# Solution for HW1

Q1. 3.5.8(3.5.10)

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy \tag{0.1}$$
$$= \frac{1}{2\pi}\exp\{-\frac{1}{2}x^2\}\left(\int_{-\infty}^{\infty}\exp\{-\frac{1}{2}y^2\}dy + \int_{-\infty}^{\infty} xy\exp(-\frac{1}{2}(x^2+2y^2-2))dy\right) \tag{0.2}$$
$$= \frac{1}{2\pi}\exp\{-\frac{1}{2}x^2\}\left(\sqrt{2\pi} + x\exp(-\frac{1}{2}(x^2-2))\int_{-\infty}^{\infty} y\exp(-\frac{1}{2}y^2)dy\right) \tag{0.3}$$
$$= \frac{1}{2\pi}\exp\{-\frac{1}{2}x^2\}\left(\sqrt{2\pi}+0\right) \tag{0.4}$$
$$= \frac{1}{\sqrt{2\pi}}\exp\{-\frac{1}{2}x^2\}. \tag{0.5}$$

where (0.3) is because $\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\{-\frac{1}{2}y^2\}dy = 1$, so $\int_{-\infty}^{\infty}\exp\{-\frac{1}{2}y^2\}dy = \sqrt{2\pi}$. And (0.4) is because for $h(y) = y\exp(-\frac{1}{2}y^2)$, $h(y) = -h(-y)$, therefore $\int_{-\infty}^{\infty} h(y)dy = 0$ (or one can use change of variable by letting $z = y^2$ to get the same result). So

$$\int_{\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\{-\frac{1}{2}x^2\} = 1.$$

To show it is a joint density we also need to show $f(x,y) \geq 0$. Since

$$e^x \geq 1+x, \quad x \geq 0,$$

we have $e^{|xy|-1} \geq 1+|xy|-1 = |xy|$, that is

$$|xy|e^{-|xy|+1} \leq 1.$$

Note $x^2+y^2 \geq 2|xy|$. Hence

$$1 + xye^{-(x^2+y^2-2)/2} \geq 1 - |xy|e^{-(x^2+y^2-2)/2} \geq 1 - |xy|e^{-|xy|+1} \geq 0.$$

Same argument as (0.4) deals the $f_Y(y)$ part. Thus the marginals are both normal distributions. However the joint distribution is not bivariate normal.

Q2. 3.5.22

a) Since $\hat{\beta} = (X'X)^{-1}X'Y$, where $Y \sim \mathcal{N}(X\beta, \sigma^2 I)$, then

$$\begin{aligned} E(\hat{\beta}) &= E((X'X)^{-1}X'Y) = (X'X)^{-1}X'E(Y) \\ &= (X'X)^{-1}X'X\beta = \beta \\ V(\hat{\beta}) &= var((X'X)^{-1}X'Y) = (X'X)^{-1}X'Var(Y)[(X'X)^{-1}X']' \\ &= (X'X)^{-1}X'(\sigma^2 I)X(X'X)^{-1} = \sigma^2(X'X)^{-1} \end{aligned}$$

$\Rightarrow \hat{\beta} \sim \mathcal{N}_p(\beta, \sigma^2(X'X)^{-1})$. This follows from Thm 3.5.2.

b) the same way as we do part a)

$$\begin{aligned} E(\hat{Y}) &= E(X\hat{\beta}) \\ &= XE(\hat{\beta}) = X\beta \\ Var(\hat{Y}) &= Var(X\hat{\beta}) \\ &= XVar(\hat{\beta})X' = (\sigma^2)[X(X'X)^{-1}X'] \end{aligned}$$

$\Rightarrow \hat{Y} \sim \mathcal{N}_n(X\beta, (\sigma^2)[X(X'X)^{-1}X'])$. This follows from Thm 3.5.2.

c) Since $\hat{e} = Y - \hat{Y}$, then

$$\begin{aligned} E(\hat{e}) &= E(Y - \hat{Y}) = E(Y) - E(\hat{Y}) \\ &= X\beta - X\beta = 0 \\ Var(\hat{e}) &= Var(Y - \hat{Y}) \\ &= (\sigma^2)(I - X(X'X)^{-1}X') \end{aligned}$$

$\Rightarrow \hat{e} \sim \mathcal{N}_n(0, (\sigma^2)(I - X(X'X)^{-1}X'))$. This follows from Thm 3.5.2.

d) Since $\hat{Y} = X\hat{\beta}, \hat{\beta} = (X'X)^{-1}X'Y, \hat{e} = Y - \hat{Y}$, then we have

$$\begin{bmatrix} \hat{Y} \\ \hat{e} \end{bmatrix} = \begin{bmatrix} X(X'X)^{-1}X' \\ I - X(X'X)^{-1}X' \end{bmatrix} Y$$

then the covariance matrix of $\begin{bmatrix} \hat{Y} \\ \hat{e} \end{bmatrix}$ is

$$\begin{bmatrix} X(X'X)^{-1}X' \\ I - X(X'X)^{-1}X' \end{bmatrix} \sigma^2 I \begin{bmatrix} X(X'X)^{-1}X' \\ I - X(X'X)^{-1}X' \end{bmatrix}' = \sigma^2 \begin{bmatrix} X(X'X)^{-1}X' & 0 \\ 0 & I - X(X'X)^{-1}X' \end{bmatrix}$$

