416: Homework 3 Due: 19/September

1. (i) Prove that the Laplacian can be written as

$$\Delta = 4\frac{\partial^2}{\partial z \partial \bar{z}}.$$

(ii) Suppose f is holomorphic. What is $\Delta|f(z)|^2$?

2. Suppose u is a real-valued harmonic function on an open disk D, and v is a harmonic conjugate of u. Prove that uv and $u^2 - v^2$ are harmonic.

3. Prove that if u is a real-valued harmonic function, then $\frac{\partial u}{\partial z}$ is holomorphic.

4. Suppose u is real-valued harmonic on an open disk D, and so is u^2. Prove that u is constant.

5. Prove that if z_1, z_2, z_3 and w_1, w_2, w_3 are both triples of distinct points in $\overline{\mathbb{C}}$, then there exists a linear fractional transformation ϕ such that $\phi(z_j) = w_j$ for $j = 1, 2, 3$. Prove that ϕ is unique.