Math 415  Assignment 7
Due Thursday, November 3,
at the beginning of class

Question 1. Consider \( \phi(x) = x^2 \) on \([0, 1]\).
(a) Compute the Fourier sine series of \( \phi \).
(b) Compute the Fourier cosine series of \( \phi \).

Question 2. Show the following identities for \( n, m \geq 1 \):
\[
\int_{-l}^{l} \cos \left( \frac{n\pi x}{l} \right) \sin \left( \frac{m\pi x}{l} \right) \, dx = 0
\]
\[
\int_{-l}^{l} \cos \left( \frac{n\pi x}{l} \right) \cos \left( \frac{m\pi x}{l} \right) \, dx = \int_{-l}^{l} \sin \left( \frac{n\pi x}{l} \right) \sin \left( \frac{m\pi x}{l} \right) \, dx = \begin{cases} l, & \text{if } n = m \\ 0, & \text{if } n \neq m \end{cases}
\]

Question 3. In class, when studying the Robin boundary condition, we encountered the eigenvalue problem
\[
X'' = -\lambda X
\]
\[
X'(0) - a_0 X(0) = 0
\]
\[
X'(l) + a_l X(l) = 0,
\]
where \( X \) is defined on \([0, l]\). Show that \( \lambda = 0 \) is an eigenvalue if and only if \( a_0 + a_l = -a_0 a_l \).

Question 4. Consider the wave equation \( u_{tt} = c^2 u_{xx} \) for \( 0 < x < l \) with Robin boundary conditions
\[
u_x(0, t) - a_0 u(0, t) = 0 \quad \text{and} \quad u_x(l, t) + a_l u(l, t) = 0.
\]
Let
\[
E(t) = \frac{1}{2} \int_{0}^{l} \left( c^{-2} u_t^2 + u_x^2 \right) \, dx + \frac{1}{2} a_l u(l, t)^2 + \frac{1}{2} a_0 u(0, t)^2.
\]
(a) Show that \( E \) is constant.
(b) (Bonus) Suppose that \( a_0, a_l \geq 0 \). Deduce uniqueness for the wave equation on \( 0 < x < l \) with Robin boundary conditions and initial conditions
\[
u(x, 0) = \phi(x) \quad \text{and} \quad u_t(x, 0) = \psi(x).
\]

Question 5. The vibrations of a tuning fork may be described by the PDE
\[
u_{tt} + c^2 u_{xxxx} = 0
\]
with boundary conditions
\[
u(0, t) = u_x(0, t) = 0 \quad \text{and} \quad u_x(l, t) = u_{xxx}(l, t) = 0.
\]
(a) Separate the variables to get the eigenvalue problem \( X''' = \lambda X \).
(b) Assume that \( \lambda > 0 \) and write \( \lambda = \beta^4 \). Show that \( \cosh(\beta l) \cos(\beta l) = -1 \).
(c) Find approximate values of the first two solutions \( \beta \) (for example using a computer).
(d) Compute the corresponding frequencies of vibration \( \omega_1 \) and \( \omega_2 \) and determine the ratio \( \omega_2 / \omega_1 \). Compare this result with the vibrating string with fixed ends.