Norm Preserving Extensions of Holomorphic Functions from Subvarieties of the Bidisk

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1 Introduction

A basic result in the theory of holomorphic functions of several complex variables is the following special case of the work of H. Cartan on the sheaf cohomology on Stein domains ([9], or see [13] or [15] for more modern treatments).

Theorem 1.1 *If V is an analytic variety in a domain of holomorphy Ω and if f is a holomorphic function on V, then there is a holomorphic function g in Ω such that $g = f$ on V.*

The subject of this paper concerns an add-on to the structure considered in Theorem 1.1 which arose in the authors recent investigations of Nevanlina-Pick interpolation on the bidisk. The definition for a general pair (Ω, V) would be as follows.

Definition 1.2 *Let V be an analytic variety in a domain of holomorphy Ω. Say V* **has the extension property** *if whenever f is a bounded holomorphic function on V, there is a bounded holomorphic function g on Ω such that*

$$g|_V = f \quad \textit{and} \quad \sup_{\Omega} |g| = \sup_{V} |f|. \tag{1.3}$$

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More generally, if $\mathrm{Hol}^\infty(V)$ *denote the bounded holomorphic functions on* V *and* $A \subseteq \mathrm{Hol}^\infty(V)$*, then we say* V **has the A-extension property** *if there is a bounded holomorphic function* g *on* Ω *such that (1.3) holds whenever* $f \in A$*.*

Before continuing we remark that in Definition 1.2 it is not essential that V be a variety: interpret f to be holomorphic on V if f has a holomorphic extension to a neighborhood of V. Also, in this paper we shall restrict our attention to the case where $\Omega = \mathbb{D}^2$. The authors intend to publish their results on more general cases in a subsequent paper. Finally, we point out that the notion in Definition 1.2 is different but closely related to extension problems studied by the group that worked out the theory of function algebras in the 60's and early 70's (see e.g. [18] and [4]). We now describe in some detail how we were led to formulate the notions in Definition 1.2.

The classical Nevanlinna-Pick Theory gives an exhaustive analysis of the following extremal problem on the disk. For data $\lambda_1, \ldots, \lambda_n \in \mathbb{D}$ and $z_1, \ldots, z_n \in \mathbb{C}$, consider

$$(1.4) \qquad \rho \;=\; \inf\,\{\sup_{\lambda \in \mathbb{D}}\, |\varphi(\lambda)| \;:\; \varphi : \mathbb{D} \overset{\text{holo}}{\longrightarrow} \mathbb{C},\; \varphi(\lambda_i) = z_i\}$$

Functions ψ for which (1.4) is attained are referred to as **extremal** and the most important fact in the whole theory is that there is only one extremal for given data. Once this fact is realized it comes as no surprise that there is a finite algebraic procedure for creating a formula for the extremal in terms of the data and the critical value ρ (calculating ρ is an eigenvalue problem) an important result, not only in function theory [12], but in the model theory for Hilbert space contractions [11] and in the mathematical theory of control [14].

Now, let us consider the associated extremal problem on the bidisk. For data $\lambda_i = (\lambda_i^1, \lambda_i^2) \in \mathbb{D}^2, 1 \le i \le n$, and $z_i \in \mathbb{C}, 1 \le i \le n$, let

$$(1.5) \qquad \rho \;=\; \inf\,\{\sup_{\lambda \in \mathbb{D}^2}\, |\varphi(\lambda)| \;:\; \varphi : \mathbb{D}^2 \overset{\text{holo}}{\longrightarrow} \mathbb{C},\; \varphi(\lambda_i) = z_i\}$$

Unlike in the case of the disk, extremals for (1.5) are not unique. The authors however have discovered the interesting fact that there is a polynomial variety in the bidisk on which the extremals are unique. Specifically, there exists a polynomial variety $V_{\lambda,z} \subseteq \mathbb{D}^2$, depending on the data, and there exists a holomorphic function f defined on $V_{\lambda,z}$ with the properties that $\lambda_1, \ldots, \lambda_n \in V_{\lambda,z}$

$$g|_{V_{\lambda,z}} = f \quad \text{and} \quad \sup_{\mathbb{D}^2} |g| = \sup_{V_{\lambda,z}} |f| \tag{1.6}$$

whenever g is extremal for (1.5). Furthermore there is a finite algebraic procedure for calculating f in terms of the data and the critical value ρ (now, calculating ρ is a problem in semi-definite programming).

Thus, it transpires that there is a unique extremal to (1.5), not defined on all of $\mathbb{D}^2$, but only on $V_{\lambda,z}$, and that the set of global extremals to (1.5) is obtained by taking the set of norm preserving extensions of this unique local extremal to the bidisk. Clearly, the nicest possible situation would arise if it were the case that $V_{\lambda,z}$ had the **polynomial extension property** (i.e. Definition 1.2 holds with $\Omega = \mathbb{D}^2, V = V_{\lambda,z}$ and $A =$ polynomials) for then the analysis of (1.5) would separate into two independent and qualitatively different problems: the analysis of the unique local extremal, and analysis of the norm preserving extensions from $V_{\lambda,z}$ to $\mathbb{D}^2$.

Thus, we see that the problem of identifying the varieties in the bidisk that have the A-extension property for a given algebra A arises naturally if one wants to understand the extremal problem (1.5). It turns out that there is also a purely operator-theoretic reason to study the A-extension property. This story begins with the famous inequality of von Neumann [24].

Theorem 1.7 *If T is a contractive operator on a Hilbert space, then*

$$\|p(T)\| \leq \sup_{\mathbb{D}} |p|$$

whenever p is a polynomial in one variable.

It was the attempt to explain this theorem that led Sz.-Nagy to discover his famous dilation theorem [21] upon which many of the pillars of modern operator theory are based (e.g. [22]). An extraordinary amount of work has been done by the operator theory community extending the inequality of von Neumann, none more elegant than the following result of Andô [6].

Theorem 1.8 *If $T = (T^1, T^2)$ is a contractive commuting pair of operators on a Hilbert space, then*

$$\|p(T)\| \leq \sup_{\mathbb{D}^2} |p| \tag{1.9}$$

whenever p is a polynomial in two variables.

We propose in this paper a refinement of Theorem 1.8 based on replacing (1.9) with an estimate

$$\|f(T)\| \leq \sup_V |f| \tag{1.10}$$

where f is allowed to be more general than a polynomial and V is a general subset of the bidisk. For $V \subseteq \mathbb{D}^2$, let $\mathrm{Hol}(V)$ denote the functions defined on V that have a holomorphic extension to a neighborhood of V and let $\mathrm{Hol}^\infty(V)$ denote the set of elements in $\mathrm{Hol}(V)$ that are bounded on V. Note that if V is a variety and f is holomorphic on V, then a baby theorem in the Cartan theory would be that there exists a neighborhood $U \supseteq V$ and a holomorphic function g on U with $f = g|V$, though such a g might well be nonunique. If we want to form $f(T)$ where T is a pair of commuting operators, one way would be to define $f(T)$ to be $g(T)$ where $g(T)$ is defined via the Taylor calculus [23]. Of course, we would need that $\sigma(T) \subseteq V$ (so that $\sigma(T) \subseteq U$) and, in addition, would want $f(T)$ to depend only on f and not on the particular extension g. This motivates the following definition which makes sense for **arbitrary** sets V.

Definition 1.11 *If $V \subseteq \mathbb{C}^2$ and T is a commuting pair of operators on a Hilbert space, say T* **is subordinate to** *V if $\sigma(T) \subseteq V$ and $g(T) = 0$ whenever g is holomorphic on a neighborhood of V and $g|V = 0$. If $f \in \mathrm{Hol}(V)$ and T is subordinate to V* **define** *$f(T)$ by setting $f(T) = g(T)$ where g is any holomorphic extension of f to a neighborhood of V. If T is subordinate to V and (1.10) holds for all $f \in \mathrm{Hol}^\infty(V)$, then we say that V* **is a spectral set for** *T. More generally, if T is subordinate to V, $A \subseteq \mathrm{Hol}^\infty(V)$ and (1.10) holds for all $f \in A$, then we say that V* **is an A-spectral set for** *T.*

Armed with Definition 1.11 it is easy to see that, modulo some simple approximations, Andô's Theorem is equivalent to the assertion that $\mathbb{D}^2$ is a spectral set for any pair of commuting contractions with $\sigma(T) \subseteq \mathbb{D}^2$. Thus the following definition seems worthy of contemplation.

Definition 1.12 *Fix $V \subseteq \mathbb{C}^2$ and let $A \subseteq \mathrm{Hol}^\infty(V)$. Say that V* **is an A-von Neumann set** *if V is an A-spectral set for T whenever T is a commuting pair of contractions subordinate to V.*

We have introduced two properties that a set $V \subseteq \mathbb{D}^2$ might have relative to a specified subset $A \subseteq \mathrm{Hol}^\infty(V)$: V might have the A-extension property as in Definition 1.2; or, it might be an A-von Neumann set as in Definition 1.12. Furthermore, we have indicated the naturalness of these properties from the appropriate perspectives. In this paper we shall show these two notions are actually the same. Specifically, we have the following result.

Theorem 1.13 *Let $V \subseteq \mathbb{D}^2$ and let $A \subseteq \mathrm{Hol}^\infty(V)$. V has the A-extension property if and only if V is an A-von Neumann set.*

This theorem will be proven in Section 2 of this paper.

