1. The shape-scale parametrization of the pdf of a Gamma random variable is given by

$$f(x; k, \theta) = \frac{1}{\Gamma(k)\theta^k} x^{k-1} e^{-x/\theta}$$

where $\Gamma(\cdot)$ denotes the gamma function.

a) For iid Gamma random variables $X_1, ..., X_n$ write down the likelihood and loglikelihood.

b) Write down the estimating equations for (k, θ) using the loglikelihood, but do not attempt to solve these for the MLEs.

a)

$$\ell(k, \theta; x_1, ..., x_n) = -n \log \Gamma(k) - nk \log \theta + (k-1)\sum_{i=1}^n \log x_i - \frac{1}{\theta}\sum_{i=1}^n x_i.$$

b)

$$0 = \frac{-nk}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^n x_i$$

$$0 = -n\frac{\Gamma'(k)}{\Gamma(k)} - n\log\theta + \sum_{i=1}^n \log x_i$$

2. Consider a random sample $X_1, ..., X_n$ from a $\texttt{Unif}(0, \theta)$ distribution for $\theta > 0$.

   a) Compute the bias of $\hat{\theta} = \max_i X_i$.

   b) Find the MLE of θ. Hint: it may help to write the pdf as

$$f(x) = \frac{1}{\theta} 1\{0 \leq x \leq \theta\}$$

   where $1\{\cdot\}$ denotes the indicator function.

a)

Let $X_{(n)}$ denote the sample maximum. Then, $P(X_{(n)} \leq x) = \left(\frac{x}{\theta}\right)^n$ and $f_{X_{(n)}}(x) = \frac{n}{\theta}\left(\frac{x}{\theta}\right)^{n-1}$. From this density, one can show

$$E(X_{(n)}) = \int_0^\theta n\left(\frac{x}{\theta}\right)^n dx = \frac{n}{\theta^n}\frac{x^{n+1}}{n+1}|_0^\theta = \frac{n}{n+1}\theta.$$

So,

$$Bias(\hat{\theta}) = E(X_{(n)}) - \theta = -\frac{1}{n+1}\theta.$$

b)

$$\ell(\theta; x_1, ..., x_n) = -n\log\theta + \sum_{i=1}^n \log 1\{0 \leq x_i \leq \theta\}.$$

The loglikelihood is $-\infty$ when θ is less than $x_{(n)}$, and for any $\theta \geq x_{(n)}$ it is equal to $-n\log\theta$, which is monotone decreasing in θ. Therefore, the MLE occurs at $X_{(n)}$.

3. Consider the following grades from a statistics exam in a class that includes both undergraduate and graduate students.

	Exam Grades									
Undergrads	66	72	74	78	81	82	83	93	93	94
Graduates	71	76	77	84	92	93	96			

Denote the undergraduates grades by $X_1, ..., X_{10}$ and assume $X_i \stackrel{iid}{\sim} N(\mu_X, \sigma_X^2)$. Similarly, denote the graduates grades by $Y_1, ..., Y_7$ and assume $Y_i \stackrel{iid}{\sim} N(\mu_Y, \sigma_Y^2)$.

a) Find a 95% confidence interval for the difference in mean grades $\mu_Y - \mu_X$.

b) Test the hypothesis of equal population variances (versus unequal) at $\alpha = 0.10$.

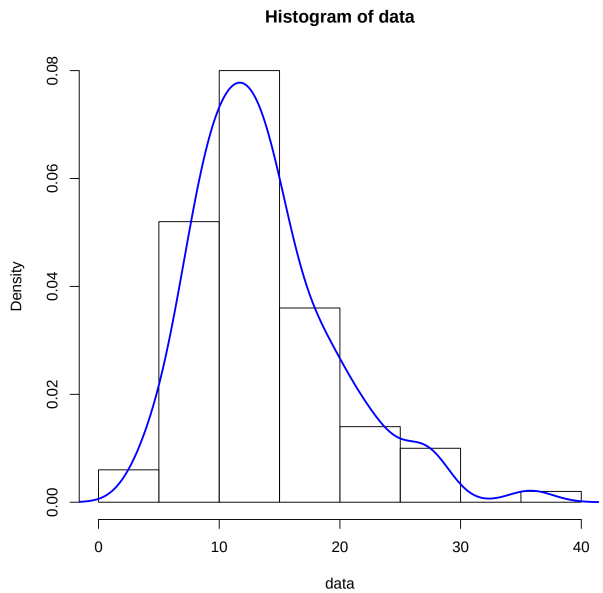
The test statistic is $t = \frac{\bar{X}-\bar{Y}}{\sqrt{S_X^2/n+S_Y^2/m}} \stackrel{H_0}{\sim} t(\nu)$ where ν equals the degrees of freedom given by the formula at the end of the practice exam. The CI is

$$(\bar{X} - \bar{Y} + t_{\alpha/2}(\nu)\sqrt{S_X^2/n + S_Y^2/m},\ \bar{X} - \bar{Y} + t_{1-\alpha/2}(\nu)\sqrt{S_X^2/n + S_Y^2/m}).$$

