subspace (in

$L^2(wd\theta))$ of analytic polynomials of degree less than n, and the space of all co-analytic polynomials with no constant term - namely that the sine of this angle must decrease more slowly than $e^{-c\sqrt{n}}$ for every positive c.

The principal result of Section 3 is the proof that the set of universal multipliers of the duals of all the $H^2(w)$'s, as w ranges over $\mathcal{W}$, is again the set of holomorphic functions whose Fourier coefficients decay like $e^{-c\sqrt{n}}$. Let us briefly explain its significance.

Let b be a function in the unit ball of H^∞, the space of bounded analytic functions in the ball. The *de Branges space* $\mathcal{H}(b)$ is defined to be the range of the operator $(1-T_bT_{\bar{b}})^{1/2}$. These spaces, originally introduced in [1], have been the object of much study recently. One question of particular interest is to determine, for fixed b, what functions multiply $\mathcal{H}(b)$ into itself - see *e.g.* [14], [20]. The question splits into two cases, depending on whether b is an extreme point of the unit ball of H^∞ (and no non-constant polynomial is a multiplier), or it isn't (and every polynomial is a multiplier) [20]. In [2], it is shown that a function m is a multiplier of every $\mathcal{H}(b)$, for b not an extreme point of the unit ball of H^∞, if and only if m is a multiplier of $H^2(w)^*$ for every w in $\mathcal{W}$; and hence Theorem 3.2 shows that these hyper-smooth functions surface again in a different context.

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1. The dual of (N^+, ρ)

We shall need three lemmata, the proofs of which are elementary, and can be found in [18, II.3.1 and II.11.2].

Lemma 1.1. *If f is in N^+ and z is in $\mathbb{D}$, then*

$$|f(z)| \leq e^{\frac{2\|f\|}{1-|z|}} - 1.$$

Lemma 1.2. *Let f be in N^+. Then its Taylor coefficients satisfy*

$$|\hat{f}(n)| \leq e^{\sqrt{8n\|f\|}(1+\epsilon_n)},$$

where ϵ_n is $o(1)$, depends only on $\|f\|$, and decreases as $\|f\|$ decreases.

Lemma 1.3. *Let c be positive, and define $f_c(z) = exp(\frac{c}{8}\frac{1+z}{1-z}) - 1$. Then $\hat{f}_c(n) = e^{c\sqrt{n}(1+o(1))}$.*

We first prove Yanagihara's theorem on the growth of Taylor coefficients [22].

Theorem 1.4. *Let f be in N^+, and let $c > 0$. Then $\hat{f}(n) = O(e^{c\sqrt{n}})$.*

Proof. By the monotone convergence theorem, $\lim_{t\to 0} \|tf\| = 0$. So choose t so that $\sqrt{8\|tf\|} \leq \frac{c}{2}$, and apply Lemma 1.2. □

Before giving our proof of Yanagihara's theorem on the dual of (N^+, ρ), we first need one more lemma, about the size of the functions f_c from Lemma 1.3.

Lemma 1.5. $\lim_{c\downarrow 0} \|f_c\| = 0$.

Proof. Let $\epsilon > 0$, and let $\delta > 0$ be such that if $|w| < \delta$, then $|e^w - 1| < \epsilon$. Let $B_c = \{z \in \mathbb{D} : |\frac{c}{8}\frac{1+z}{1-z}| \geq \delta\}$.

For any $0 < r < 1$, the integral

$$\frac{1}{2\pi}\int_{\mathbb{T}} log(1 + |f_c(re^{i\theta})|)d\theta = \frac{1}{2\pi}\int_{\mathbb{T}} log(1 + |exp\frac{c}{8}\frac{1+re^{i\theta}}{1-re^{i\theta}} - 1|)d\theta \tag{1.6}$$

decomposes into the integral over the arc where $re^{i\theta}$ is not in B_c, which can contribute at most $log(1+\epsilon)$, and the complementary arc, $I(c, r)$.