Theorem 1.13 provides a powerful set of tools for investigating the extension property, namely the techniques of operator dilation theory. Specifically, in Section 3 of this paper we shall show that if V is polynomially convex and V is an A-spectral set for a commuting pair of 2×2 matrices, then the induced contractive algebra homomorphism of $\mathrm{Hol}^\infty(V)$ is in fact completely contractive (Proposition 3.1). It will then follow via Arveson's Dilation Theorem [7], operator model theory, and concrete H^2-arguments that V must satisfy a purely geometric property: V must be balanced.

To define this notion of balanced, we first recall for the convenience of the reader some simple notions from the theory of complex metrics (see [16] for an excellent discussion).

(1.14) $$C_U(\lambda_1, \lambda_2) \;=\; \sup\{d(F(\lambda_1), F(\lambda_2)) \;:\; F : U \to \mathbb{D},\; F \,\text{is holomorphic}\}$$

and

(1.15)
$$K_U(\lambda_1, \lambda_2) \;=\; \inf\{d(\mu_1, \mu_2) \;:\; \varphi : \mathbb{D} \to U,\; \varphi(\mu_i) = \lambda_i,\; \varphi \,\text{is holomorphic}\}$$

where $d(\mu_1, \mu_2) = \left|\frac{\mu_1 - \mu_2}{1 - \bar{\mu}_1 \mu_2}\right|$ is the pseudo-hyperbolic metric on the disk. Here, (1.14) is referred to as the **Carathéodory extremal problem** and functions for which the supremum in (1.14) is attained are referred to as **Carathéodory extremals**. Furthermore, $C_U(\lambda_1, \lambda_2)$ is always a metric, the **Carathéodory metric**. Likewise, (1.15) is the **Kobayashi extremal problem** and functions for which the infimum is attained are **Kobayashi extremals**. However, $K(\lambda_1, \lambda_2)$ is in general not a metric though the beautiful theorem of Lempert [17] asserts that if U is convex, then in fact K_U is a metric, and indeed $K_U = C_U$. In the simple case when $U = \mathbb{D}^2$ both (1.14) and (1.15) are easily solved to yield the formulas[1]

$$C_{\mathbb{D}^2}(\lambda_1, \lambda_2) = K_{\mathbb{D}^2}(\lambda_1, \lambda_2) = \max\{d^1, d^2\} \tag{1.16}$$

where $d^1 = d(\lambda_1^1, \lambda_2^1)$ and $d^2 = d(\lambda_1^2, \lambda_2^2)$.

The formulas (1.16) allow one to see that the description of the extremal functions for (1.14) and (1.15) in the case when $U = \mathbb{D}^2$ splits naturally into three cases according as $d^1 > d^2, d_1 = d_2$, or $d^1 < d^2$. If $d^1 > d^2$, then the extremal function for (1.14) is unique: $F(\lambda) = \lambda^1$. However when $d^1 > d^2$, there is not a unique extremal for (1.15): any function $\varphi(z) = (z, f(z))$ where $f : \mathbb{D} \to \mathbb{D}$ satisfies $f(\lambda_i^1) = \lambda_i^2$ will do. Likewise when $d^1 < d^2$ the Carathéodory extremal is the unique function $F(\lambda) = \lambda^2$ and any function $\varphi(z) = (f(z), z)$ where $f : \mathbb{D} \to \mathbb{D}$ solves $f(\lambda_2^2) = \lambda_2^1$ is a Kobayashi extremal. Thus, when $d^1 \neq d^2$, the Carathéodory extremal is unique and the Kobayashi is not. When $d^1 = d^2$, the reverse is true, the Carathéodory extremal is not unique and the Kobashi extremal is: either $F(\lambda) = \lambda^1$ or $F(\lambda) = \lambda^2$ is extremal for (1.14) while $\varphi(z) = (z, f(z))$ where f is the unique Möbius mapping $f : \mathbb{D} \to \mathbb{D}$ satisfying $f(\lambda_i^1) = \lambda_i^2$ is the unique extremal for (1.15). These considerations prompt the following definition.

Definition 1.17 *If $\lambda = (\lambda_1, \lambda_2) = ((\lambda_1^1, \lambda_1^2), (\lambda_2^1, \lambda_2^2))$ is a pair of points in $\mathbb{D}^2$, say λ is a* **balanced pair** *if $d(\lambda_1^1, \lambda_2^1) = d(\lambda_1^2, \lambda_2^2)$.*

Thus, the Kobayashi extremal for a pair of points λ is unique if and only if λ is a balanced pair. Now, if λ is pair of points in $\mathbb{D}^2$, and φ is extremal for the Kobayashi problem, it is easy to check that $D = \operatorname{ran}\varphi$ is a totally geodesic one dimensional complex submanifold of $\mathbb{D}^2$. Conversely, if $D = \operatorname{ran}\varphi$ is an analytic disk in $\mathbb{D}^2$ and D is totally geodesic, then φ is a Kobayashi extremal for any pair of points in D. Thus, we may assert, based on the observation following Definition 1.17, that there exists a unique totally geodesic disk D_λ passing through a pair of points λ in $\mathbb{D}^2$ if and only if λ is a balanced pair. Concretely, it is the set

$$D_\lambda = \{(z, f(z)) : z \in \mathbb{D}\}$$

where $f : \mathbb{D} \to \mathbb{D}$ is the unique mapping satisfying $f(\lambda_i^1) = \lambda_i^2$.

We now are able to give the promised definition of a balanced subset of $\mathbb{D}^2$.

[1] We shall use superscripts to denote coordinates in $\mathbb{D}^2$, subscripts to distinguish points, or to denote coordinates of a vector.

Definition 1.18 *If $V \subseteq \mathbb{D}^2$, say V* **is balanced** *if $D_\lambda \subseteq V$ whenever λ is a balanced pair of points in V.*

Note that if D is either an analytic disk or a totally geodesic disk in $\mathbb{D}^2$, then D is balanced in the sense of Definition 1.18 if and only if $D = D_\lambda$ for some balanced pair λ. For this reason we refer to D_λ as **the balanced disk passing through λ_1 and λ_2**. Note that if D is a balanced disc, then every pair of points in V is balanced and also $D = D_\lambda$ for each pair of points $\lambda \in D \times D$. The significance of balanced sets in the context of the extension property on the bidisk will be revealed in Section 3 where we shall exploit Theorem 1.13 to give an operator-theoretic proof of the following result.

Theorem 1.19 *Let $V \subseteq \mathbb{D}^2$ and assume that V is relatively polynomially convex (i.e. $V^\wedge \cap \mathbb{D}^2 = V$ where $V^\wedge$ denotes the polynomially convex hull of V). If V has the polynomial extension property, then V is balanced.*

It turns out that the property of being balanced is much more rigid than one might initially suspect. In Section 4 we shall investigate this phenomenon by establishing several geometric properties of balanced sets. Finally, in Section 5 of this paper we shall combine this geometric rigidity of balanced sets with the elementary observation that subsets of the bidisk with an extension property must be H^∞-varieties to obtain the following result which gives a complete classification of the subsets V of the bidisk with the polynomial extension property (at least in the case when V is relatively polynomially convex).

Theorem 1.20 *Let V be a nonempty relatively polynomially convex subset of $\mathbb{D}^2$. V has the polynomial extension property if and only if V has one of the following forms.*

(i) $V = \{\lambda\}$ *for some* $\lambda \in \mathbb{D}^2$

(ii) $V = \mathbb{D}^2$

(iii) $V = \{(z, f(z)|z \in \mathbb{D}\}$ *for some holomorphic* $f : \mathbb{D} \to \mathbb{D}$

(iv) $V = \{f(z), z)|z \in \mathbb{D}\}$ *for some holomorphic* $f : \mathbb{D} \to \mathbb{D}$

After this paper was submitted, Pascal Thomas devised an elegant function theoretic proof of Theorem 1.19. We include his proof in an Appendix at the end of the paper.

2 The Equivalence of the von Neumann Inequality and the Extension Property

In this section we shall prove Theorem 1.13 from the introduction. Accordingly, fix a set $V \subseteq \mathbb{D}^2$ and a set $A \subseteq \mathrm{Hol}^\infty(V)$.

One side of Theorem 1.13 is straightforward. Thus, assume that V has the A-extension property and fix a commuting pair of contractions T such that T is subordinate to V. If $f \in A$ and $g \in H^\infty(\mathbb{D}^2)$ with $g|V = f$ and $\|g\|_{\mathbb{D}^2} = \|f\|_V$, then

$$\|f(T)\| = \|g(T)\| \leq \|g\|_{\mathbb{D}^2} = \|f\|_V.$$

Hence, since f was arbitrarily chosen, V is an A-spectral set for T. Hence since T was arbitrarily chosen, V is an A-von Neumann set.

The reverse direction of Theorem 1.13 is much more subtle and will rely on some basic facts about Nevanlinna-Pick interpolation on the bidisk. For n distinct points in the bidisk $\lambda_1, \ldots, \lambda_n$, let $\mathcal{K}_\lambda$ denote the set of $n \times n$ strictly positive definite matrices, $[k_i(j)]_{i,j=1}^n$, such that $k_i(i) = 1$ for each i,

$$\left[\left(1 - \overline{\lambda_i^1}\lambda_j^1\right) k_i(j)\right] \geq 0 \quad \text{and} \quad \left[\left(1 - \overline{\lambda_i^2}\lambda_j^2\right) k_i(j)\right] \geq 0. \tag{2.1}$$

For a proof of the next result see the papers [10], [8] or [2], or the book [3].