For the given data $\bar{x} = 81.6$, $\bar{y} = 84.14$, $s_X^2 = 81.38$, $s_Y^2 = 95.14$ and $\nu = 12.89$ which we should round down to 12 df. Then, using the table, the 90% CI is $(-11.04, 5.95)$ (95% CI is about (-12.93, 7.84)) and we would not reject H_0; the test statistic value is -0.53.

For b) we have $S_Y^2/S_X^2 = 1.0412$. Compare this to $F_{0.95}(6, 9) = 3.37$, and do not reject the null hypothesis of equal population variances.

4. Consider the following histogram of 100 lifetimes (in days) of worker bees during peak summer season.



a) Given $\sum_{i=1}^{100} x_i = 1376.027$ and $\sum_{i=1}^{100} x_i^2 = 22324.95$ find the Normal distribution best fitting this data based on the MLEs of μ and σ^2.

b) Find a 90% CI for the median $\tilde{X}$ of the distribution based on the Normal distribution in a). Hint: the mean and median are the same for a normal distribution.

c) Let $F(j,k;p) := \sum_{\ell=j}^{k-1} \binom{n}{\ell} p^\ell (1-p)^{n-\ell}$, where $n = 100$. Find a nonparametric CI for $\tilde{X}$ using the following information:

j	k	p	F(j,k;p)
40	60	0.5	0.954
41	59	0.5	0.927
42	58	0.5	0.889
43	57	0.5	0.837

And, $X_{(40)} = 11.680$, $X_{(41)} = 11.681$, $X_{(42)} = 11.769$, $X_{(43)} = 11.844$, $X_{(57)} = 13.403$, $X_{(58)} = 13.528$, $X_{(59)} = 13.864$, $X_{(60)} = 14.239$

Comment on which CI, from b) or c), you feel to be more appropriate. Why?

a) The MLEs are the ordinary sample mean and sample variance times $n-1/n$ $\bar{x} = 13.76$ and $(n-1/n)S^2 = \frac{1}{n-1}[\sum x_i^2 - \frac{1}{n}(\sum x_i)^2] = 34.25 * 100/99 = 34.6$.

b) Since the mean and median are the same, use the CI of the mean for the median, which is

$$(\bar{X} + t_{.05}(99)S/\sqrt{100},\ \bar{X} + t_{.95}(99)S/\sqrt{100})$$

$$= (12.79,\ 14.73)$$

Since the table provided only has t quantiles up to 30 df you may use the standard normal quantiles.

c) From Section 4.4.2, if we have

$$P(X_{(i)} < x_p < X_{(j)}) \geq 1 - \alpha$$

then the sample quantiles can be used as an approximate CI for the p^{th} quantile $(x_{(i)}, x_{(j)})$. In this case we have

$$P(X_{(41)} < x_{0.5} < X_{(59)}) \geq 0.9$$

so use (11.681, 13.864) as a CI for the median. Since the distribution is skewed positive the CI using the normal distribution is to the right of the nonparametric CI. The normal distribution is not a great fit (due to skew) so it's better to use the nonparametric CI.

5. Let X_1, X_2 be independent r.v.'s both having density $f(x;\theta) = \frac{1}{\theta}e^{-x/\theta}$ for $x > 0$. Suppose we reject $H_0 : \theta = 2$ versus $H_1 : \theta = 1$ if

$$\frac{f(x_1;2)f(x_2;2)}{f(x_1;1)f(x_2;1)} \leq 1/2.$$

Find the probability of type 1 error and the power if we consider the parameter space to be $\theta \in \{1,2\}$.