But the integrand is always less than or equal to $2 + \Re\frac{c}{8}\frac{1+re^{i\theta}}{1-re^{i\theta}}$, so the contribution from $I(c, r)$ is at most $\frac{1}{\pi}|I(c, r)| + \frac{c}{8}$. As c decreases to zero, $|I(c, r)|$ gets small uniformly in r, so c can be chosen to make all the integrals in (1.6), for all $0 < r < 1$, arbitrarily small. □

To characterise the dual of (N^+, ρ), notice that, because $f(rz)$ tends to $f(z)$ as r increases to 1, the polynomials are dense in N^+, and so any continuous linear functional is uniquely defined by knowing its action on the monomials z^n. We shall let A denote the disk algebra, those functions analytic on $\mathbb{D}$ and continuous on the closure.

Theorem 1.7. *Let Γ be a linear functional, defined on polynomials by $\Gamma(\sum_n a_n z^n) = \sum_n a_n \gamma_n$. Then Γ extends to be continuous on (N^+, ρ) if and only if, for some $c > 0$, $\gamma_n = O(e^{-c\sqrt{n}})$.*

Proof. (Sufficiency) By Theorem 1.4, $\Gamma(f(z) \sim \sum_{n=0}^{\infty} a_n z^n) := \sum_{n=0}^{\infty} a_n \gamma_n$ defines a linear functional on all of N^+. Moreover it is continuous, because if f_k tends to zero, and $\epsilon > 0$ is given, choose N large enough that $\sum_{n=N}^{\infty} e^{-c\sqrt{n}/2} < \frac{\epsilon'}{2}$, where $\epsilon' = \epsilon / \sup_n \{\gamma_n e^{c\sqrt{n}}\}$; and choose K large enough that, for $k \geq K$, if $n \geq N$ then $|\hat{f}_k(n)| \leq e^{c\sqrt{n}/2}$ (can do this by Lemma 1.2), and if $n < N$ then $|\hat{f}_k(n)| \leq \frac{\epsilon'}{2N}$ (can do this by Lemma 1.1). Then $|\Gamma(f_k)| \leq \epsilon$, for $k \geq K$.

(Necessity) Define a map T from polynomials into A by $T(\sum_{n=0}^{\infty} a_n z^n) = \sum_{n=0}^{\infty} a_n \gamma_n z^n$. The norm of $T(p(z))$ in the disk algebra is

$$\|T(p(z))\|_A = \sup_{|\lambda|=1} |\Gamma(p(\lambda z))|.$$

As $\|.\|$ is rotation invariant, and Γ is continuous, T is a continuous map from N^+ into A; in particular, there is some $d > 0$ such that the ball of radius d in N^+ gets mapped into the ball of radius 1 in A. Therefore we can conclude that if f is in N^+, and $\|f\| < d$, then $|\hat{f}(n)\gamma_n| \leq 1$.

Now, consider the functions f_c of Lemma 1.3. These functions are not in N^+, but $f_c(rz)$ is in N^+ for all $0 \leq r < 1$, and, by Lemma 1.5, c can be chosen so that $\|f_c(rz)\| < d$ for all r in this range. Letting $r = 2^{-1/n}$, we get

$$|\gamma_n| \leq \frac{1}{r^n} e^{-c\sqrt{n}(1+o(1))} = O(e^{-c\sqrt{n}}),$$

as required. □

It is easy to show that (N^+, ρ) is actually complete (see *e.g.* [6, p.123] or [23]). This makes the proof of sufficiency above even easier - just apply the uniform boundedness principle [3, p.52] to the partial sums. The uniform boundedness principle also proves that the bound in Theorem 4 cannot be replaced by $O(e^{c_n\sqrt{n}})$ for any sequence c_n decreasing to zero, for otherwise $\Gamma(z^n) = \frac{1}{(n+1)^2} e^{-c_n\sqrt{n}}$ would extend to a continuous linear functional that doesn't satisfy Theorem 5. Finally, the same ideas yield that if a sequence $\{\gamma_n\}_{n=0}^{\infty}$ multiplies N^+ into H^p for some $p > 0$ (*i.e.* $\sum_{n=0}^{\infty} a_n z^n$ in N^+ implies $\sum_{n=0}^{\infty} a_n \gamma_n z^n$ is in H^p), then, because Taylor coefficients of functions in H^p can grow only polynomially fast [4, p.98], $\{\gamma_n\}_{n=0}^{\infty}$ actually multiplies N^+ into A^∞, and must decay like $e^{-c\sqrt{n}}$.

