Math 415 Assignment 7
Due Thursday, November 3, at the beginning of class

Question 1. Consider $\phi(x) = x^2$ on $[0, 1]$.

(a) Compute the Fourier sine series of ϕ.
(b) Compute the Fourier cosine series of ϕ.

Question 2. Show the following identites for $n, m \geq 1$:

$$\int_{-l}^{l} \cos\left(\frac{n\pi x}{l}\right) \sin\left(\frac{m\pi x}{l}\right) dx = 0$$

$$\int_{-l}^{l} \cos\left(\frac{n\pi x}{l}\right) \cos\left(\frac{m\pi x}{l}\right) dx = \int_{-l}^{l} \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{m\pi x}{l}\right) dx = \begin{cases} l, & \text{if } n = m \\ 0, & \text{if } n \neq m. \end{cases}$$

Question 3. In class, when studying the Robin boundary condition, we encountered the eigenvalue problem

$$\begin{aligned} X'' &= -\lambda X \\ X'(0) - a_0 X(0) &= 0 \\ X'(l) + a_l X(l) &= 0, \end{aligned}$$

where X is defined on $[0, l]$. Show that $\lambda = 0$ is an eigenvalue if and only if $a_0 + a_l = -a_0 a_l$.

Question 4. Consider the wave equation $u_{tt} = c^2 u_{xx}$ for $0 < x < l$ with Robin boundary conditions

$$u_x(0, t) - a_0 u(0, t) = 0 \quad \text{and} \quad u_x(l, t) + a_l u(l, t) = 0.$$

Let

$$E(t) = \frac{1}{2} \int_0^l (c^{-2} u_t^2 + u_x^2)\, dx + \frac{1}{2} a_l u(l, t)^2 + \frac{1}{2} a_0 u(0, t)^2.$$

(a) Show that E is constant.
(b) (Bonus) Suppose that $a_0, a_l \geq 0$. Deduce uniqueness for the wave equation on $0 < x < l$ with Robin boundary conditions and initial conditions

$$u(x, 0) = \phi(x) \quad \text{and } u_t(x, 0) = \psi(x).$$

Question 5. The vibrations of a tuning fork may be described by the PDE

$$u_{tt} + c^2 u_{xxxx} = 0$$

with boundary conditions

$$u(0, t) = u_x(0, t) = 0 \quad \text{and} \quad u_{xx}(l, t) = u_{xxx}(l, t) = 0.$$

(a) Separate the variables to get the eigenvalue problem $X'''' = \lambda X$.
(b) Assume that $\lambda > 0$ and write $\lambda = \beta^4$. Show that $\cosh(\beta l) \cos(\beta l) = -1$.
(c) Find approximate values of the first two solutions β (for example using a computer).
(d) Compute the corresponding frequencies of vibration ω_1 and ω_2 and determine the ratio ω_2/ω_1. Compare this result with the vibrating string with fixed ends.