This ratio simplifies to $\frac{1}{4}e^{x_1/2+x_2/2}$. Then, for Type 1 error probability we must compute $P(X_1 + X_2 < 2\log 2)$:

$$\int_0^{2\log 2} \frac{1}{2}e^{-x_2/2}(1 - e^{-\log 2 + x_2/2})dx_2$$

$$= 0.15342.$$

For power we have

$$\begin{aligned} &P(X_1 + X_2 < 2\log 2 | X_1, X_2 \stackrel{iid}{\sim} Exp(1)) \\ &= \int_0^{2\log 2} e^{-x_2} \int_0^{2\log 2 - x_2} e^{-x_1} dx_1 dx_2 \\ &= 1 - 1/4 - 1/2\log 2 \\ &\approx 0.403 \end{aligned}$$

6. The following data are reproduced from the 1996 General Social Survey from the National Opinion Research Center:

	Religious Beliefs			
Highest Degree	Fundamentalist	Moderate	Liberal	Total
Less than High School	178	138	108	424
High School or Junior College	570	648	442	1660
Bachelor or Graduate	138	252	252	642
Total	886	1038	802	2726

Test the null hypothesis of independence of religious beliefs and education level using the appropriate Chi Squared test and $\alpha = 0.01$.

Here's some R code I used to compute $\sum_{i=1}^{KJ}(O_i - E_i)^2/E_i = \sum_{k=1}^{K}\sum_{j=1}^{J}\frac{(n_{kj}-n_{j+}n_{+k}/n)^2}{n_{j+}n_{+k}/n}$ where n_{jk} are the cell counts in the table, n is the total, and n_{+k} and n_{j+} and column and sum totals. The number of parameters is $KJ - 1 = 9 - 1 = 8$ and the number of estimated parameters is $K - 1 + J - 1 = 4$ so we get $df = 4$. The cutoff value for the test is the quantile $\chi^2_4(.99) = 13.28$ so reject the null hypothesis.

```
> Oi_Ei2_Ei <- c(  ((178 - 424*886/2726)^2)/(424*886/2726),
+     ((138 - 424*1038/2726)^2)/(424*1038/2726),
+     ((108 - 424*802/2726)^2)/(424*802/2726),
+     ((570 - 1660*886/2726)^2)/(1660*886/2726),
+     ((648 - 1660*1038/2726)^2)/(1660*1038/2726),
+     ((442 - 1660*802/2726)^2)/(1660*802/2726),
+     ((138 - 642*886/2726)^2)/(642*886/2726),
+     ((252 - 642*1038/2726)^2)/(642*1038/2726),
+     ((252 - 642*802/2726)^2)/(642*802/2726)
+ )
> Oi_Ei2_Ei
[1] 11.7222324  3.4059544  2.2471145  1.7207437  0.4004124  4.4043135 23.9290905
[8]  0.2326050 21.0942930
> sum(Oi_Ei2_Ei)
[1] 69.15676
>
```

Formulas

Two-sample t-test and confidence intervals degrees of freedom:

$X_1, ..., X_n \overset{iid}{\sim} N(\mu_X, \sigma_X^2)$, $Y_1, ..., Y_m \overset{iid}{\sim} N(\mu_Y, \sigma_Y^2)$

$$df = \frac{(\frac{s_X^2}{n} + \frac{s_Y^2}{m})^2}{\frac{s_X^4}{n^2(n-1)} + \frac{s_Y^4}{m^2(m-1)}}$$

Chi-squared tests: Let O_i and E_i denote the observed and expected cell counts in a frequency table. Then, $\sum_{i=1}^k \frac{(O_i - E_i)^2}{E_i} \dot{\sim} \chi^2(df)$ where $df =$ number of free parameters - number of estimated parameters.

Beta distribution: $f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$. $E(X) = \frac{\alpha}{\alpha+\beta}$.

$\Gamma(n) = (n-1)!$ when n is a positive integer.

Table II
Normal Distribution

The following table presents the standard normal distribution. The probabilities tabled are

$$P(Z \leq z) = \Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\, dw.$$

Note that only the probabilities for $z \geq 0$ are tabled. To obtain the probabilities for $z < 0$, use the identity $\Phi(-z) = 1 - \Phi(z)$. At the bottom of the table, some useful quantiles of the standard normal distribution are displayed. The R function `normaltable.s` generates this table.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998
3.5	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998

α	0.400	0.300	0.200	0.100	0.050	0.025	0.020	0.010	0.005	0.001
z_α	0.253	0.524	0.842	1.282	1.645	1.960	2.054	2.326	2.576	3.090
$z_{\alpha/2}$	0.842	1.036	1.282	1.645	1.960	2.241	2.326	2.576	2.807	3.291

Table III
t-Distribution

The following table presents selected quantiles of the t-distribution, i.e., the values t such that

$$P(T \leq t) = \int_{-\infty}^{t} \frac{\Gamma[(r+1)/2]}{\sqrt{\pi r}\,\Gamma(r/2)(1+w^2/r)^{(r+1)/2}}\,dw,$$

for selected degrees of freedom r. The last row gives the standard normal quantiles.

