Lecture 01/17/20

Dr. Syring

Multivariate Normal and linear transformations

Theorem 3.5.2: the family of multivariate normal distributions is closed under linear transformations.

Let $X \sim N_p(\mu, \Sigma)$. Let $Y = AX + b$, A is an $m \times p$ matrix and b is an $m \times 1$ vector.

Then,

$$M_Y(t) = E(e^{t^\top Y}) = E(e^{t^\top (AX+b)})$$

$$= e^{t^\top b} E(e^{(A^\top t)^\top X})$$

$$= e^{t^\top b} e^{(A^\top t)^\top \mu + \frac{1}{2}(A^\top t)^\top \Sigma (A^\top t)}$$

$$= e^{t^\top (A\mu + b) + (1/2) t^\top A \Sigma A^\top t}$$

which is the MGF of $N_m(A\mu + b, A\Sigma A^\top)$.

Independence of multivariate normal r.v.'s

Theorem 3.5.3: Let $X \sim N_n(\mu, \Sigma)$ and partition $X = (X_1, X_2)$, $\mu = (\mu_1, \mu_2)$ and

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}.$$

If $\Sigma_{12} = 0$ then $X_1 \perp X_2$.

Using the partition the MGF of X can be writted

$$M_{X_1,X_2}(t_1, t_2) = e^{t_1^\top \mu_1 + t_2^\top \mu_2 + \frac{1}{2}[t_1^\top \Sigma_{11} t_1 + t_2^\top \Sigma_{21} t_1 + t_1^\top \Sigma_{12} t_2 + t_2^\top \Sigma_{22} t_2]} .$$

Then, putting $\Sigma_{12} = 0_{p\times q}$ which implies $\Sigma_{21} = 0_{q\times p}$ we have

$$M_{X_1,X_2}(t_1, t_2) = e^{t_1^\top \mu_1 \frac{1}{2} t_1^\top \Sigma_{11} t_1} e^{t_2^\top \mu_2 \frac{1}{2} t_2^\top \Sigma_{22} t_2}$$

$$= M_{X_1}(t_1) M_{X_2}(t_2)$$

proving independence.

Student t Distribution

Let $U \sim N(\mu, \sigma^2)$ and $V \sim \chi 2(k)$ and $U \perp V$. Then, $T = U/\sqrt{V/k}$ has a Student t distribution with k degrees of freedom. The density function is

$$f(t) = \frac{\Gamma(\frac{k+1}{2})}{\sqrt{k\pi}\Gamma(\frac{k}{2})}(1 + \frac{t^2}{k})^{-\frac{k+1}{2}}.$$

Statistics application

Student's Theorem: Let $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$,
$S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2$. Then,
a) $\bar{X}_n \sim N(\mu, \sigma^2/n)$
b) $\bar{X}_n \perp S^2$
c) $(n-1)S^2/\sigma^2 \sim \chi^2(n-1)$
d) $T = \frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t(n-1)$

Part a) $\bar{X}_n \sim N(\mu, \sigma^2/n)$

$$E(e^{t\bar{X}}) = \prod_{i=1}^{n} \int_{\mathbb{R}} e^{\frac{1}{n}x_i t} \phi(x_i; \mu, \sigma^2) dx_i$$

$$= \prod_{i=1}^{n} e^{\frac{t}{n}\mu + \frac{1}{2}\frac{t^2}{n^2}\sigma^2}$$

$$= e^{t\mu + \frac{1}{2}t\sigma^2/n}$$

which is the MGF of $N(\mu, \sigma^2/n)$

Theorem 3.5.2 similarly shows that linear transformations of normal r.v.'s are normally distributed.

Part b) $\bar{X}_n \perp S^2$

Let $v^\top = (1/n, 1/n, \ldots, 1/n)_{1\times n}$, I_n be the $n \times n$ identity matrix and 1_n be an $n - vector$ of 1's. Then,

$$W := (\bar{X}_n, X - \bar{X}_n) = (v^\top X, (I_n - 1_n v^\top)X).$$

The covariance of W is

$$Cov(W) = \sigma^2 \begin{bmatrix} 1/n & 0_n^\top \\ 0_n & I_n - 1_n v^\top \end{bmatrix}.$$

Since the covariance of $\bar{X}_n$ and $X - \bar{X}_n$ is zero, they are independent (Theorem 3.5.3).

Part c) $(n-1)S^2/\sigma^2 \sim \chi^2(n-1)$

Let $V = \sum_{i=1}^{n} (\frac{X_i - \mu}{\sigma})^2$. Then $V \sim \chi^2(n)$ because the summands are independent $\chi^2(1)$.

Next, decompose V as

$$V = \sum_{i=1}^{n} \left(\frac{X_i - \bar{X}_n + \bar{X}_n - \mu}{\sigma} \right)^2$$

$$= \sum_{i=1}^{n} (\frac{X_i - \bar{X}_n}{\sigma})^2 + (\frac{X_i - \mu}{\sigma/\sqrt{n}})^2$$

$$= \frac{(n-1)S^2}{\sigma^2} + (\frac{X_i - \mu}{\sigma/\sqrt{n}})^2.$$

The LHS is $\chi^2(n)$ as we said previously. The second term on the RHS is $\chi^2(1)$ and the two terms on the RHS are independent. Therefore, the first term on the RHS must be $\chi^2(n-1)$.

Part d) $T = \frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t(n-1)$

The conclusion is now immediate by expressing T as

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}}$$

$$= \frac{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{(n-1)S^2}{(n-1)\sigma^2}}}$$

is the ratio of a standard normal r.v. and independent chi-squared r.v. divided by its df.