Math 415 Assignment 3
Due Thursday, September 22, at the beginning of class

The wave equation is $u_{tt} = c^2 u_{xx}$.

Question 1. Solve the wave equation with the following initial conditions.

(a) $u(x,0) = \cos(x), u_t(x,0) = x$.
(b) $u(x,0) = x^2, u_t(x,0) = \frac{1}{1+x^2}$.

Question 2. Consider the PDE

$$u_{xx} - 5u_{xy} + 6u_{yy} = 0$$

(a) Determine if the PDE is elliptic, parabolic or hyperbolic.
(b) Find the general solution of the PDE (*Hint: Make an appropriate change of variables first*).

Question 3. For a solution u of the wave equation with $c = 1$, the energy density is defined as $e = \frac{1}{2}(u_t^2 + u_x^2)$ and the momentum density as $p = u_t u_x$.

(a) Verify that

$$\frac{\partial e}{\partial t} = \frac{\partial p}{\partial x} \quad \text{and} \quad \frac{\partial p}{\partial t} = \frac{\partial e}{\partial x}.$$

(b) Verity that e and p also satisfy the wave equation.

Question 4. Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is said to be *even* if $f(-x) = f(x)$ for all $x \in \mathbb{R}$. Consider the wave equation with initial conditions

$$u(x,0) = \varphi(x) \quad \text{and} \quad u_t(x,0) = \psi(x).$$

Show that if φ and ψ are even functions, then the solution $u(x,t)$ is an even function in x for every t.

Question 5. The dampened wave equation is

$$u_{tt} - c^2 u_{xx} + r u_t = 0,$$

where $r > 0$ and $c^2 = \frac{T}{\rho}$. Recall from class that the energy of the wave at time t is

$$E(t) = \frac{1}{2} \int_{-\infty}^{\infty} (\rho u_t^2 + T u_x^2)\, dx.$$

Show that $E(t)$ is descreasing (assuming that for fixed t, the function u vanishes outside some interval).