

Putnam Practice Problems November 20, 2015

0 [Warm-up - from Car talk] You're one of a hundred people standing in line to get onto an airplane that has 100 seats. There's a seat for every person who's in line, and each of you has a boarding pass for your assigned a seat. The first person to walk onto the plane drops his boarding pass and, instead of picking it up, decides, "I'm just going to sit anyplace." He takes a seat at random.

Now, every other passenger will take either his assigned seat or, if that seat is taken, that passenger will take any seat at random. You are the last passenger to walk onto the plane. Obviously, there's going to be one seat left, because everyone else is sitting in his correct seat, or not.

The question is: What are the chances that you get to sit in your assigned seat?

2008-A-2 Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

2011-A-4 For which positive integers n is there an $n \times n$ matrix with integer entries such that every dot product of a row with itself is even, while every dot product of two different rows is odd?

2011-B-4 In a tournament, 2011 players meet 2011 times to play a multiplayer game. Every game is played by all 2011 players together and ends with each of the players either winning or losing. The standings are kept in two 2011×2011 matrices, $T = (T_{hk})$ and $W = (W_{hk})$. Initially, $T = W = 0$. After every game, for every (h, k) (including for $h = k$), if players h and k tied (that is, both won or both lost), the entry T_{hk} is increased by 1, while if player h won and player k lost, the entry W_{hk} is increased by 1 and W_{kh} is decreased by 1.

Prove that at the end of the tournament, $\det(T + iW)$ is a non-negative integer divisible by 2^{2010} .

2010-B-6 Let A be an $n \times n$ matrix of real numbers for some $n \geq 1$. For each positive integer k , let $A^{[k]}$ be the matrix obtained by raising each entry to the k^{th} power. Show that if $A^k = A^{[k]}$ for $k = 1, 2, \dots, n + 1$, then $A^k = A^{[k]}$ for all $k \geq 1$.

2009-A-3 Let d_n be the determinant of the $n \times n$ matrix whose entries, from left to right and then from top to bottom, are $\cos 1, \cos 2, \dots, \cos n^2$. (For example,

$$d_3 = \begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}.$$

The argument of \cos is always in radians, not degrees.) Evaluate $\lim_{n \rightarrow \infty} d_n$.

2009-B-4 Say that a polynomial with real coefficients in two variables, x, y , is *balanced* if the average value of the polynomial on each circle centered at the origin is 0. The balanced polynomials of degree at most 2009 form a vector space V over \mathbb{R} . Find the dimension of V .

2005-A-4 Let H be an $n \times n$ matrix all of whose entries are ± 1 and whose rows are mutually orthogonal. Suppose H has an $a \times b$ submatrix whose entries are all 1. Show that $ab \leq n$.

1996-B-4 For any square matrix A , we can define $\sin A$ by the usual power series:

$$\sin A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1}.$$

Prove or disprove: there exists a 2×2 matrix A with real entries such that

$$\sin A = \begin{pmatrix} 1 & 1996 \\ 0 & 1 \end{pmatrix}.$$

1995-B-3 To each positive integer with n^2 decimal digits, we associate the determinant of the matrix obtained by writing the digits in order across the rows. For example, for $n = 2$, to the integer 8617 we associate $\det \begin{pmatrix} 8 & 6 \\ 1 & 7 \end{pmatrix} = 50$. Find, as a function of n , the sum of all the determinants associated with n^2 -digit integers. (Leading digits are assumed to be nonzero; for example, for $n = 2$, there are 9000 determinants.)

2003-B-1 Do there exist polynomials $a(x), b(x), c(y), d(y)$ such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$

holds identically?

- 2006-B-4 Let Z denote the set of points in \mathbb{R}^n whose coordinates are 0 or 1. (Thus Z has 2^n elements, which are the vertices of a unit hypercube in \mathbb{R}^n .) Given a vector subspace V of \mathbb{R}^n , let $Z(V)$ denote the number of members of Z that lie in V . Let k be given, $0 \leq k \leq n$. Find the maximum, over all vector subspaces $V \subseteq \mathbb{R}^n$ of dimension k , of the number of points in $V \cap Z$.
- 2002-A-4 In Determinant Tic-Tac-Toe, Player 1 enters a 1 in an empty 3×3 matrix. Player 0 counters with a 0 in a vacant position, and play continues in turn until the 3×3 matrix is completed with five 1's and four 0's. Player 0 wins if the determinant is 0 and player 1 wins otherwise. Assuming both players pursue optimal strategies, who will win and how?
- 1999-B-5 For an integer $n \geq 3$, let $\theta = 2\pi/n$. Evaluate the determinant of the $n \times n$ matrix $I + A$, where I is the $n \times n$ identity matrix and $A = (a_{jk})$ has entries $a_{jk} = \cos(j\theta + k\theta)$ for all j, k .