Math 415 Assignment 10
Due Thursday, December 1, at the beginning of class

Question 1. Let

$$S_N(x) = \sum_{k=0}^{N} \frac{2}{(2k+1)\pi} \sin((2k+1)x)$$

be the partial sums of the Fourier series of the step function which we studied in class. Moreover, let

$$F_N(x) = \frac{1}{N+1} \sum_{n=0}^{N} S_n(x)$$

be the corresponding arithmetic means. For $N = 5, 10, 20$, sketch S_N and F_N, for example with the help of a computer. Can you see the Gibbs phenomenon for F_N as well?

Question 2. Let V be a vector space over the complex numbers, let $\langle \cdot, \cdot \rangle$ be an inner product on V and let $||v|| = \sqrt{\langle v, v \rangle}$ for $v \in V$.

(a) Show that $|\langle v, w \rangle| \leq ||v|| \, ||w||$ for all $v, w \in V$.
(*Hint: Expand* $||v + tw||^2$ *for scalars* t *and choose* $t = -\frac{\langle v,w \rangle}{\langle w,w \rangle}$ *if* $w \neq 0$).

(b) Show that if $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are sequences of complex numbers, then

$$\sum_{n=1}^{\infty} |a_n b_n| \leq \left(\sum_{n=1}^{\infty} |a_n|^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} |b_n|^2 \right)^{1/2}.$$

(*Hint: Use part (a) for the usual inner product on* $\mathbb{C}^N$ *and let* $N \to \infty$).

Question 3. Use Parseval's equality to show that if f is a C^1 function on $[-\pi, \pi]$ which satisfies $f(-\pi) = f(\pi)$ and $\int_{-\pi}^{\pi} f(x)\, dx = 0$, then

$$\int_{-\pi}^{\pi} |f(x)|^2 \, dx \leq \int_{-\pi}^{\pi} |f'(x)|^2 \, dx.$$

Question 4. Let $p(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial in a complex variable $z = x + iy$ with complex coefficients. Write

$$p(x + iy) = u(x, y) + iv(x, y),$$

where u and v are functions from $\mathbb{R}^2$ to $\mathbb{R}$.

(a) Show that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

(b) Deduce that u and v satisfy the Laplace equation.

Continued on the back.

Question 5. Let f be a C^1 function on $\mathbb{R}$ which is periodic with period 2π and let

$$\sum_{n=-\infty}^{\infty} C_n e^{inx}$$

be the Fourier series of f in complex form. The goal of this question is to give another proof of the fact that this series converges to f pointwise. We assume that $f(0) = 0$ and we only consider the Fourier series at $x = 0$. (One can show that this is enough, by adding a constant to f and by applying a translation.)

(a) Define a function g on $[-\pi, \pi] \setminus \{0\}$ by $g(x) = \frac{f(x)}{e^{ix}-1}$. Show that g can be extended continuously to the point $x = 0$.
(b) Let D_n be the complex Fourier coefficients of g. Show that $D_n \to 0$ as $n \to \infty$ and $n \to -\infty$.
(c) Show that $C_n = D_{n-1} - D_n$ for all $n \in \mathbb{Z}$.
(d) Deduce that the Fourier series of f at $x = 0$ converges to zero.