2. Convergence of Fourier series in $\mathcal{H}$

In [24] Yanagihara studied the space F^+, which consists of those functions $f(z) \sim \sum_{k=0}^{\infty} a_k z^k$ for which $a_k = O(exp(o\sqrt{k}))$, topologised by the family of norms

$$\|f\|_c = \sum_{k=0}^{\infty} |a_k| e^{-c\sqrt{k}},$$

where c is a positive constant. This can be made into a metric topology by putting

$$d(f,g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|f-g\|_{1/\sqrt{n}}}{1+\|f-g\|_{1/\sqrt{n}}}.$$

With respect to this topology F^+ is a complete, locally convex metric space, *i.e.* a Fréchet space. He showed that F^+ is actually the containing Fréchet space of (N^+, ρ), *i.e.* (N^+, ρ) embeds densely into F^+, under the natural inclusion, and $(F^+, d)^* = (N^+, \rho)^*$.

As d is weaker than ρ, $(N^+, d)^* = (N^+, \rho)^*$. So $\mathcal{H}$ and d are both locally convex topologies on N^+ that give rise to the same dual. Now, on any space there is a strongest locally convex topology with a given dual, called the Mackey topology for that dual pair. Certain conditions on a topology force it to coincide with the Mackey topology; in particular, being barrelled, as $\mathcal{H}$ is, or being metrizable, as d is, both make the topologies coincide with the Mackey topology [21, 10-1.9],so $\mathcal{H}$ and d must be the same. In particular, the Helson topology must be metrizable, and so have a countable neighborhood base, and d must be barrelled.

(Recall that a locally convex topological vector space is called *barrelled* if every closed, absolutely convex, absorbing set is a neighborhood of zero. Every Banach space is barrelled, and the inductive limit of barrelled spaces is barrelled, so $(N^+, \mathcal{H})$ is barrelled. The importance of being barrelled is that every pointwise bounded family of continuous linear functionals is equicontinuous. See *e.g* [21, 9-3.4] or [12, p. 141].)

We shall say that a topology on N^+ is *Szegö* if the functionals $g \mapsto \hat{g}(k)$ are continuous (equivalently, the forward and backward shifts are continuous). All natural topologies on N^+ are Szegö.

Theorem 2.1. *Let τ be a barrelled Szegö topology on N^+. Then the following are equivalent:*

(i) $\tau = H$

(ii) $\tau = d$

(iii) *Fourier series converge in (N^+, τ)*

Proof. $(ii) \iff (i)$ follows from the above remarks.

$(i) \implies (iii)$: Let $g \sim \sum_{k=0}^{\infty} a_k z^k$ be in N^+. Set

$$R_N = N \sum_{k=N+1}^{\infty} a_k z^k.$$

Let Φ be in $(N^+, H)^*$. By [15], $\Phi(z^k) = O(e^{-c\sqrt{k}})$, for some $c > 0$. Therefore by Theorem 1.4 (applied to $c/2$) the set

$$\{\Phi(R_N)\}_{N=0}^{\infty}$$

is bounded, so the set $\{R_N\}$ is weakly bounded. Because (N^+, H) is locally convex, the set $\{R_N\}$ must therefore be H-bounded [**19**,p. 68]. So $\frac{1}{N} R_N$ must tend to zero (H), so

$$\sum_{k=0}^{k=N} a_k z^k \to g.$$

$(iii) \implies (i)$: Suppose Φ is in $(N^+, \tau)^*$, $\Phi(z^k) = \alpha_k$. If $g \sim \sum_{k=0}^{\infty} a_k z^k$ is in N^+,

$$\begin{aligned} \Phi(g) &= \Phi(\lim_{N\to\infty} \sum_{k=0}^{k=N} a_k z^k) \\ &= \lim_{N\to\infty} \sum_{k=0}^{k=N} \alpha_k a_k. \end{aligned}$$

Thus $\sum_{k=0}^{\infty} \alpha_k a_k$ converges for all $g \sim \sum_{k=0}^{\infty} a_k z^k$ in N^+; so by the Banach-Steinhaus theorem [21, 9-3.7], Φ is in $(N^+, H)^*$.

