1. Let $z_1, z_2 \in \mathbb{C}$, and let $\rho(z_1, z_2)$ be the Euclidean distance between their stereographic images on the unit sphere. Prove

$$
\rho(z_1, z_2) = \frac{2|z_1 - z_2|}{\sqrt{1 + |z_1|^2} \sqrt{1 + |z_2|^2}}.
$$

2. The extended complex plane is $\mathbb{C} \cup \{\infty\}$, where $\infty$ is called the point at infinity, and the stereographic projection maps it to the north pole. Prove

$$
\rho(z, \infty) = \frac{2}{\sqrt{1 + |z|^2}}.
$$

3. Prove that

$$
\frac{\partial}{\partial z} z^m \overline{z}^n = mz^{m-1} \overline{z}^n
$$

$$
\frac{\partial}{\partial \overline{z}} z^m \overline{z}^n = nz^{m-1} \overline{z}^{n-1}.
$$

4. Prove that the polar form of the Cauchy-Riemann equations are

$$
r \frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta}, \quad \frac{\partial u}{\partial \theta} = -r \frac{\partial v}{\partial r}.
$$

Hint:

$$
\nabla u = (\cos \theta \frac{\partial u}{\partial r} - \frac{1}{r} \sin \theta \frac{\partial u}{\partial \theta}, \sin \theta \frac{\partial u}{\partial r} + \frac{1}{r} \cos \theta \frac{\partial u}{\partial \theta})^t.
$$

5. Let $f$ be holomorphic on $\mathbb{D}$. Prove that each of the following conditions force $f$ to be constant.

(i) $f'(z) = 0 \ \forall \ z \in \mathbb{D}$.

(ii) $f$ is real-valued in $\mathbb{D}$.

(iii) $|f|$ is constant in $\mathbb{D}$.

(iv) $\arg(f)$ is constant in $\mathbb{D}$.

6. Suppose $f$ is holomorphic on the open set $\Omega$. Prove that $g(z) = \overline{f(\overline{z})}$ is holomorphic on $\Omega^* = \{z : \overline{z} \in \Omega\}$. 