Math 415 Assignment 9
Due Thursday, November 17, at the beginning of class

Question 1. Consider the eigenvalue problem $X'' + \lambda X = 0$ on $(0, l)$ with Robin boundary conditions

$$X'(0) - a_0 X(0) = 0 \quad \text{and} \quad X'(l) + a_l X(l) = 0.$$

In Section 4.3, we showed directly that all eigenvalues λ are non-negative if $a_0 \geq 0$ and $a_l \geq 0$. Give a new (and shorter) proof of this fact by using the results in Section 5.3.

A series of functions $\sum_{n=0}^{\infty} f_n$ *converges uniformly to* f *in* (a, b) *if*

$$\lim_{N\to\infty} \sup_{a<x<b} \left| f(x) - \sum_{n=0}^{N} f_n(x) \right| = 0.$$

Question 2. Consider the geometric series $\sum_{n=0}^{\infty} x^n$.

(a) Show that the series converges pointwise in $(-1, 1)$ and find the limit of the series.
(b) Show that the series does not converge uniformly in $(-1, 1)$.
(c) Show that for each $r \in (0, 1)$, the series converges uniformly in $[-r, r]$.
(d) Does the series converge in L^2 in $(-1, 1)$?

Question 3. For each of the following functions, determine if the Fourier sine series converges pointwise, uniformly and in L^2. (*Hint: You do not have to compute the series explicitely.*)

(a) $f(x) = x^2 - x$ on $(0, 1)$.
(b) $f(x) = x^4$ on $(0, 1)$.

Question 4. (a) Compute the Fourier sine series of $f(x) = x$ on $(0, 1)$.
(b) Use Parseval's equality and the result of (a) to determine the value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Question 5. The Gram-Schmidt process for the L^2 inner product works in the same way as the Gram-Schmidt process for the dot product in $\mathbb{R}^n$ or $\mathbb{C}^n$. That is, given functions $f_0, \ldots, f_n$, we define recursively

$$\begin{aligned}
g_0 &= f_0 \\
g_1 &= f_1 - \frac{\langle f_1, g_0 \rangle}{\langle g_0, g_0 \rangle} g_0 \\
g_2 &= f_2 - \frac{\langle f_2, g_0 \rangle}{\langle g_0, g_0 \rangle} g_0 - \frac{\langle f_2, g_1 \rangle}{\langle g_1, g_1 \rangle} g_1 \\
&\vdots \\
g_n &= f_n - \sum_{k=0}^{n-1} \frac{\langle f_n, g_k \rangle}{\langle g_k, g_k \rangle} g_k.
\end{aligned}$$

The resulting functions $g_1, \ldots, g_n$ are in particular orthogonal.

Apply the Gram-Schmidt process to the functions

$$f_0(x) = 1, f_1(x) = x, f_2(x) = x^2, f_3(x) = x^3,$$

defined on $(-1, 1)$.