Conversely, another application of the Banach-Steinhaus theorem (using that τ is barrelled and Szegö) gives that $(N^+, \tau)^*$ actually equals $(N^+, H)^*$. Since both topologies are barrelled, hence Mackey, they must be the same. $\square$

We remark that various other conditions can also be proved to be equivalent; for example, requiring that τ be metrizable and that the completion of (N^+, τ) be contained in the space of functions analytic on the open unit disk.

3. Weighted Hardy Spaces

It is well-known that a weight w is in $\mathcal{W}$ if and only if there is an outer function h in H^2 with $|h|^2 = w$ almost everywhere - see *e.g* [4] or [7]. Thus $(N^+, \mathcal{H})^*$ consists of those functionals in $H^2(w)^*$ for every w in $\mathcal{W}$. If w is in $\mathcal{W}$, any sequence of polynomials that converges in $H^2(w)$ must converge almost uniformly on $\mathbb{D}$, and hence converges to an analytic function on the disk (whose radial limits agree almost everywhere with the limit in $H^2(w)$). Thus evaluating Fourier coefficients extends continuously to $H^2(w)$; knowing how rapidly the Fourier coefficients of a function in N^+ can grow enables us to put a bound on the norms of these functionals. We believe that this "asymptotic Szegö theorem" is important, but it does not appear to have been noticed previously. Note that the bounds are for general w: for a specific w (*e.g.* $w = 1$), obviously the bounds can be improved.

Theorem 3.1. *Let w be in $\mathcal{W}$, and $c > 0$. Then there exists a constant C_1 such that*

$$\sup_{\int_{\mathbb{T}} |p|^2 w \, d\theta \leq 1} |\hat{p}(n)| \leq C_1 e^{c\sqrt{n}}.$$

Moreover, $e^{c\sqrt{n}}$ cannot be replaced by $e^{c_n\sqrt{n}}$ for any sequence c_n converging to zero.

Proof. Let f_n be the functional in $(N^+, \mathcal{H})^*$ with $\hat{f}_n(k) = \delta_{nk} e^{-c\sqrt{n}}$ (δ_{nk} is the Kronecker delta). These functionals are pointwise bounded, so they constitute an equicontinuous family, *i.e.* there is an open neighborhood of zero U in $(N^+, \mathcal{H})$ such that $| < g, f_n > |$ is less than one for all g in U and all f_n. For any w in $\mathcal{W}$, $U \cap H^2(w)$ contains some open ball around zero, of radius ϵ say. Putting $C_1 = \frac{1}{\epsilon}$, we get that

$$\sup_{\int_{\mathbb{T}} |p|^2 w \, d\theta \leq 1} |\hat{p}(n)| \leq C_1 e^{c\sqrt{n}}.$$

as desired.

That these bounds are best possible follows immediately from the existence, for each sequence c_n converging to zero, of a function f in some $H^2(w)$ for which $\hat{f}(n) \neq O(e^{c_n\sqrt{n}})$. $\square$

For any element f in the dual of $H^2(w)$ we will write $f = \sum_{k=0}^{\infty} \hat{f}(k) z^k$ to mean $< z^k, f >= \overline{\hat{f}(k)}$, so for any polynomial $p = \sum_{k=0}^{N} a_k z^k$, the action of f on p is given by

$$< p, f >= \int_{\mathbb{T}} p\overline{f} d\theta = \sum_{k=0}^{N} a_k \overline{\hat{f}(k)}.$$

We say that g is a multiplier of $H^2(w)^*$ if, for every f in $H^2(w)^*$, gf is in $H^2(w)^*$. The following theorem says that the universal multipliers are precisely $(N^+, H)^*$.