Theorem 2.2 *If $\lambda_1, \ldots, \lambda_n$ are n distinct points in $\mathbb{D}^2$ and $z_1, \ldots, z_n \in \mathbb{C}$ then there exists $\varphi \in H^\infty(\mathbb{D}^2)$ with $\|\varphi\|_{\mathbb{D}^2} \leq 1$ and $\varphi(\lambda_i) = z_i$ for each i if and only if*

$$\left[(1 - \overline{z}_i z_j)\, k_i(j)\right] \geq 0, \qquad \text{for all } k \in \mathcal{K}_\lambda. \tag{2.3}$$

Theorem 2.2 allows us via a simple argument to dualize the extremal problem (1.5) in the following form.

Theorem 2.4 *If distinct points $\lambda_1, \ldots, \lambda_n \in \mathbb{D}^2$ and points $z_1, \ldots, z_n \in \mathbb{C}$ are given and ρ is as in (1.5), then*

$$\rho = \inf\left\{\sigma \,\middle|\, \left[(\sigma^2 - \overline{z}_i z_j) k_i(j)\right] \geq 0 \quad \text{for all } k \in \mathcal{K}_\lambda\right\}.$$

Finally, Theorem 2.4 yields the following result which shall provide the key ingredient for completing the proof of Theorem 1.13.

Lemma 2.5 *If ρ is as in (1.5) and ψ is extremal for (1.5), then there exist a commuting pair of contractions $T = (T_1, T_2)$ subordinate to $\{\lambda_1, \dots, \lambda_n\}$ and such that $\|\psi(T)\| = \rho$*

Proof: We first claim that $\mathcal{K}_\lambda$ is compact as a subset of the selfadjoint $n \times n$ complex matrices equipped with the matrix norm. To see this we show that $\mathcal{K}_\lambda$ is both bounded and closed. That $\mathcal{K}_\lambda$ is bounded follows by noting that the normalization condition $k_i(i) = 1$ implies that if $k \in \mathcal{K}_\lambda$, then

$$\|k\| \leq \text{tr } K = \sum k_i(i) = n.$$

To see that $\mathcal{K}_\lambda$ is closed, we argue by contradiction. Thus, assume that $\{k^\ell\}$ is a sequence in $\mathcal{K}_\lambda, k^\ell \to k$ as $\ell \to \infty$, and $k \notin \mathcal{K}_\lambda$. By continuity, $k_i(i) = 1$ for each i. Also by continuity, condition (2.1) holds. Hence since $k \notin \mathcal{K}_\lambda$ it must be the case that k is not strictly positive definite. Choose a vector $v = (v_i)$ with $kv = 0$ and $v \neq 0$. Letting Λ^1 and Λ^2 denote the diagonal matrices whose $(i,i)^{th}$ entries are λ_i^1 and λ_i^2 we deduce from (2.1), that if $r = 1$ or $r = 2$, then,

$$\begin{aligned} 0 &\leq \sum_{i,j} (1 - \lambda_i^{\overline{r}} \lambda_j^r) k_i(j) v_j \overline{v}_i \\ &= \sum_{i,j} k_i(j) v_j \overline{v}_i - \sum_{i,j} k_i(j) (\lambda_j^r v_j) \overline{(\lambda_i^r v_i)} \\ &= < kv, v > - < k\Lambda^r v, \Lambda^r v > . \end{aligned}$$

Now, $kv = 0$ and k, by continuity, is positive semidefinite. Hence both $\Lambda^1 v$ and $\Lambda^2 v$ are in the kernel of k. Continuing, we deduce by induction that if $m = (m_1, m_2)$ is a multi-index, then $\Lambda^m v = (\Lambda^1)^{m_1} (\Lambda^2)^{m_2} v$ is in the kernel of k. Finally, if p is any polynomial in two variables we deduce that

$$k\ p(\Lambda) v = 0. \tag{2.6}$$

Now $v \neq 0$ so that there exists i s.t. $v \neq 0$. On the other hand $p(\Lambda)$ is the diagonal operator whose $j - j^{th}$ entry is $p(\lambda_j)$ and $\lambda_1, \dots \lambda_n$ are assumed distinct so that there is a polynomial p such that $p(\lambda_i) = 1$ and $p(\lambda_j) = 0$ for $j \neq i$. Hence from (2.6) we see that $p(\Lambda)v = v_i e_i \in \ker k$ which contradicts the fact that $k_i(i) \neq 0$. This contradiction establishes that $\mathcal{K}_\lambda$ is closed and completes the proof that $\mathcal{K}_\lambda$ is compact.

As an immediate consequence of the compactness of $\mathcal{K}_\lambda$ and Theorem 2.4 there exists $k \in \mathcal{K}_\lambda$ such that

$$\left[(\rho^2 - \overline{z}_i z_j)k_i(j)\right] \geq 0 \qquad \text{and} \tag{2.7}$$

$$\exists\ w \in \mathbb{C}^n \text{ with } w \neq 0 \text{ and } \sum_{i,j}(\rho^2 - \overline{z}_i z_j)k_i(j)\bar{w}_i\ w_j = 0. \tag{2.8}$$

To define T we first define a pair $X = (X^1, X^2)$. Choose vectors $k_1, \ldots, k_n \in \mathbb{C}^n$ such that $< k_i, k_j > = k_i(j)$ for all i and j. Since k is strictly positive definite, the formulas

$$X^r k_i = \overline{\lambda_i^r} k_i \qquad 1 \leq i \leq n,\ r = 1, 2$$

uniquely define a commuting pair of $n \times n$ matrices $X = (X^1, X^2)$. Set $T = (X^{1*}, X^{2*})$. Since X^1 and X^2 share a set of n eigenvectors with corresponding eigenvalues $\overline{\lambda}_1, \ldots \overline{\lambda}_n$ it is clear that X is subordinate to $\{\overline{\lambda}_1, \ldots, \overline{\lambda}_n\}$. Hence T is subordinate to $\{\lambda_1, \ldots, \lambda_n\}$. Noting that (2.1) implies that both X^1 and X^2 are contracting we see that T is a contractive pair. Finally, note that if ψ is extremal for (1.5) and $\breve{\psi}$ is defined by $\breve{\psi}(\lambda) = \overline{\psi(\overline{\lambda})}$, then

$$\breve{\psi}(X)k_i = \breve{\psi}(\overline{\lambda}_i)k_i = \overline{\psi(\lambda_i)}k_i = \overline{z}_i k_i.$$

Hence (2.7) implies that $\|\breve{\psi}(X)\| \leq \rho$ and (2.8) implies that $\|\breve{\psi}(X)\| \geq \rho$. Hence $\|\breve{\psi}(X)\| = \rho$. But $\breve{\psi}(X) = \psi(T)^*$ so that so also $\|\psi(T)\| = \rho$. This establishes Lemma 2.5. $\square$

We now are ready to complete the proof of Theorem 1.13. Thus, assume that V is an A-von Neumann set and fix $f \in A$. We need to show that there exists $g \in H^\infty(\mathbb{D}^2)$ with $g|_V = f|_V$ and $\|g\|_{\mathbb{D}^2} = \|f\|_V$.

Choose a dense sequence $\{\lambda_i\}_{i=1}^\infty$ in V. For each $n \geq 1$ consider the extremal problem

$$\rho_n = \inf_{\varphi:\mathbb{D}^2 \to \mathbb{D}} \{\|\varphi\|_{\mathbb{D}^2}\ :\ \varphi(\lambda_i) = f(\lambda_i) \text{ for } i \leq n,\ \varphi \text{ is holomorphic}\}. \tag{2.9}$$

If ψ_n is chosen extremal for (2.9), then

$$\begin{aligned} \|\psi_n\|_{\mathbb{D}^2} &\overset{(i)}{=} \rho_n \\ &\overset{(ii)}{=} \|\psi_n(T_n)\| \\ &\overset{(iii)}{=} \|f(T_n)\| \\ &\overset{(iv)}{\leq} \|f\|_V. \end{aligned}$$

Here, (i) holds since ψ_n is extremal for (2.9), T_n is the commuting pair of contractions subordinate to $\{\lambda_1, \ldots, \lambda_n\}$ whose existence is guaranteed by Lemma 2.5, (ii) holds by Lemma 2.5, (iii) holds since T_n is subordinate to $\{\lambda_1, \ldots, \lambda_n\}$ and $\psi_n(\lambda_i) = f(\lambda_i)$ for $i \leq n$, and (iv) holds from the assumption that V is an A-von Neumann set and the fact that T_n is a contractive pair (T_n is subordinate to V since T_n is subordinate to $\{\lambda_1, \ldots, \lambda_n\} \subseteq V$).

Summarizing, in the previous paragraph we have shown that for each $n \geq 1$, there exists $\psi_n \in H^\infty(\mathbb{D}^2)$ with

$$\|\psi_n\|_{\mathbb{D}^2} \leq \|f\|_V \qquad \text{and} \tag{2.10}$$

$$\psi_n(\lambda_i) = f(\lambda_i) \quad i \leq n. \tag{2.11}$$

Evidently, either by a uniform family argument or a weak-* compactness in H^∞ argument, (2.10) implies that there exists $g \in H^\infty(\mathbb{D}^2)$ with $\psi_n \to g$ pointwise on $\mathbb{D}^2$ and $\|g\|_{\mathbb{D}^2} \leq \|f\|_V$. By (2.11) we also have that $g(\lambda_i) = f(\lambda_i)$ for all i. Since $\{\lambda_i\}_{i=1}^\infty$ was chosen dense in V it follows that $g|V = f$. We have shown that g exists with the desired properties and the proof of Theorem 1.13 is complete. □

3 Sets with the Polynomial Extension Property are Balanced

In this section we shall prove Theorem 1.19 from the introduction. In the statement of our first result, note that λ^1 and λ^2 are the coordinate functions, and λ_1 and λ_2 are points in $\mathbb{D}^2$. We use V^- to denote the closure of V.

