Math 415  Assignment 9  
Due Thursday, November 17,  
at the beginning of class

**Question 1.** Consider the eigenvalue problem $X'' + \lambda X = 0$ on $(0, l)$ with Robin boundary conditions 

$X'(0) - a_0 X(0) = 0$ and $X'(l) + a_l X(l) = 0$.

In Section 4.3, we showed directly that all eigenvalues $\lambda$ are non-negative if $a_0 \geq 0$ and $a_l \geq 0$. Give a new (and shorter) proof of this fact by using the results in Section 5.3.

A series of functions $\sum_{n=0}^{\infty} f_n$ converges uniformly to $f$ in $(a, b)$ if 

$$
\lim_{N \to \infty} \sup_{a < x < b} \left| f(x) - \sum_{n=0}^{N} f_n(x) \right| = 0.
$$

**Question 2.** Consider the geometric series $\sum_{n=0}^{\infty} x^n$.

(a) Show that the series converges pointwise in $(-1, 1)$ and find the limit of the series.

(b) Show that the series does not converge uniformly in $(-1, 1)$.

(c) Show that for each $r \in (0, 1)$, the series converges uniformly in $[-r, r]$.

(d) Does the series converge in $L^2$ in $(-1, 1)$?

**Question 3.** For each of the following functions, determine if the Fourier sine series converges pointwise, uniformly and in $L^2$. (*Hint: You do not have to compute the series explicitly.*)

(a) $f(x) = x^2 - x$ on $(0, 1)$.

(b) $f(x) = x^4$ on $(0, 1)$.

**Question 4.**

(a) Compute the Fourier sine series of $f(x) = x$ on $(0, 1)$.

(b) Use Parseval’s equality and the result of (a) to determine the value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

**Question 5.** The Gram-Schmidt process for the $L^2$ inner product works in the same way as the Gram-Schmidt process for the dot product in $\mathbb{R}^n$ or $\mathbb{C}^n$. That is, given functions $f_0, \ldots, f_n$, we define recursively

$$
g_0 = f_0
$$

$$
g_1 = f_1 - \frac{\langle f_1, g_0 \rangle}{\langle g_0, g_0 \rangle} g_0
$$

$$
g_2 = f_2 - \frac{\langle f_2, g_0 \rangle}{\langle g_0, g_0 \rangle} g_0 - \frac{\langle f_2, g_1 \rangle}{\langle g_1, g_1 \rangle} g_1
$$

$$
\vdots
$$

$$
g_n = f_n - \sum_{k=0}^{n-1} \frac{\langle f_n, g_k \rangle}{\langle g_k, g_k \rangle} g_k.
$$

The resulting functions $g_1, \ldots, g_n$ are in particular orthogonal.

Apply the Gram-Schmidt process to the functions

$$
f_0(x) = 1, f_1(x) = x, f_2(x) = x^2, f_3(x) = x^3,
$$

defined on $(-1, 1)$. 