Theorem 3.2. *The function g is a multiplier of $H^2(w)^*$ for every w in $\mathcal{W}$ if and only if g is in $(N^+, H)^*$.*

Proof. Necessity is clear, as 1 is in every $H^2(w)^*$, so every multiplier must itself be in $H^2(w)^*$.

For sufficiency, fix w in $\mathcal{W}$, and f in $H^2(w)^*$. Let g be given, with $|\hat{g}(k)| \leq C_1 e^{-c\sqrt{k}}$ for some positive c. We must show fg is in $H^2(w)^*$, *i.e.* there exists a constant K such that, for all polynomials p,

$$* = |\int_{\mathbb{T}} p\overline{(gf)} d\theta|^2 \leq K \int_{\mathbb{T}} |p|^2 w d\theta.$$

Writing $p = \sum_{k=0}^{N} a_k z^k$, $*$ becomes

$$\begin{aligned} * &= \sum_{j,k=0}^{n} a_k \overline{a}_j \overline{\widehat{gf}(k)} \widehat{gf}(j) \\ &= \sum_{j,k=0}^{n} a_k \overline{a}_j \overline{\sum_{l=0}^{\infty} \hat{g}(l) \hat{f}(k-l)} \sum_{m=0}^{\infty} \hat{g}(m) \hat{f}(j-m) \\ &= \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \overline{\hat{g}(l)} \hat{g}(m) \sum_{j,k=0}^{n} a_k \overline{a_j} \overline{\hat{f}(k-l)} \hat{f}(j-m) \end{aligned}$$

where the interchange of summation signs is justified because $\{\hat{f}(k-l)\}$ is zero for $l > k$, so the sums are really finite.

Now,

$$\sum_{k=0}^{N} a_k \overline{\hat{f}(k-l)} = \int_{\mathbb{T}} \overline{z}^l p(z) \overline{f}(z) d\theta,$$

so if $k(w,l,f)$ is defined by

$$k(w,l,f) = sup\{|\int_{\mathbb{T}} \overline{z}^l p(z)\overline{f}(z)d\theta| : p \text{ a polynomial, and } \int_{\mathbb{T}} |p|^2 w d\theta \leq 1\},$$

then

$$* \leq \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \overline{\hat{g}(l)}\hat{g}(m)k(w,l,f)k(w,m,f)\int_{\mathbb{T}} |p|^2 w d\theta.$$

Write $\overline{z}^l p(z) = F + P$, where F is a polynomial in z, and P is a polynomial of degree less than or equal to l in $\overline{z}$, with no constant term. Then

$$\begin{aligned} k(w,l,f)^2 &= \sup_p \frac{|\int_{\mathbb{T}}(F+P)\overline{f}d\theta|^2}{\int_{\mathbb{T}}|F+P|^2 w d\theta} \\ &= \sup_p \frac{|\int_{\mathbb{T}} F\overline{f}d\theta|^2}{\int_{\mathbb{T}}|F+P|^2 w d\theta}. \end{aligned}$$

Because f is in $H^2(w)^*$, there is a constant C_2 such that

$$k(w,l,f)^2 \leq C_2 \sup_p \frac{\int_{\mathbb{T}}|F|^2 w d\theta}{\int_{\mathbb{T}}|F+P|^2 w d\theta}.$$

By lemma 3.3, this is less than or equal to $C_2 C_3 e^{c\sqrt{l}}$. Therefore

$$\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \overline{\hat{g}(l)}\hat{g}(m)k(w,l,f)k(w,m,f) \leq (C_1)^2 C_2 C_3 \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} e^{-c\sqrt{l}}e^{-c\sqrt{m}}e^{\frac{c}{2}\sqrt{l}}e^{\frac{c}{2}\sqrt{m}}$$

and the right hand side sums to a finite value, C_4 say. Therefore

$$* = |\int_{\mathbb{T}} p\overline{(fg)}d\theta|^2 \leq C_4 \int_{\mathbb{T}} |p|^2 w d\theta,$$

as desired. □

Lemma 3.3. *Let w be in $\mathcal{W}$, l be a positive integer, and let $c > 0$ be given. Then there is a constant C_3, depending only on c and w, such that for any analytic polynomial F, and any co-analytic polynomial P of degree l and with no constant term,*

$$\frac{\int_{\mathbb{T}}|F|^2 w d\theta}{\int_{\mathbb{T}}|F+P|^2 w d\theta} \leq C_3 e^{c\sqrt{l}}.$$

