Let
\[
\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2} \, dp.
\]
You may use without proof that
\[
\lim_{x \to \infty} \text{erf}(x) = 1.
\]

**Question 1.** Solve the diffusion equation \( u_t = u_{xx} \) on the whole line with initial condition \( u(x,0) = e^{2x} \).

**Question 2.** Solve the diffusion equation \( u_t = u_{xx} \) on the half line \( x > 0 \) with boundary condition \( u(0,t) = 0 \) and initial condition \( u(x,0) = e^{-x} \). Express your solution in terms of erf.

**Question 3.** A wave \( f(x+ct) \) travels along a semi-infinite string \( (0 < x < \infty) \) for \( t < 0 \). Find the vibrations \( u(x,t) \) of the string for \( t > 0 \) if the end \( x = 0 \) is fixed.

**Question 4.** Let \( u \) be a solution of the diffusion equation \( u_t = ku_{xx} \) for \( 0 < x < 1 \) and \( 0 < t < 1 \) with \( u(0,t) = u(1,t) = 0 \) and \( u(x,0) = x(1-x) \). Show that \( u(x,t) \leq \frac{1}{2} \) for all \( 0 < x < 1 \) and \( 0 < t < 1 \).

**Question 5.** Let \( u \) be a solution of the diffusion equation \( u_t = ku_{xx} \) for \( -1 < x < 1 \) and \( 0 < t < 1 \) with boundary condition \( u(-1,t) = u(1,t) = g(t) \) and initial condition \( u(x,0) = \varphi(x) \). Show that if \( \varphi \) is an even function, then \( u(x,t) \) is an even function in \( x \) for every \( t \).

**Hint:** Consider \( v(x,t) = u(x,t) - u(-x,t) \) and use uniqueness of solutions.