Proposition 3.1 *Let $V \subseteq \mathbb{D}^2$, assume that V has the polynomial extension property, and let $\lambda_1, \lambda_2 \in V$ with $\lambda_1 \neq \lambda_2$. If τ is a contractive 2- dimensional representation of $P(V^-)$ with $(\tau(\lambda^1), \tau(\lambda^2)) = (\lambda_1, \lambda_2)$, then τ is completely contractive.*

Proof: Let Ω denote the unit ball of the complex $n \times n$ matrices and fix an $n \times n$ matrix p of polynomials in two variables with

$$\|p(\lambda)\| < 1 \quad \text{for all } \lambda \in V. \tag{3.2}$$

Let $T = (\tau(\lambda^1), \tau(\lambda^2))$, a pair of commuting contractions on $\mathbb{C}^2$. Proposition 3.1 will be established if we can show that

$$\|p(T)\| \leq 1. \tag{3.3}$$

We claim that

$$d_\Omega(p(\lambda_1), p(\lambda_2)) \leq d_{\mathbb{D}^2}(\lambda_1, \lambda_2), \tag{3.4}$$

where d_Ω (respectively, $d_{\mathbb{D}^2}$) is the Carathéodory metric in Ω (resp., $\mathbb{D}^2$). To see (3.4) fix $\epsilon > 0$. Choose a polynomial q such that $q : \Omega \to \mathbb{D}$ and $d_\mathbb{D}(q(p(\lambda_1)), q(p(\lambda_2))) > d_\Omega(p(\lambda_1), p(\lambda_2)) - \epsilon$. Thus $q \circ p$ is a polynomial and (3.2) implies that $\sup\limits_V |q \circ p| \leq 1$. Since V has the polynomial extension property, there exists $\varphi \in H^\infty(\mathbb{D}^2)$ with $\sup\limits_{\mathbb{D}^2} |\varphi| \leq 1$ and $\varphi(\lambda_i) = q(p(\lambda_i))$. Hence, for each $\varepsilon > 0$,

$$\begin{aligned} d_\Omega(p(\lambda_1), p(\lambda_2)) - \epsilon &< d_\mathbb{D}(q(p(\lambda_1)), q(p(\lambda_2))) \\ &\leq d_{\mathbb{D}^2}(\lambda_1, \lambda_2), \end{aligned}$$

which establishes (3.4).

To see (3.3), choose points $z_1, z_2 \in \mathbb{D}$ such that $d(z_1, z_2) = d_\Omega(p(\lambda_1), p(\lambda_2))$ and let $f : \mathbb{D} \to \Omega$ with $f(z_1) = p(\lambda_1)$ and $f(z_2) = p(\lambda_2)$. Evidently, (3.4) implies that there exists $\psi : \mathbb{D}^2 \to \mathbb{D}$ such that $\psi(\lambda_1) = z_1$ and $\psi(\lambda_2) = z_2$. Hence, since T is a pair of commuting contractions (τ is assumed contractive), it follows from Andô's Theorem (Theorem 1.8) that $\psi(T)$ is a contraction. Since $\psi(T)$ is a contraction and $f : \mathbb{D} \to \Omega$, it follows that $f(\psi(T))$ is a contraction. (This follows from the Sz.-Nagy Dilation Theorem [21], as observed by Arveson [7]: in modern language, the disk is a complete spectral set for any contraction). But by construction, $f(\psi(T)) = p(T)$. Thus, (3.3) holds and the proof of Proposition 3.1 is complete. $\square$

Our next result exploits the Arveson Extension theorem [7], the Stinespring Representation Theorem [20], and the Sarason interpretation of semi-invariant subspaces [19] to interpret the completely contractive representation of Proposition 3.1. We shall say that a subnormal pair S has extension spectrum in Γ if there is a commuting normal pair N whose spectral measure is supported by Γ and such that the restriction of N to an invariant subspace is unitarily equivalent to S.

Proposition 3.5 *Let $V \subseteq \mathbb{D}^2$ and let Γ denote the Shilov boundary of $P(V^-)$. Let $\mathcal{H}$ be a Hilbert space and let $\tau : P(V^-) \to \mathcal{L}(\mathcal{H})$ be a completely contractive representation. There exists a Hilbert space $\mathcal{K}$ and a subnormal pair S with extension spectrum in Γ such that for all $\varphi \in P(V^-), \mathcal{K} \ominus \mathcal{H}$ is invariant for $\varphi(S)$ and $\tau(\varphi) = P_{\mathcal{H}}\varphi(S)|\mathcal{H}$.*

Proof: By the theorems of Arveson and Stinespring, there is a Hilbert space $\mathcal{G}$ containing $\mathcal{H}$ and a representation $\pi : C(\Gamma) \to \mathcal{L}(\mathcal{G})$ such that

$$\tau(\phi) \;=\; P_{\mathcal{H}}\,\pi(\phi)\,|_{\mathcal{H}} \qquad \forall\; \phi \,\in\, P(V^-).$$

Let $N = (\pi(\lambda^1), \pi(\lambda^2))$. Then the spectrum of N is contained in Γ. As τ is a representation, it follows from Sarason's lemma that $\mathcal{H}$ is semi-invariant for $\pi(P(V^-))$. This means that there exists a superspace $\mathcal{K} \supseteq \mathcal{H}$ such that $\mathcal{K}$ and $\mathcal{K} \ominus \mathcal{H}$ are both invariant for $\pi(P(V^-))$. Let S be $N\,|_{\mathcal{K}}$. $\square$

Armed with Propositions we are now ready to commence the proof of Theorem 1.19. Accordingly, fix $V \subseteq \mathbb{D}^2$ and assume that

$$V^\wedge \cap \mathbb{D}^2 = V \tag{3.6}$$

and

(3.7) $\quad V$ has the polynomial extension property.

Fix $d > 0$ and a pair of points $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_i \in V$ and with the property that $d(\lambda_1^1, \lambda_2^1) = d = d(\lambda_1^2, \lambda_2^2)$. Define a pair of operators $T = (T_1, T_2)$ by the formulas, $T_r k_i = \overline{\lambda}_i^r k_i, i = 1, 2$ and $r = 1, 2$, where $k_i, i = 1, 2$ is a pair of unit vectors in $\mathbb{C}^2$ with the property that $| < k_1, k_2 > |^2 = 1 - d^2$. Noting that $d = d(\lambda_1, \lambda_2)$, we see from [1] that

$$\|\breve{\varphi}(T)\| = 1, \tag{3.8}$$

whenever φ is an extremal for the Carathéodory Problem for the pair λ. (Recall that $\breve{\varphi}$ is defined by $\breve{\varphi}(\lambda) = \overline{\varphi(\overline{\lambda})}$.)

Now observe that $\|T_1\| = 1$ and $\|T_2\| = 1$. Hence, since (3.7) holds we see from Theorem 1.13 that V is a polynomial von Neumann set. Consequently, if we define τ by the formula $\tau(\psi)^* = \breve{\psi}(T)$, then τ is a contractive representation of $P(V^-)$. It follows from Proposition 3.1 that τ is completely contractive. Consequently, if we let Γ denote the Shilov boundary of $P(V^-)$ we obtain from Proposition 3.5 a commuting subnormal pair of operators S with extension spectrum in Γ and a 2-dimensional invariant subspace $\mathcal{M}$ for S such that $T_i \cong S_i^*|\mathcal{M}$.

Now observe that since $\|T_1\| = 1$, there exists a vector $\gamma \in \mathcal{M}$ with $\|\gamma\| = 1$ such that

$$P_{\mathcal{M}} S_1(S_1^*\gamma) = T_1^* T_1 \gamma = \gamma.$$

Since $\|S_1\| \leq 1$, it follows that $S_1^*\gamma \in \mathcal{M}$. Thus both γ and $S_1^*\gamma$ are in $\mathcal{M}$ and it certainly cannot be the case that they are linearly dependent, for otherwise T_1 would have a unimodular eigenvalue, contradicting the fact that $\sigma(T_1) = \{\lambda_1^1, \lambda_2^1\} \subseteq \mathbb{D}^2$. It follows that if we let $\mathcal{N}$ denote the closed linear span of $\{S_1^{n_1} S_2^{n_2}(S_1^*\gamma) : n_1, n_2 \geq 0\}$ and set $R_i = S_i|_{\mathcal{N}}$, then $R = (R_1, R_2)$ is a commuting subnormal pair with extension spectrum in Γ. Furthermore, R is cyclic with cyclic vector $S_1^*\gamma, R_1(S_1^*\gamma) = \gamma, \mathcal{M} \subseteq \mathcal{N}, \mathcal{M}$ is invariant for R_1^* and R_2^*, and $T_i = R_i^*|_{\mathcal{M}}$.

The facts in the preceding paragraph imply that there is a model for T and R of the following type: for some probability measure μ with support on $\Gamma, \mathcal{N} = H^2(\mu)$, the closure of the polynomials in $L^2(\mu)$; $R_1 = M_{\eta^1}$ and $R_2 = M_{\eta^2}$, multiplication by the coordinate functions; evaluation at λ_1 and evaluation at λ_2 on polynomials extend by continuity to continuous linear functionals on $H^2(\mu)$ represented, say, by the vectors k_{λ_1} and $k_{\lambda_2}; \mathcal{M} = span\{k_{\lambda_1}, k_{\lambda_2}\}; S_1^*\gamma = 1; T_r k_{\lambda_1} = \overline{\lambda_i^r} k_{\lambda_i}$. Also, note that (3.8) in this model asserts that

$$\|M_\varphi^*|\mathcal{M}\| = 1 \tag{3.9}$$

whenever φ is an extremal for the Carathéodory Problem for the pair λ.

