Name:

Instructions: You have 2 hours to take this exam. You may use one 4 x 6 notecard. Calculators are not allowed.

Show your work.

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1. Use the limit definition of the derivative to find the derivatives of the following functions:

(a) \( f(x) = x^2 - 3x \)

\[
\lim_{h \to 0} \frac{(x+h)^2 - 3(x+h) - (x^2 - 3x)}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 3x - 3h - x^2 + 3x}{h} = \lim_{h \to 0} (2x + h - 3) = 2x - 3
\]

(b) \( g(x) = \frac{2}{\sqrt{x}} + 1 \)

\[
\lim_{h \to 0} \frac{\left(\frac{2}{\sqrt{x+h}} + 1\right) - \left(\frac{2}{\sqrt{x}} + 1\right)}{h} = \lim_{h \to 0} \frac{2}{h} \left(\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}\right) = \lim_{h \to 0} \frac{2}{h} \left(\frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x+h} \sqrt{x}}\right) \left(\frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}}\right) = \lim_{h \to 0} \frac{2}{h} \left(\frac{x - (x+h)}{\sqrt{x+h} \sqrt{x} (\sqrt{x} + \sqrt{x+h})}\right) = \lim_{h \to 0} \frac{-2}{\sqrt{x+h} \sqrt{x} (\sqrt{x} + \sqrt{x+h})} = \frac{-2}{2 \sqrt{x}} = \frac{-1}{\sqrt{x}} = \sqrt{x^{-3/2}}
\]
2. Find the derivatives of the following functions:

(a) \( y = 2x^{33} - \frac{33}{x} \)

\[ y' = 66x^{32} + \frac{33}{x^2} \]

(b) \( y = \sin(x^2) \)

\[ y' = 2x \cos(x^2) \]

(c) \( y = e^{3x-1} \)

\[ y' = 3e^{3x-1} \]

(d) \( y = \ln(x^5) \)

\[ y' = \frac{5x^4}{x^5} = \frac{5}{x} \]

(e) \( y = (\ln x)^5 \)

\[ y' = 5(\ln x)^4 \left( \frac{1}{x} \right) = \frac{5}{x} (\ln x)^4 \]
3. Find the derivatives of the following functions:

(a) \( f(x) = x \sin(e^{x^2}) \)

\[
\begin{align*}
  f'(x) &= \sin(e^{x^2}) + x(e^{x^2}) (2xe^{x^2}) \\
       &= [\sin(e^{x^2}) + 2xe^{x^2} \cos(e^{x^2})]
\end{align*}
\]

(b) \( g(x) = \frac{\ln(3x+1)}{(e^{x^2}-1)^2} \)

\[
\begin{align*}
  g'(x) &= \frac{3}{3x+1} (e^{x^2-1})^{3/2} - \frac{\ln(3x+1)}{2} (e^{x^2-1})^{3/2} (2e^{x^2}) \\
       &= \frac{3}{3x+1} (e^{x^2-1}) - 8e^{x^2} \ln(3x+1) \\
       &= \frac{3e^{x^2-1} - 8e^{x^2} \ln(3x+1)}{(3x+1)(e^{x^2}-1)^3}
\end{align*}
\]
4. (a) Using implicit differentiation, find \( \frac{dy}{dx} \) for the curve \( 2xy = (x^2 + y^2)^{3/2} \).

\[
2y + 2xy' = \frac{3}{2} (x^2 + y^2)^{1/2} (2x + 2yy')
\]

\[
2xy' - \frac{3}{2} (x^2 + y^2)^{1/2} (2yy') = \frac{3}{2} (x^2 + y^2)^{1/2} (2x) - 2y
\]

\[
(2x - 3y\sqrt{x^2 + y^2})y' = 3x\sqrt{x^2 + y^2} - 2y
\]

\[
y' = \frac{3x\sqrt{x^2 + y^2} - 2y}{2x - 3y\sqrt{x^2 + y^2}}
\]

(b) Find the equation of the tangent line to this curve at the point \( x = \frac{\sqrt{2}}{2}, y = \frac{\sqrt{2}}{2} \).

\[
y' = \frac{3\left(\frac{\sqrt{2}}{2}\right)\sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} - 2\left(\frac{\sqrt{2}}{2}\right)}{2\left(\frac{\sqrt{2}}{2}\right) - 3\left(\frac{\sqrt{2}}{2}\right)\sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2}} = -1
\]

\[
y - \frac{\sqrt{2}}{2} = -1\left(x - \frac{\sqrt{2}}{2}\right)
\]

\[y = -x + \sqrt{2}\]
5. Sketch a graph of the function

\[ y = -\frac{2}{9}x^3 + \frac{1}{3}x^2 + \frac{4}{3}x - \frac{2}{9}. \]

Clearly label (1) all maxima and minima; (2) regions where the function is increasing and decreasing; (3) regions where the function is concave up and concave down.

\[ y' = -\frac{2}{3}x^2 + \frac{2}{3}x + \frac{4}{3} \]

Critical points: \( x^2 - x - 2 = 0 \)
\( (x-2)(x+1) = 0 \)
\( x = 2, -1 \)

Increasing/decreasing:
\[-\frac{2}{3}x^2 + \frac{2}{3}x + \frac{4}{3} > 0 \]
\( x^2 - x - 2 < 0 \)
\(-1 < x < 2 \) increasing
\( x < -1, x > 2 \) decreasing