$\Rightarrow$ Since the variance-covariance matrix only has diagonal entries, the random vectors $\hat{Y}$ and $\hat{e}$ are independent. This follows from Thm 3.5.3.

e) To get the $\beta$ that minimizes $||Y - X\beta||^2$, we take the derivative w.r.t $\beta$, and get

$$\begin{aligned} \frac{\partial}{\partial \beta}||Y - X\beta||^2 &= (-2)X'(Y - X\beta), \text{ then set the derivative to zero and get} \\ &\Rightarrow 2X'Y = 2X'X\beta \\ &\Rightarrow X'X\beta = X'Y \\ &\Rightarrow \hat{\beta} = (X'X)^{-1}X'Y \text{ if } (X'X) \text{ is invertible} \end{aligned}$$

$\Rightarrow$ Thus, $\hat{\beta}$ minimizes $||Y - X\beta||^2$.

Q5. 3.6.10 (3.6.11)

Since $T = \frac{W}{\sqrt{V/r}}$, where $W \sim \mathcal{N}(0, 1)$ and $V \sim \chi_r^2$,

then $W^2 \sim \chi_1^2$.

Then we have $T^2 = \frac{W^2/1}{V/r}$.

Since $W^2 \sim \chi_1^2$ , $V \sim \chi_r^2$ and $W$ and $V$ are independent variables, based on the definition(3.6.7) on page 192 of the textbook, $T^2 \sim F_{1,r}$.

Q4. 3.6.5

```
## theoretical value of prob + simulations
#a) standard normal
pnorm(2, mean=0, sd=1, lower.tail=FALSE)*2
```

```
## [1] 0.04550026
```

```
set.seed(1)
n=100000
x1=rnorm(n,0,1)
(prop1=2*(length(which(x1>=2)))/n)
```

```
## [1] 0.04702
```

```
#b) t-dist with 1 d.f.
pt(2, df=1, lower.tail=FALSE)*2
```

```
## [1] 0.2951672
```

```
x2=rt(n,1)
(prop2=2*(length(which(x2>=2)))/n)
```

```
## [1] 0.29692
```

```
#c) t-dist with 3 d.f.
pt(2, df=3, lower.tail=FALSE)*2
```

```
## [1] 0.139326
```

```
x3=rt(n,3)
(prop3=2*(length(which(x3>=2)))/n)
```

```
## [1] 0.13988
```

```
#d) t-dist with 10 d.f.
pt(2, df=10, lower.tail=FALSE)*2
```

```
## [1] 0.07338803
```

```
x4=rt(n,10)
(prop4=2*(length(which(x4>=2)))/n)
```

```
## [1] 0.07328
```

```
#e) t-dist with 30 d.f.
pt(2, df=30, lower.tail=FALSE)*2
```

```
## [1] 0.05462504
```

```
x5=rt(n,30)
(prop5=2*(length(which(x5>=2)))/n)
```

```
## [1] 0.0554
```

We can see that the theoretical values are close to the simulated values. This should be the case because of the law of large numbers.

1.9.20

$$\begin{aligned}
M_X(t) &= \int_{\mathbb{R}} e^{tx}\frac{1}{2}e^{-|x|}dx \\
&= \frac{1}{2}\int_0^\infty e^{-x(1-t)}dx + \frac{1}{2}\int_{-\infty}^0 e^{tx-x}dx \\
&= \frac{1}{2}\int_0^\infty e^{-x(1-t)}dx + \frac{1}{2}\int_0^\infty e^{-x(1+t)}dx \\
&= \frac{1}{2(1-t)}\int_0^\infty (1-t)e^{-x(1-t)}dx + \frac{1}{2(1+t)}\int_0^\infty (1+t)e^{-x(1+t)}dx \\
&= \frac{1}{2(1-t)} + \frac{1}{2(1+t)} \\
&= \frac{1}{1-t^2}
\end{aligned}$$

Note 1: We must have $|t| < 1$ in order for the integral in the first line to exist.
Note 2: $\int_0^\infty \theta e^{-x\theta}dx = 1$ because the integrand is an exponential density.

For the second part, note that a commonly-used Maclaurin series from your first course in calculus is the following:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

for $x \in (-1, 1)$. Then, use the fact that

$$\frac{1}{1-x^2} = \frac{1/2}{1-x} + \frac{1/2}{1+x}$$

to express the Maclaurin series of $(1-x^2)^-1$ as a sum of half of the Maclaurin series' of $(1-x)^{-1}$ and $(1+x)^{-1}$:

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \cdots .$$

But, since $(1-x^2)^{-1}$ is a MGF we also know that

$$\frac{1}{1-x^2} = 1 + \frac{M^{(1)}(0)}{1!}x^1 + \frac{M^{(2)}(0)}{2!}x^2 + \frac{M^{(3)}(0)}{3!}x^3 + \cdots$$

which shows that

$$E(X^k) = 0 \text{ for k odd and } k! \text{ for k even.}$$