Now let $D_\lambda = \{(z, m(z)) : z \in \mathbb{D}\}$ be a parametrization of the balanced disk passing through the points λ_1 and λ_2. If functions φ_1 and φ_2 are defined by the formulas $\varphi_1(\eta) = \eta^1$ and $\varphi_2(\eta) = m^{-1}(\eta^2)$, then φ_1 and φ_2 are extremal for the Carathéodory Problem for the pair λ and have the additional property that

$$\varphi_1(\lambda_i) = \lambda_i^1 = \varphi_2(\lambda_i) \quad \text{for} \quad i = 1, 2. \tag{3.10}$$

Now, since φ_1 is extremal, (3.9) and the fact that $S_1^*\gamma = 1$ imply that

$$\int |\varphi_1|^2 d\mu = \|M_{\varphi_1} 1\|^2 = 1.$$

Hence, $|\varphi_1(\eta)| = 1$ μ a.e. But (3.10) implies that $M_{\varphi_1}^*|_{\mathcal{M}} = M_{\varphi_2}^*|_{\mathcal{M}}$ so that M_{φ_2} also attains its norm on 1. Hence we can also deduce that $|\varphi_2(\eta)| = 1$ μ a.e. Since $|\varphi_1|$, $|\varphi_2|$, and $\left|\frac{\phi_1+\phi_2}{2}\right|$ are all 1 μ a.e., we have that μ is actually supported on ∂D.

Summarizing, we have shown that if $\lambda = (\lambda_1, \lambda_2)$ is a balanced pair of points in V and D_λ is the balanced disk passing through the pair, then there is a probability measure μ supported on $\Gamma \cap \partial D_\lambda$ such that λ_1 and λ_2 are bounded point evaluations for $H^2(\mu)$. Changing variables to $\mathbb{D}$ and invoking the classical result of Kolmogorov, Szegő, and Krein yields that in fact every point in D_λ is a bounded point evaluation of $H^2(\mu)$. As bounded point evaluations are necessarily in the polynomial convex hull of the support of μ, we conclude via (3.6) that $D_\lambda \subseteq V$ and the proof of Theorem 1.19 is complete. □

4 Some Geometric Properties of Balanced Sets

In this section we shall derive several simple facts about balanced subsets of the bidisk. The main point is that if there is more than one balanced disk in a balanced set, then the set must fill in to the entire bidisk. This will provide us with a lot of mileage in Section 5 when we prove the classification theorem (Theorem 1.20) from Section 1.

Lemma 4.1 *Let B be a balanced subset of $\mathbb{D}^2$. If D_1 is a balanced disk in B and $\lambda \in B \setminus D_1$, then there exists a balanced disk D_2 in B with $\lambda \in D_2$ and $D_1 \cap D_2 \neq \emptyset$.*

Proof: Let $D_1 = \{(z, f(z)) : z \in \mathbb{D}\}$ be a parametrization of D_1 with f an automorphism of the disc. If we define a function ρ on $\mathbb{D}$ by the formula

$$\rho(z) = d(\lambda^1, z) - d(\lambda^2, f(z)),$$

then the facts that $\lambda \notin D_1$ and f is an isometry guarantee that $\rho(\lambda^1) < 0$ and $\rho(f^{-1}(\lambda^2)) > 0$. Consequently, the connectedness of $\mathbb{D}$ implies that there exists a point $z_0 \in \mathbb{D}$ with the property that $\rho(z_0) = 0$. But since $\rho(z_0) = 0, ((\lambda^1, \lambda^2), (z_0, f(z_0)))$ is a balanced pair of points in B. Since B is assumed balanced, it follows that if we let D_2 denote the balanced disk that passes through these points, then D_2 lies in B. Finally, noting that $(z_0, f(z_0)) \in D_1 \cap D_2$ completes the proof of Lemma 4.1. □

Lemma 4.2 *For $\tau \in \partial\mathbb{D}$ and $\alpha \in \mathbb{D}$, let $m_{\tau,\alpha}$ denote the automorphism of $\mathbb{D}$ defined by $m_{\tau,\alpha}(z) = \tau\,\frac{z-\alpha}{1-\bar{\alpha}z}$. If we view $m_{\tau,\alpha}$ as a mapping from $\mathbb{D}^-$ into $\mathbb{D}^-$, then exactly one of the following four cases occurs.*

(i) $\tau = 1$ *and* $\alpha = 0$.

(ii) $\tau \neq 1$ *and* $|1-\tau| = 2|\alpha|$.

(iii) $|1-\tau| < 2|\alpha|$.

(iv) $|1-\tau| > 2|\alpha|$.

Furthermore,

(i) holds if and only if every point in $\mathbb{D}^-$ is a fixed point for $m_{\tau,\alpha}$.

(ii) holds if and only if $m_{\tau,\alpha}$ has exactly one fixed point on $\partial\mathbb{D}^-$.

(iii) holds if and only if $m_{\tau,\alpha}$ has exactly two fixed point on $\partial\mathbb{D}^-$.

(iv) holds if and only if $m_{\tau,\alpha}$ has exactly one fixed point in $\mathbb{D}$.

We eschew the proof of Lemma 4.2.

Lemma 4.3 *Let B be a subset of $\mathbb{D}^2$ with the balanced point property. If D_1 and D_2 are two distinct balanced discs in B and $\lambda \in D_1 \cap D_2$, then there exists a set U in $\mathbb{C}^2$ such that U is open in $\mathbb{C}^2$ and $\lambda \in U \subseteq B$.*

Proof: Let B be a balanced subset of $\mathbb{D}^2$ and let D_1 and D_2 be as in the lemma. By composing with an appropriate automorphism of $\mathbb{D}^2$ we can reduce the lemma to the special case when $(0,0) \in D_1 \cap D_2, D_1 = \{(z,z) : z \in \mathbb{D}\}$, and for some $\omega \in \partial\mathbb{D}, D_2 = \{(z, \omega z) : z \in \mathbb{D}\}$. For $\tau \in \partial\mathbb{D}$ and $\alpha \in \mathbb{D}$, let $m_{\tau,\alpha}$ denote the automorphism of $\mathbb{D}$ defined by $m_{\tau,\alpha}(z) = \tau\frac{z-\alpha}{1-\bar{\alpha}z}$ and let $D_{\tau,\alpha}$ denote the balanced disk defined by $D_{\tau,\alpha} = \{(z, m_{\tau,\alpha}(z)) : z \in \mathbb{D}\}$. Evidently, what we would like to show is that for each sufficiently small $\lambda \in \mathbb{D}^2$, there exist $\tau \in \partial\mathbb{D}$ and $\alpha \in \mathbb{D} \setminus \{0\}$ such that $\lambda \in D_{\tau,\alpha}, D_{\tau,\alpha} \cap D_1 \neq \emptyset$, and $D_{\tau,\alpha} \cap D_2 \neq \emptyset$. Equivalently, we want to show that for each sufficiently small $\lambda \in \mathbb{D}^2$, there exists $\tau \in \partial\mathbb{D}$ and $\alpha \in \mathbb{D} \setminus \{0\}$ such that

$$m_{\tau,\alpha}(\lambda^1) = \lambda^2, \tag{4.4}$$

$$m_{\tau,\alpha}(z) = z \quad \text{for some} \quad z \in \mathbb{D}, \tag{4.5}$$

and

$$m_{\tau,\alpha}(z) = \omega z \quad \text{for some} \quad z \in \mathbb{D}, \tag{4.6}$$

Now observe that (4.5) asserts that $m_{\tau,\alpha}$ has a fixed point in $\mathbb{D}$. Similarly, (4.6) asserts that $\overline{\omega} m_{\tau,\alpha} = m_{\overline{\omega}\tau,\alpha}$ has a fixed point in $\mathbb{D}$. Furthermore, the general $m_{\tau,\alpha}$ satisfying (4.4) is given in terms of a free unimodular constant, e^{it}, by

$$\tau = \frac{e^{it} - \overline{\lambda}^1 \lambda^2}{1 - e^{it}\lambda^1 \overline{\lambda}^2} \quad \text{and} \quad \alpha = \frac{e^{it}\lambda^1 - \lambda^2}{e^{it} - \overline{\lambda}^1 \lambda^2}. \tag{4.7}$$

An examination of Case (iv) in Lemma 4.2 thus reveals that Lemma 4.3 will follow if we can show that for all sufficiently small $\lambda \in \mathbb{D}^2$, there exists e^{it} such that if τ and α are defined as in (4.7), then $|1 - \tau| > 2|\alpha| > 0$ and $|1 - \overline{\omega}\tau| > 2|\alpha| > 0$. Noting that if λ is small, so also is α and that if λ is small, τ and $\overline{\omega}\tau$ are close to e^{it} and $\overline{\omega}e^{it}$, we see that the lemma follows by choosing e^{it} so that neither of these latter quantities are close to 1, and so that $e^{it}\lambda^1 \neq \lambda^2$. □

Lemma 4.8 *If B is a balanced subset of $\mathbb{D}^2$ and B has nonempty interior, then $B = \mathbb{D}^2$.*

Proof: Since automorphisms of $\mathbb{D}^2$ map balanced sets to balanced sets, we can assume that $(0,0)$ is in the interior of B. Let $E = \{\lambda \in \mathbb{D}^2 \ : \ |\lambda^1| = |\lambda^2|\}$. We claim that

$$E \subseteq B. \tag{4.9}$$

To see (4.9) fix $\lambda \in E$. Since $(0,0)$ is in the interior of B, there exists $\varepsilon > 0$ such that $\varepsilon\lambda \in B$. Now, B is balanced and $((0,0), \varepsilon\lambda)$ is a balanced pair. Hence $D = \left\{ \left(z, \frac{\lambda^2}{\lambda^1} z \right) \mid z \in \mathbb{D} \right\} \subseteq B$. In particular, $\lambda \in B$ and we have shown that (4.9) holds.