	$P(T \leq t)$					
r	0.900	0.950	0.975	0.990	0.995	0.999
1	3.078	6.314	12.706	31.821	63.657	318.309
2	1.886	2.920	4.303	6.965	9.925	22.327
3	1.638	2.353	3.182	4.541	5.841	10.215
4	1.533	2.132	2.776	3.747	4.604	7.173
5	1.476	2.015	2.571	3.365	4.032	5.893
6	1.440	1.943	2.447	3.143	3.707	5.208
7	1.415	1.895	2.365	2.998	3.499	4.785
8	1.397	1.860	2.306	2.896	3.355	4.501
9	1.383	1.833	2.262	2.821	3.250	4.297
10	1.372	1.812	2.228	2.764	3.169	4.144
11	1.363	1.796	2.201	2.718	3.106	4.025
12	1.356	1.782	2.179	2.681	3.055	3.930
13	1.350	1.771	2.160	2.650	3.012	3.852
14	1.345	1.761	2.145	2.624	2.977	3.787
15	1.341	1.753	2.131	2.602	2.947	3.733
16	1.337	1.746	2.120	2.583	2.921	3.686
17	1.333	1.740	2.110	2.567	2.898	3.646
18	1.330	1.734	2.101	2.552	2.878	3.610
19	1.328	1.729	2.093	2.539	2.861	3.579
20	1.325	1.725	2.086	2.528	2.845	3.552
21	1.323	1.721	2.080	2.518	2.831	3.527
22	1.321	1.717	2.074	2.508	2.819	3.505
23	1.319	1.714	2.069	2.500	2.807	3.485
24	1.318	1.711	2.064	2.492	2.797	3.467
25	1.316	1.708	2.060	2.485	2.787	3.450
26	1.315	1.706	2.056	2.479	2.779	3.435
27	1.314	1.703	2.052	2.473	2.771	3.421
28	1.313	1.701	2.048	2.467	2.763	3.408
29	1.311	1.699	2.045	2.462	2.756	3.396
30	1.310	1.697	2.042	2.457	2.750	3.385
∞	1.282	1.645	1.960	2.326	2.576	3.090

Table I
Chi-Square Distribution

The following table presents selected quantiles of chi-square distribution, i.e., the values x such that

$$P(X \leq x) = \int_0^x \frac{1}{\Gamma(r/2)2^{r/2}} w^{r/2-1} e^{-w/2} \, dw,$$

for selected degrees of freedom r. The R function `chistable.s` generates this table.

	$P(X \leq x)$							
r	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990
1	0.000	0.001	0.004	0.016	2.706	3.841	5.024	6.635
2	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210
3	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345
4	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277
5	0.554	0.831	1.145	1.610	9.236	11.070	12.833	15.086
6	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812
7	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475
8	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090
9	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666
10	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209
11	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725
12	3.571	4.404	5.226	6.304	18.549	21.026	23.337	26.217
13	4.107	5.009	5.892	7.042	19.812	22.362	24.736	27.688
14	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141
15	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578
16	5.812	6.908	7.962	9.312	23.542	26.296	28.845	32.000
17	6.408	7.564	8.672	10.085	24.769	27.587	30.191	33.409
18	7.015	8.231	9.390	10.865	25.989	28.869	31.526	34.805
19	7.633	8.907	10.117	11.651	27.204	30.144	32.852	36.191
20	8.260	9.591	10.851	12.443	28.412	31.410	34.170	37.566
21	8.897	10.283	11.591	13.240	29.615	32.671	35.479	38.932
22	9.542	10.982	12.338	14.041	30.813	33.924	36.781	40.289
23	10.196	11.689	13.091	14.848	32.007	35.172	38.076	41.638
24	10.856	12.401	13.848	15.659	33.196	36.415	39.364	42.980
25	11.524	13.120	14.611	16.473	34.382	37.652	40.646	44.314
26	12.198	13.844	15.379	17.292	35.563	38.885	41.923	45.642
27	12.879	14.573	16.151	18.114	36.741	40.113	43.195	46.963
28	13.565	15.308	16.928	18.939	37.916	41.337	44.461	48.278
29	14.256	16.047	17.708	19.768	39.087	42.557	45.722	49.588
30	14.953	16.791	18.493	20.599	40.256	43.773	46.979	50.892

