# MATH 494 Lecture 01/13/20

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# Moment Generating Function (MGF)

Let $X$ be a continuous random variable (r.v.) with density function $f$ and define

$$M_X(t) = \int_{\mathbb{R}} e^{tx} f(x)dx.$$

Then, $M_X(t)$ is the "moment-generating function" of $X$ where it exists, that is, for all $t$ such that the above integral is finite.

# MGF Examples

1. Bernoulli: $1 - p + pe^t$
2. Binomial: $(1 - p + pe^t)^n$
3. Poisson: $e^{\lambda(e^t - 1)}$
4. Exponential: $(1 - t/\lambda)^{-1}$
5. Normal: $e^{t\mu + \frac{1}{2}\sigma^2 t^2}$
6. Multivariate Normal: $e^{t^\top \mu + \frac{1}{2} t^\top \Sigma t}$

# Derivation of Bernoulli MGF

Recall that $P(X=1) := p$ and $P(X=0) = 1-p$ , then

$$E(e^{tX}) = pe^t + (1-p)e^0 = 1-p+pe^t$$

# Derivation of Binomial MGF

Let $X$ and $Y$ be iid Bernoulli $p$ r.v.'s. Then

$$M_{X+Y}(t) = E(e^{t(X+Y)}) = \sum_{x,y} e^{t(x+y)} p(x,y)$$

$$= \sum_{x,y} e^{tx} e^{ty} p(x,y)$$

$$= \sum_{x,y} e^{tx} e^{ty} p(x)p(y)$$

$$= \sum_{x} e^{tx} p(x) \sum_{y} e^{ty} p(y)$$

$$= (1-p+pe^t)(1-p+pe^t)$$

$$= (1-p+pe^t)^2$$

And, therefore, the MGF of a Binomial $(n,p)$ r.v. is $(1-p+pe^t)^n$ .

# Uses of MGFs

To compute (generate) moments!
If $X$ has MGF $M_X(t)$ that exists in a neighborhood of $0$ , i.e. it exists for all $t \in (-h, h)$ for some constant $h > 0$ , then $E(X^k) = M_X^{(k)}(0)$ , that is, the $k^{th}$ moment of $X$ is the $k^{th}$ derivative of the MGF, evaluated at $t = 0$ .
Example: The mean and variance of a normal r.v.:

$$E(X) = \frac{d}{dt} e^{t\mu + \frac{1}{2}\sigma^2 t^2}|_{t=0} = (\mu + t\sigma_2) e^{t\mu + \frac{1}{2}\sigma^2 t^2}|_{t=0}$$

$$= (\mu + 0)e^0 = \mu$$

$$E(X^2) = \frac{d^2}{dt^2} e^{t\mu + \frac{1}{2}\sigma^2 t^2}|_{t=0}$$

$$= (\mu + t\sigma_2)^2 e^{t\mu + \frac{1}{2}\sigma^2 t^2} + \sigma^2 e^{t\mu + \frac{1}{2}\sigma^2 t^2}|_{t=0}$$

$$= \mu^2 + \sigma^2.$$

Then, $V(X) = M''(0) - [M'(0)]^2$ .

# Uses of MGFs

Theorem: For two r.v.'s $X$ and $Y$ , if $M_X(t)$ and $M_Y(t)$ exist and are equal for all $t \in (-h, h)$ for some $h > 0$ , then $X$ and $Y$ have the same distribution. This can be proved by showing the uniqueness of the characteristic function (Fourier Transform) which would be appropriate for a graduate course in probability or analysis.
Theorem: For a sequence of r.v.'s $X_1, \ldots, X_n$ $n \to \infty$ , if the MGFs $M_{X_n}(t)$ have a limit $M_X(t)$ (that exists) then the r.v.'s converge in distribution to the r.v. with the distribution implied by $M_X(t)$ . This is Levy's Continuity Theorem.

Example: Poisson approximation to Binomial.

# Poisson approximation to Binomial

Recall that if $lim_{n\to\infty} np = \lambda$ then the Poisson with mean $\lambda$ is a good approximation to $Bin(n, p)$ . Justification?

Lemma: If $a_n \to a$ then $\lim_{n\to\infty}(1 + \frac{a_n}{n})^n = e^a$ .
Easiest Proof: Rewrite the limit as

$$[\lim_{n\to\infty}(1 + \frac{1}{s_n})^{s_n}]^{\lim_{n\to\infty} a_n}$$

where $s_n = n/a_n$ . This "separating" of limits is allowed because both limits exist and do not admit an indeterminate form. Clearly, the exponent limit is $a$ , and $s_n \to \infty$ such that for every $M > 0$ there is an $N > 0$ with $s_n > N$ . Then, the base limit is $e$ and the whole limit expression is $e^a$ .

Then, working from MGF of Binomial:

$$\lim_{n\to\infty}(1 - p + pe^t)^n$$

$$= \lim_{n\to\infty}(1 - \frac{np_n}{n} + \frac{np_n}{n}e^t)^n$$

$$= \lim_{n\to\infty}[1 + \frac{np_n}{n}(e^t - 1)]^n$$

$$= e^{\lambda(e^t-1)}$$

which is MGF of $Pois(\lambda)$ !

# Nonexistence?

Cauchy distribution.

A Cauchy r.v. $X$ has density $f(x) = \frac{1}{\pi(1+x^2)}$ .

Then,

$$\int \frac{e^{tx}}{\pi(1+x^2)} dx$$

is infinite!

Let $t > 0$ :

$$\int_{\mathbb{R}} e^{tx} \frac{1}{\pi(1+x^2)} dx \geq \int_0^\infty e^{tx} \frac{1}{\pi(1+x^2)} dx$$

$$\geq \int_0^\infty \frac{tx}{\pi(1+x^2)} dx$$

$$= \frac{t}{\pi} \int_0^\infty \frac{1}{x+\frac{1}{x}} dx$$

$$\geq \frac{t}{\pi} \int_1^\infty \frac{1}{x+1} dx$$

$$= \frac{t}{\pi} \log(1+x)|_1^\infty$$
$$= \infty$$

for all $t > 0$ . Hence, the MGF does not exist in a neighborhood of $0$ . Note, in the above we use the fact that $e^y \geq y$ for all $y \geq 0$ .

A generalization of the MGF, the characteristic function, always exists

$$\phi(x) = \int e^{itx} F(dx)$$

but its study is the subject of a graduate course in measure-theoretic probability.