To see that $B = \mathbb{D}^2$, fix $\lambda \in \mathbb{D}^2 \setminus E$, choose an automorphism $m : \mathbb{D} \to \mathbb{D}$ such that $m(\lambda^1) = \lambda^2$ and let $D = \{(z, m(z)) \mid z \in \mathbb{D}\}$. Evidently, since $\lambda \in D$, Lemma 4.8 will follow if we can show that $D \subseteq B$. In turn, $D \subseteq B$ will follow from (4.9) and the fact that B is balanced if we can show that there exists 2 distinct points in $D \cap E$. Now, since $\lambda \notin E, m(0) \neq 0$. Hence sometimes $|z| < |m(z)|$ and sometimes $|z| > |m(z)|$. It follows from the Intermediate Value Theorem that there is in fact a continuum of points z such that $|z| = |m(z)|$ i.e. $(z, m(z)) \in D \cap E$.

This completes the proof of Lemma 4.8. $\square$

Proposition 4.10 *If B is a balanced subset of $\mathbb{D}^2$ and B contains a balanced pair of points, then either B is a balanced disk or $B = \mathbb{D}^2$.*

Proof: B contains a balanced pair, say, λ. Since B is balanced, $D_\lambda \subseteq B$. If $B = D_\lambda$, then B is a balanced disc. Thus, the proposition will follow if it were the case that the assumption $B \neq D_\lambda$ implies $B = \mathbb{D}^2$.

Accordingly, assume that $B \setminus D_\lambda \neq \emptyset$. By Lemma 4.1, there exists a balanced disc D with $D \neq D_\lambda$ and $D \cap D_\lambda \neq \emptyset$. By Lemma 4.3, B has nonempty interior. Hence Lemma 4.8 implies that $B = \mathbb{D}^2$ and the proof of Proposition 4.10 is complete. $\square$

5 Which Sets Have the Extension Property?

In this section we shall prove Theorem 1.20 from Section 1 of this paper.

Lemma 5.1 *Let $V \subseteq \mathbb{D}^2$ and assume that $V^\wedge \cap \mathbb{D}^2 = V$. If V has the polynomial extension property, then either V consists of a single point or V has no isolated points.*

Proof: The conclusion of the lemma is logically equivalent to the assertion that if V has an isolated point then V is a point. Accordingly, assume $\lambda_0 \in V$ is an isolated point of V. Now, the maximal ideal space of $P(V^-)$ is $V^\wedge$ and by hypothesis $V^\wedge \cap \mathbb{D}^2 = V$. Hence since λ_0 is an isolated point in V, λ_0

is an isolated point in the maximal ideal space of $P(V^-)$. It follows from the Shilov Idempotent Theorem (see *e.g.* [5]) that there exists $e \in P(V^-)$ with $e(\lambda_0) = 1$ and $e(\lambda) = 0$ for all $\lambda \in V^\wedge \setminus \{\lambda_0\}$. By approximating e we may find a polynomial p with $|p(\lambda_0)| > \frac{1}{2}$ and $|p| < \frac{1}{2}$ on $V^\wedge \setminus \{\lambda_0\}$. Now, necessarily $\sup_V |p| = |p(\lambda_0)|$. Hence the polynomial extension property implies that there exists $g \in H^\infty(\mathbb{D}^2)$ with $\sup\limits_{\mathbb{D}^2} |g| = |g(\lambda_0)|$. It follows from the maximum principle that g is constant. Thus, p is constant on V and $V^\wedge \setminus \{\lambda_0\}$ is empty. This means that, indeed, $V = \{\lambda_0\}$ and concludes the proof of Lemma 5.1. $\square$

Before continuing we remark that an alternate proof of Lemma 5.1 can be obtained by showing that the Shilov boundary of $P(V^-)$ must necessarily lie in the boundary of $\mathbb{D}^2$ if V has the polynomial extension property and then invoking the Hugo Rossi Local Maximum Principle. Either way it is curious that there appears to be no "elementary" proof of the lemma.

Lemma 5.2 *Let $V \subseteq \mathbb{D}^2$ and assume that $V^\wedge \cap \mathbb{D}^2 = V$. If V has the polynomial extension property and $\lambda_0 \in \mathbb{D}^2 \setminus V$, then there exists $h \in H^\infty(\mathbb{D}^2)$ with $h(\lambda_0) \neq 0$ and $h|V = 0$.*

Proof: Fix positive ε with $\varepsilon < 1$. Since $\lambda_0 \in \mathbb{D}^2 \setminus V, \lambda_0 \notin V^\wedge$. Hence there exists a polynomial p with $p(\lambda_0) = 1$ and $\sup\limits_V |p| \leq \varepsilon$. As V has the poynomial extension property, there exists G in $H^\infty(\mathbb{D}^2)$ with $g|_V = p$ and $\sup\limits_{\mathbb{D}^2} |g| \leq \varepsilon$. Setting $h = p - g$ we see immediately that $h \in H^\infty(\mathbb{D}^2), h(\lambda_0) \neq 0$ and $h|V = 0$. This establishes Lemma 5.2 $\square$

We recall for the reader that the intersection of an arbitrary collection of zero sets of holomorphic functions defined on a domain of holomorphy D is in fact analytic in D (see eg. [13]). Hence Lemma 5.2 has as an immediate corollary that any relatively polynomially convex subset of $\mathbb{D}^2$ with the polynomial extension property is in fact an analytic variety.

In the following lemma for $z \in \mathbb{C}$ and $\varepsilon > 0$ we let $\Delta_z(r) = \{w \in \mathbb{C} \mid |w - z| < \varepsilon\}$. The lemma may be regarded as an immediate consequence of the local descriptive theory of analytic varieties and we therefore omit its proof. We use $Zer(h)$ to denote the zero-set of h.

Lemma 5.3 *Let $h \in H^\infty(\mathbb{D}^2)$, assume that $h \not\equiv 0$, and fix $\lambda_0 \in \mathbb{D}^2$ with $h(\lambda_0) = 0$. There exists $n \geq 0$ and $\varepsilon_1, \varepsilon_2 > 0$ such that for all*

$z \in \Delta_{\lambda_0^1}(\varepsilon_1) \setminus \{\lambda_0^1\}$ *there exists a neighborhood* U *with* $U \subseteq \Delta_{\lambda_0^1}(\varepsilon_1) \setminus \{\lambda_0^1\}$ *of* z *and* n *distinct holomorphic functions* $f_1, \ldots f_n$ *defined on* U *with the property that*

$$Zer(h) \cap \Big(U \times \Delta_{\lambda_0^2}(\varepsilon_2)\Big) = \bigcup_{\ell=1}^{n} \{(z, f_\ell(z)) | z \in U\}.$$

Thus, the zero set $Zer(h)$ of an H^∞ function near a point λ_0 consists of n horizontally stretched "sheets" as described in Lemma 5.3. There may or may not be a vertical sheet $\{(\lambda_0^1, z) : z \in \mathbb{D}\}$ in $Zer(h)$, though if there isn't necessarily $h > 0$.

Armed with the preceding three lemmas we now are ready to commence the proof of Theorem 1.20 from the introduction. Accordingly, fix a nonempty relatively polynomially convex subset V of $\mathbb{D}^2$. If V has one of the forms (i) - (iv) of Theorem 1.20 then V is a retract, i.e. there exists a holomorphic mapping $\rho : \mathbb{D}^2 \to \mathbb{D}^2$ with $\rho \circ \rho = \rho$ and $\operatorname{ran} \rho = V$. Clearly if p is a polynomial, $p \circ \rho$ yields a norm preserving extension of p to $\mathbb{D}^2$ and thus V has the polynomial extension property.

To see the reverse direction assume V has the polynomial extension property. If V consists of a single point, then V is as in (i) and we are done. Also, since by Theorem 1.19, V is balanced, Proposition 4.10 implies that if V contains a balanced pair of points then either B is a balanced disk (in which case V is as in (iii) as well as (iv)) or V is $\mathbb{D}^2$ (in which case V is as in (ii)). Thus, we may make the following assumptions.

$$V \quad \text{contains more than one point.} \tag{5.4}$$

$$V \quad \text{contains no pair of balanced points.} \tag{5.5}$$

At this point we are able to describe how the proof of Theorem 1.20 will be consummated. Say a set $E \subseteq \mathbb{D}^2$ is an *extremal disc* if E has one of the forms

$$E = \{(z, f(z)) \mid z \in \mathbb{D}\}, \tag{5.6}$$

or

$$E = \{(f(z), z) \mid z \in \mathbb{D}\} \tag{5.7}$$

for some holomorphic mapping $f : \mathbb{D} \to \mathbb{D}$. If E is an extremal disk we say E *is type 1* if (5.6) holds and we say E *is type 2* if (5.7) holds. Extremal discs are so named because they are precisely the ranges of extremal functions for the Kobayashi extremal problems. Also, an extremal disk is both type 1 and type 2 if and only if it is a balanced disc.