Table IV
F-Distribution
Upper 0.05 Critical Points

The following table presents selected 0.95 and 0.99 quantiles of the F-distribution, i.e., for $\alpha = 0.05, 0.01$, the values $F_\alpha(r_1, r_2)$ such that

$$\alpha = P(X \geq F_\alpha(r_1, r_2)) = \int_{F_\alpha(r_1,r_2)}^{\infty} \frac{\Gamma[(r_1+r_2)/2](r_1/r_2)^{r_1/2} w^{r_1/2-1}}{\Gamma(r_1/2)\Gamma(r_2/2)(1+r_1w/r_2)^{(r_1+r_2)/2}}\, dw,$$

where r_1 and r_2 are the numerator and denominator degrees of freedom, respectively. The R function `fp1.r` generates this table.

$F_{0.05}(r_1, r_2)$									
	r_1								
r_2	1	2	3	4	5	6	7	8	9
1	161.45	199.50	215.71	224.58	230.16	233.99	236.77	238.88	240.54
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88

Table IV
F-Distribution, Continued
Upper 0.05 Critical Points

Generated by the R function fp2.r.

	$F_{0.05}(r_1, r_2)$								
	r_1								
r_2	10	15	20	25	30	40	60	120	∞
1	241.88	245.95	248.01	249.26	250.10	251.14	252.20	253.25	254.31
2	19.40	19.43	19.45	19.46	19.46	19.47	19.48	19.49	19.50
3	8.79	8.70	8.66	8.63	8.62	8.59	8.57	8.55	8.53
4	5.96	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	4.74	4.62	4.56	4.52	4.50	4.46	4.43	4.40	4.36
6	4.06	3.94	3.87	3.83	3.81	3.77	3.74	3.70	3.67
7	3.64	3.51	3.44	3.40	3.38	3.34	3.30	3.27	3.23
8	3.35	3.22	3.15	3.11	3.08	3.04	3.01	2.97	2.93
9	3.14	3.01	2.94	2.89	2.86	2.83	2.79	2.75	2.71
10	2.98	2.85	2.77	2.73	2.70	2.66	2.62	2.58	2.54
11	2.85	2.72	2.65	2.60	2.57	2.53	2.49	2.45	2.40
12	2.75	2.62	2.54	2.50	2.47	2.43	2.38	2.34	2.30
13	2.67	2.53	2.46	2.41	2.38	2.34	2.30	2.25	2.21
14	2.60	2.46	2.39	2.34	2.31	2.27	2.22	2.18	2.13
15	2.54	2.40	2.33	2.28	2.25	2.20	2.16	2.11	2.07
16	2.49	2.35	2.28	2.23	2.19	2.15	2.11	2.06	2.01
17	2.45	2.31	2.23	2.18	2.15	2.10	2.06	2.01	1.96
18	2.41	2.27	2.19	2.14	2.11	2.06	2.02	1.97	1.92
19	2.38	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	2.35	2.20	2.12	2.07	2.04	1.99	1.95	1.90	1.84
21	2.32	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	2.30	2.15	2.07	2.02	1.98	1.94	1.89	1.84	1.78
23	2.27	2.13	2.05	2.00	1.96	1.91	1.86	1.81	1.76
24	2.25	2.11	2.03	1.97	1.94	1.89	1.84	1.79	1.73
25	2.24	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71
26	2.22	2.07	1.99	1.94	1.90	1.85	1.80	1.75	1.69
27	2.20	2.06	1.97	1.92	1.88	1.84	1.79	1.73	1.67
28	2.19	2.04	1.96	1.91	1.87	1.82	1.77	1.71	1.65
29	2.18	2.03	1.94	1.89	1.85	1.81	1.75	1.70	1.64
30	2.16	2.01	1.93	1.88	1.84	1.79	1.74	1.68	1.62
40	2.08	1.92	1.84	1.78	1.74	1.69	1.64	1.58	1.51
60	1.99	1.84	1.75	1.69	1.65	1.59	1.53	1.47	1.39
120	1.91	1.75	1.66	1.60	1.55	1.50	1.43	1.35	1.25
∞	1.83	1.67	1.57	1.51	1.46	1.39	1.32	1.22	1.00