Proof. Without loss of generality, we can assume that $\int_{\mathbb{T}} |F|^2 w d\theta = 1$, and that $\int_{\mathbb{T}} |F+P|^2 w d\theta \leq \frac{1}{4}$, and $\int_{\mathbb{T}} |P|^2 w d\theta \geq \frac{1}{4}$.
Write $P(z) = \sum_{k=1}^{l} \alpha_k \overline{z}^k$. Then

$$\begin{aligned} \int_{\mathbb{T}} |P|^2 w d\theta &= \sum_{j,k=1}^{l} \alpha_k \overline{\alpha}_j \int_{\mathbb{T}} z^{k-j} w d\theta \\ &\leq \sum_{j,k=1}^{l} |\alpha_k||\alpha_j|\|w\|_1. \end{aligned}$$

So there exists some m between 1 and l for which

$$\begin{aligned} |\alpha_m| &\geq \frac{\|P\|_{H^2(w)}}{l\sqrt{\|w\|_1}} \\ &\geq \frac{1}{2l\|w\|_1}. \end{aligned}$$

But $\int_{\mathbb{T}} |F+P|^2 w d\theta = \int_{\mathbb{T}} |z^l(F+P)|^2 w d\theta$, and $z^l(F+P)$ is an analytic polynomial with one of its first l coefficients greater than or equal to $\frac{1}{2l\|w\|_1}$, so by Theorem 3.1 there is a constant C_1 such that $\int_{\mathbb{T}} |z^l(F+P)|^2 w d\theta \geq C_1 e^{-c\sqrt{l}}$. This gives the desired conclusion. □

We remark that in [**10**], Helson and Szegö found necessary and sufficient conditions for a weight w in $\mathcal{W}$ to satisfy

$$\sup_{F,P} \frac{\int_{\mathbb{T}} |F|^2 w d\theta}{\int_{\mathbb{T}} |F+P|^2 w d\theta} < \infty$$

where F is an analytic polynomial and P is a co-analytic polynomial with no constant term. The condition is that $log(w)$ be writable as $u + \tilde{v}$, where u and v are in $L^\infty(\mathbb{T}, \theta)$, $\|v\|_\infty < \frac{\pi}{2}$, and $\tilde{v}$ is the harmonic conjugate of v. For weights which do not satisfy the Helson-Szegö condition, Lemma 3.3 can be thought of as an asymptotic version.

4. Conclusion

Given a uniform algebra, with a distinguished homomorphism with (say) a unique representing measure, one can define a Smirnov class, and most of the classical theorems go through - see [6], where it is called the Hardy algebra. In particular, the Smirnov class can be represented as a union of Hilbert spaces [6, V.4.4], and one can therefore define an inductive limit topology as above. How far can the results of this paper be extended to the abstract setting? For example, if the kernel of the homomorphism is singly generated by some function F, when can one represent the Smirnov class as convergent power series in F?

What is the Bass stable rank of N^+? It is easy to see that the topological stable rank of $(N^+, \mathcal{H})$ is 2, but because the invertible elements are not open, there is no *a priori* reason that this should dominate the Bass stable rank. Determining the Bass stable rank of H^∞ is hard - see [13], where a lot of progress has been made - but, because all outer functions are invertible in N^+, the problem here might be somewhat easier.

A final remark, for topological vector space enthusiasts: $(N^+, \mathcal{H})$ is a natural example of an ultrabornological space (*i.e.* the inductive limit of Banach spaces) that is not quasicomplete (for this, plus the fact that the dual is reflexive, would force $(N^+, \mathcal{H})$ itself to be reflexive - [**11**, p. 303]).

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