We shall show in Lemma 5.8 below, that if (5.4) and (5.5) hold, then given any point $\lambda \in V$, there exists extremal discs $E \subseteq V$ that are arbitrarily close to λ. Since (5.4) implies via Lemma 5.1 that V has no isolated points, it will then follow that V is a union of extremal discs (Lemma 5.17). Finally, we shall show that V cannot contain more than one extremal disk (Lemma 5.20) and the proof of Theorem 1.20 will be complete.

Lemma 5.8 *Let $V \subseteq \mathbb{D}^2$, assume that $V^\wedge \cap \mathbb{D}^2 = V$, that V has the polynomial extension property, and that (5.4) and (5.5) hold. If $\varepsilon > 0$ and $\lambda_0 \in V$, then there exists $z_0 \in \mathbb{D}$ $\mu_0 \in V$ and holomorphic $f : \mathbb{D} \to \mathbb{D}$ with $\|\mu_0 - \lambda_0\| < \varepsilon$ and such that either*

$$(i) \quad \{(z, f(z)) \mid z \in \mathbb{D}\} \subseteq V \, and \, (z_0, f(z_0)) = \mu_0.$$

or

$$(ii) \quad \{(f(z), z) \mid z \in \mathbb{D}\} \subseteq V \, and \, (f(z_0, z_0) = \mu_0.$$

Proof: Assume that V satisfies the hypotheses of the lemma, let $\varepsilon > 0$ and fix $\lambda_0 \in V$. By Lemma 5.2 there exists $h \in H^\infty(\mathbb{D}^2)$ with $h \neq 0$ and $V \subseteq Zer(h)$. An application of Lemma 5.3 to h yields $n \geq 0$ and $\varepsilon_1, \varepsilon_2 > 0$ such that for all $z_0 \in \Delta_{\lambda_0^1}(\in_1) \setminus \{\lambda_0^1\}$ there exists a neighborhood U_{z_0} of z_0 with $U_{z_0} \subseteq \Delta_{\lambda_0^1}(\in_1) \setminus \{\lambda_0^1\}$ and n distinct holomorphic functions $f_1, \ldots, f_n$ on U_{z_0} with the property that

$$(5.9) \qquad Zer(h) \cap \Big(U_{z_0} \times \Delta_{\lambda_0^2}(\in_2)\Big) = \bigcup_{\ell=1}^{n} \{(z, f_\ell(z)) \mid z \in U_{z_0}\}.$$

Now, by Lemma 5.1, λ_0 is not an isolated point of V. Thus, either

(5.10) there is a sequence $\{w_n\} \subseteq \mathbb{D}$ with $w_n \to \lambda_0^2$ and $(\lambda_0^1, w_n) \in V$ $\forall n$

or,

(5.11) there is a point $\mu_0 \in V$ with $\mu_0^1 \neq \lambda_0^1$ and $\|\mu_0 - \lambda_0\| < \min\{\varepsilon, \varepsilon_1, \varepsilon_2\}$

If (5.10) obtains, then it is easy to see that the conclusion of Lemma 5.8 holds. Simply note that if $g \in H^\infty(\mathbb{D}^2)$ and $g|V = 0$, then (5.10) implies that $g(\lambda_0^1, w_n) = 0$ for all n. Hence since $w_n \to \lambda_0^2 \in \mathbb{D}, g(\lambda_0^1, w) = 0$ for all $w \in \mathbb{D}$. But this implies via Lemma 5.2 that $(\lambda_0^1, w) \in V$ for all $w \in \mathbb{D}$. But this implies that (ii) holds with $f(z) \equiv \lambda_0^1, \mu_0 = \lambda_0$, and $z_0 = \lambda_0^2$ and establishes Lemma 5.8 in the case when (5.10) holds.

To handle the case when (5.11) holds is more subtle. First note that since $\mu_0^1 \neq \lambda_0^1$ and $\|\mu_0 - \lambda_0\| < \min\{\varepsilon, \varepsilon_1, \varepsilon_2\}$, we have $\mu_0^1 \in \Delta_{\lambda_0^1}(\varepsilon_1) \setminus \{\lambda_0^1\}$ and $\mu_0 \in Zer(h) \cap (U_{\mu_0^1} \times \Delta_{\lambda_0^1}(\varepsilon_1))$. Thus, noting that $V \subseteq Zer(h)$ and that Lemma 5.1 implies μ_0 is not an isolated point of V, we see from (5.9) (with $z_0 = \mu_0^1$), that there exists ℓ and a sequence $\{z_i\}$ with $z_i \to \mu_0^1, f_\ell(z_i) \to \mu_0^2$ and such that both $z_i \in U_{\mu_0^1}$ and $(z_i, f_\ell(z_i)) \in V$.

Let G_0 denote the component of $U_{\mu_0^1}$ containing μ_0^1. We claim that $(z, f_\ell(z)) \in V$ for all $z \in G_0$. To see this we shall use Lemma 5.2. Thus, let $g \in H^\infty(\mathbb{D}^2)$ with $g|_V = 0$. Define a holomorphic function φ on G_0 by $\varphi(z) = g(z, f_\ell(z))$. Indeed, φ is a well defined holomorphic function since (5.9) implies that $(z, f_\ell(z)) \subseteq \mathbb{D}^2$ if $z \in U_{\mu_0^1} \supseteq G_0$.

Since $z_i \to \mu_0^1, z_i \in G_0$ for sufficiently large i. Also, $(z_i, f_\ell(z_i)) \in V$ so $\varphi(z) = g(z, f(z)) = 0$ for i sufficiently large. As φ is holomorphic and G_0 is connected these facts imply that $\varphi(z) = 0$ for all $z \in G_0$. Therefore, if $g \in H^\infty(\mathbb{D}^2)$ and $g|_V = 0$, then $g(z, f_\ell(z)) = 0$ whenever $z \in G_0$. Thus, we see from Lemma 5.2 that indeed $(z, f_\ell(z)) \in V$ for all $z \in V$.

Recapping, we have shown that there is a $\mu_0 \in V$, a connected neighborhood G_0 of μ_0^1, and a holomorphic $f_\ell : G_0 \to \mathbb{D}$ with such that $\|\mu_0 - \lambda_0\| < \varepsilon$ and such that

$$\{(z, f(z)) \mid z \in G_0\} \subseteq V$$

and

$$\left(\mu_0^1, f_\ell(\mu_0^1)\right) = \mu_0.$$

Notice how similar this is to (i) in the conclusion of Lemma 5.8 (with $z_0 = \mu_0^1$). Indeed, we would have established the lemma if it were the case that f_ℓ had an analytic continuation $\tilde{f}_\ell$ to the entire disk with $\operatorname{ran} \tilde{f} \subseteq \mathbb{D}$. This however does not have to be the case in general. The next part of the proof of Lemma 5.8 addresses this issue.

Define a set S in $\mathbb{C}^2$ by letting $S = \{(z, w) \mid z, w \in G_0 \text{ and } z \neq w\}$ and define a real valued function ρ on S by letting

$$\rho(z, w) = d(z, w) - d(f_\ell(z), f_\ell(w)).$$

Since G_0 is connected and open, S is also connected. Furthermore, notice that since $(z, f_\ell(z)) \in V$ whenever $z \in G_0$, (5.5) implies that $\rho(z,w) \neq 0$ for all $(z,w) \in S$. Since ρ is continuous it follows from the Intermediate Value Theorem that either

$$\rho(z,w) > 0 \quad \text{for all} \quad (z,w) \in S, \tag{5.12}$$

or

$$\rho(z,w) < 0 \quad \text{for all} \quad (z,w) \in S. \tag{5.13}$$

Now, if (5.12) holds then, indeed, f_ℓ does extend to a holomorphic mapping $f : \mathbb{D} \to \mathbb{D}$. To see this consider function elements (G, f_G), with G a connected open subset of $\mathbb{D}$ such that $G_0 \subseteq G$ and $f_G : G \to \mathbb{D}$ a holomorphic function with $f_G \mid_{G_0} = f_\ell$.

We first claim that

$$(z, f_G(z)) \in V \quad \text{for all} \quad z \in G. \tag{5.14}$$

To see (5.14), let G_1 be the component of $\{z \in G \mid (z, f_G(z)) \in V\}$ that contains G_0. The proof of (5.14) will follow if we can show that (5.14) is both open and closed as a subset of G. That G_1 is open follows from Lemmas 5.2 and 5.3. That G_1 is closed follows from the fact that $V^\wedge \cap \mathbb{D}^2 = V$.

Next we claim that if (G, f_G) is a function element then

(5.15) $\quad f_G$ is strictly contractive on G in the pseudo hyperbolic metric.

To prove (5.15) note that we are assuming that (5.12) holds i.e. f_G is contractive on G_0. Just as in the proof that either (5.12) or (5.13) hold we see that $\rho > 0$ on the set $\{(z,w) \mid (z,w) \in G \text{ and } z \neq w\}$. Hence (5.15) obtains.

We next claim that if $(G_1, f_{G_1}), (G_2, f_{G_2})$ are two function elements, then

$$f_{G_1} = f_{G_2} \quad \text{on} \quad G_1 \cap G_2. \tag{5.16}$$

To see (5.16) we argue by contradiction. Thus, assume that $z_1 \in G_1 \cap G_2$ with $f_{G_1}(z_1) \neq f_{G_2}(z_1)$. Choose a curve $z : [0,1] \to G_2$ with $z(0) \in G_0$ and $z(1) = z_1$. Define $\psi : [0,1] \to \mathbb{R}$ by

$$\psi(t) \;=\; d(z(t), z_1) - d(f_{G_2} z(t), f_{G_1}(z_1)).$$

When $t = 0, f_{G_2}(z_t) = f_{G_2}(z_0) = f_{G_1}(z_0)$ since $f_{G_1} = f_{G_2}$ on G_0 and $z_0 \in G_0$. Hence $\psi(0) > 0$ by (5.15). On the other hand, $\psi(1) < 0$ since $z(1) = z_1$ and $f_{G_1}(z_1) \neq f_{G_2}(z_1)$. Hence there exists t_0 such that $\psi(t_0) = 0$. But this says that the pair of points $(z(t_0), f_{G_2}(z(t_0)))$ and $(z_1, f_{G_1}(z_1))$, which by (5.14) are in V, is a balanced pair. This contradiction to (5.5) establishes (5.16).

Finally, note that (5.16) allows us to construct a maximal function element (G, f_G). We claim that $G = \mathbb{D}$. Indeed, if $w \in \partial G \cap \mathbb{D}$, one of two things must have happened. Either

$$\lim_{G \ni z \to w} |f(z)| = 1,$$

which would contradict 5.15; or f_G has a singularity at w. By Lemmas 5.2 and 5.3, this singularity is isolated. Since isolated singularities of bounded holomorphic functions are removable, $\partial G \cap \mathbb{D} = \emptyset$.

We are finally able to complete the proof of Lemma 5.8. If (5.12) holds, then take $f = f_G$ where Then G is as in the previous paragraph. (i) holds by (5.14). If (5.13) holds then execute the argument just carried out under the assumption that (5.12) held but with the coordinates interchanged, *i.e.* G_0 replaced with an appropriate connected neighborhood of μ_0^2, f_{G_0} replaced with f_ℓ^{-1} and the manifolds $(z, f_G(z))$ replaced by $(f_G(z), z)$. This yields that (ii) holds and completes the proof of Lemma 5.8. □

Lemma 5.17 *If V satisfies the hypotheses of Lemma 5.8, then V is a union of extremal discs.*

Proof: Fix $\lambda \in V$. By Lemma 5.8, there exists sequences $\{z_n\}, \{\lambda_n\}$, and $\{f_n\}$ such that $\lambda_n \to \lambda$ and such that either

$$\{(z, f_n(z)) | z \in \mathbb{D}\} \subseteq V \quad \text{and} \quad (z_n, f_n(z_n)) = \lambda_n \quad \text{for all } n, \tag{5.18}$$

or

$$\{(f_n(z), z) | z \in \mathbb{D}\} \subseteq V \quad \text{and} \quad (f_n(z_n), z_n) = \lambda_n \quad \text{for all } n. \tag{5.19}$$

Suppose (5.18) holds. Evidently, $z_n \to \lambda^1$ and $f_n(z_n) \to \lambda^2$. Furthermore, since $\sup |f_n| \leq 1$, by passing to a subsequence if necessary we may assume that $f_n \to f$ uniformly on compact subsets of $\mathbb{D}$. Necessarily,

$$\{(z, f(z)) \mid z \in \mathbb{D}\} \subseteq V$$

(since V is a closed subset of $\mathbb{D}^2$) and obviously, $(\lambda^1, f(\lambda^1)) = \lambda$. Thus, λ is in a subset of V which is an extremal disk of type 1. Likewise, if we assume that (5.19) holds, then λ is in a subset of V which is an extremal disk of type 2. This completes the proof of Lemma 5.17. $\square$

In light of Lemma 5.17 the proof of Theorem 1.20 will be complete once we have established our final lemma.

Lemma 5.20 *If V satisfies the hypotheses of Lemma 5.8, then V cannot contain more than one extremal disc.*

Proof: There are two cases to rule out: V contains two extremal discs of the same type and V contains two extremal discs of different type.

First suppose V contain two distinct discs of type 1

$$E_f = \{(z, f(z)) | z \in \mathbb{D}\} \text{ and } E_g = \{(z, g(z)) | z \in \mathbb{D}\}.$$

We claim that

$$d(z, w) < d(f(z), g(w)) \text{ whever } z \neq w \text{ or } f(z) \neq g(w). \tag{5.21}$$

To prove (5.21) let $S = \{(z, w) | z \neq w \text{ or } f(z) \neq g(w)\}$, define $\rho : S \to \mathbb{R}$ by $\rho(z, w) = d(f(z), g(w)) - d(z, w)$. Since $E_f \neq E_g, f \neq g$ and we see that for some $z_0, (z_0, z_0) \in S$ and $\rho(z_0, z_0) > 0$. Also, by (5.5), $\rho(z, w) \neq 0$ for all $(z, w) \in S$. It follows using the fact that S is connected and ρ is continuous that $\rho > 0$ on all of S i.e. (5.21) holds.

Armed with (5.21) we are able to show that $f = g$ in the following way. Alternately fixing w and letting $z \to \partial\mathbb{D}$ and fixing z and letting $w \to \partial\mathbb{D}$ immediately implies via (5.21) that f and g are inner. Furthermore since $|f|$ and $|g|$ have unimodular limits along all paths to $\partial\mathbb{D}$ in fact f and g are finite Blaschke products. Observe that if z is a zero of f and w is a zero of g, then (5.21) implies that $z = w$. Thus, f and g are automorphisms of $\mathbb{D}$. Letting $w_n \to g^{-1}(f(z))$ in such a way that $w_n \neq z$ and $w_n \neq g^{-1}(f(z))$ reveals via (5.21) that $g^{-1}(f(z)) = z$ i .e. $f = g$. Thus, we see that E_f and E_g are not distinct after all, a contradiction that establishes U does not contain two extremal discs of type 1. Of course a nearly identical argument shows that V cannot contain two extremal discs of type 2 as well.

Now suppose V contains two distinct extremal discs of opposite type say $E_f = \{(z, f(z)) | z \in \mathbb{D}\}$ and $E_g = \{(g(z), z) | z \in \mathbb{D}\}$. Arguing as in the previous part of the proof gives that either

$$d(z, g(w)) < d(f(z), w) \text{ whenever } z \neq g(w) \text{ or } w \neq f(z) \tag{5.22}$$

or

$$d(f(z)w) < d(z, g(w)) \text{ whenever } z \neq g(w) \text{ or } w \neq f(z) \tag{5.23}$$

If (5.22) holds, then choosing $w_n \to f(z)$ in such a way that $w_n \neq f(z)$ and $g(w_n) \neq z$ gives that $g(f(z)) = z$. But this implies that $E_f = E_g$ a contradiction. Likewise, if (5.23) holds, we reach a contradiction and Lemma 5.20 is established. □

6 Appendix: By Pascal J. Thomas

The following direct function theoretic proof of Theorem 1.19 is due to Pascal J. Thomas, of Université Paul Sabatier. It is based in part on an argument, in the context of the unit ball, that Jean-Pierre Rosay showed Thomas in a conversation in 1993.

Proof: First, note that (p_1, p_2) is a balanced pair of points if and only if there exist $m_1, m_2 \in \mathcal{M}$ two automorphisms of the unit disk such that $m_1(p_1^1) = m_2(p_2^1) = 0$ and $m_1(p_1^2) = m_2(p_2^2)$. Polynomials themselves are not invariant under automorphisms of the bidisk, but any composition of a polynomial by such an automorphism is easily seen to be uniformly approximable by polynomials on the closed bidisk. So we might reduce ourselves to the case $p_1 = (0,0), p_2 = (\lambda, \lambda)$. In that case, the unique totally geodesic complex disk going through both points is the diagonal $\Delta := \{(z,z) : z \in \mathbb{D}\}$.

By the hypothesis that $\overline{V}$ is polynomially convex, to prove that $\Delta \subset V$, it will be enough to prove that $\overline{V} \supset \partial\Delta := \{(z,z) : z \in \partial\mathbb{D}\}$. Suppose not, then there exists θ_0 and $\delta > 0$ such that $\overline{V} \subset \overline{\mathbb{D}^2} \setminus D(e^{i\theta_0}, \delta)^2 =: K$.

Consider the holomorphic retraction from $\mathbb{D}^2$ onto Δ (or rather, the unit disk parametrizing Δ) given by $\pi(z,w) := \frac{1}{2}(z+w)$. By strict convexity of the unit disk, $\pi^{-1}\{e^{i\theta_0}\} \cap \overline{\mathbb{D}^2} = \{(e^{i\theta_0}, e^{i\theta_0})\}$, so $(e^{i\theta_0}, e^{i\theta_0}) \notin \pi(K)$ and by compactness, there exists $\epsilon > 0$ so that $\pi(V) \subset \mathbb{D} \setminus D(e^{i\theta_0}, \epsilon) =: \Omega$.

Now there exists a holomorphic function f on Ω such that $f(\Omega) \subset \mathbb{D}$, $f(0) = 0$ and $|f(\lambda)| > |\lambda|$; take for instance f the conformal mapping from Ω to $\mathbb{D}$, the property is then simply the Schwarz Lemma applied to the map f^{-1} at the point $f(\lambda)$. I claim that the map $(f \circ \pi)|_V$, which has uniform norm bounded by 1, cannot extend to any $\tilde{f} \in H^\infty(\mathbb{D}^2)$ with $\|\tilde{f}\|_\infty \leq 1$.

Indeed, if such an $\tilde{f}$ existed, then $f_1(z) := \tilde{f}(z,z)$ would give a function from the unit disk to itself such that $f_1(0) = 0$, but by the extension property $f_1(\lambda) = f(\lambda)$, which violates the Schwarz lemma.

To see that the *polynomial* extension property is violated, it is enough to approximate f by a polynomial P (by Mergelyan's theorem), so that $\sup_{z\in\Omega} |P(z)| \leq 1$ and $|P(\lambda)| > |\lambda|$, and then run the same argument with $(P \circ \pi)|_V$. $\square$

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