Concavity:
\[ y'' = -\frac{4}{3}x + \frac{2}{3} > 0 \]
\( \frac{2}{3} > \frac{4}{3}x \)
\( x < \frac{1}{2} \) concave up
\( x > \frac{1}{2} \) concave down

\( x = 2 \) local max
\( x = -1 \) local min

\[ x = 2, y = -\frac{2}{9} \cdot 2 + \frac{1}{3} \cdot 4 + \frac{4}{3} \cdot 2 - \frac{2}{9} = \frac{-16}{9} + \frac{4}{3} + \frac{8}{3} - \frac{2}{9} = \frac{2}{3} \]

\[ x = -1, y = \frac{2}{9} + \frac{1}{3} - \frac{4}{3} - \frac{2}{9} = -1 \]
6. Find the dimensions of the rectangle of greatest area which can be inscribed under the curve $y = 1 - x^2$ (and above the $x$-axis). Be sure to verify that you have found the maximum area.

\[
\text{area} = 2x(1-x^2) = 2x - 2x^3
\]

\[
2 - 6x^2 = 0
\]

\[
x^2 = \frac{1}{3}
\]

\[
x = \frac{\sqrt{3}}{3}, \quad \text{discard negative root}
\]

\[
y = 1 - \left(\frac{1}{3}\right)^2 = \frac{2}{3}
\]

\[
\text{dimension: } \frac{2}{\sqrt{3}} \times \frac{2}{3}
\]

Verify local max: \( (\text{area})'' = -12x \)

\[
x = \frac{\sqrt{3}}{3} \rightarrow (\text{area})'' < 0 \rightarrow \text{local max}
\]
7. Evaluate the following integrals:

(a) \( \int (x^2 + x + 1) \, dx \)
- \( \frac{1}{3} x^3 + \frac{1}{2} x^2 + x + C \)

(b) \( \int 3e^x \, dx \)
- \( 3e^x + C \)

(c) \( \int \left( \frac{12}{x} - \sin x \right) \, dx \)
- \( 12 \ln x + C \)

(d) \( \int_1^4 (3\sqrt{x} + \frac{4}{\sqrt{x}}) \, dx \)
- \( 3 \left( \frac{2}{3} x^{3/2} + 4 \cdot 2 x^{1/2} \right) \bigg|_1^4 = (2(\sqrt{4}) + \sqrt{4}) - (2(1) + \sqrt{1}) = 22 \)

(e) \( \int_2^3 \frac{4}{x} \, dx \)
- \( \frac{4}{3} x^{-3} \bigg|_2^3 = \frac{-4}{3(3)} - \frac{-4}{3(2)} = \frac{4}{3} \left( \frac{1}{2^3} - \frac{1}{3^3} \right) \)
8. Evaluate the following integrals:

(a) \( \int \frac{x^2}{x^3 - 2} \, dx \)

\[ u = x^3 - 2 \]
\[ du = 3x^2 \, dx \]
\[ \int \frac{x^2}{x^3 - 2} \, dx = \frac{1}{3} \int \frac{1}{u} \, du = \frac{1}{3} \ln u + C \]
\[ = \frac{1}{3} \ln (x^3 - 2) + C \]

(b) \( \int_{\ln 2}^{\ln 3} e^{x} \sqrt{e^{x} + 1} \, dx \)

\[ u = e^{x} + 1 \]
\[ x = \ln 3 \rightarrow u = 4 \]
\[ x = \ln 2 \rightarrow u = 3 \]
\[ \int_{\ln 2}^{\ln 3} e^{x} \sqrt{e^{x} + 1} \, dx = \int_{3}^{4} u^{1/2} \, du = \frac{2}{3} u^{3/2} \bigg|_{3}^{4} \]
\[ = \frac{2}{3} (4^{3/2} - 3^{3/2}) \]
9. A rocket is launched at $t = 0$ from a platform 10 meters above the ground. It accelerates upward at 20 m/sec$^2$ (roughly 2 g's).

(a) If its initial velocity is zero, find its velocity as a function of time.

$$v(t) = \int v_0 \, dt = v_0 t + C$$

$$0 = v_0(0) + C \rightarrow C = 0$$

$$\boxed{v(t) = 20t}$$

(b) Find its height as a function of time.

$$h(t) = \int v_0 t \, dt = 10t^2 + C$$

$$10 = 10(0)^2 + C$$

$$C = 10$$

$$\boxed{h(t) = 10t^2 + 10}$$
10. (a) $100$ is invested in an account that pays $3\%$ annual interest, compounded continuously. How long must we wait to get $200$ in the account? (Express your answer in as simple a form as possible.)

\[ 200 = 100 \, e^{0.03t} \]
\[ 2 = e^{0.03t} \]
\[ 0.03t = \ln(2) \]
\[ t = \frac{\ln(2)}{0.03} \]

(b) In a separate account that also pays $3\%$ annual interest, compounded continuously, a continuous income stream is added at a rate of $100$ per year. How long must we wait to get $1000$ in this account? (Again, express your answer in as simple a form as possible.)

\[ 1000 = \int_0^T 100 \, e^{0.03(T-t)} \, dt = \int_0^{T} 100 \, e^{u} \, du = \frac{100}{0.03} e^{u} \bigg|_{u=0}^{T} = \frac{100}{0.03} (1 - e^{0.03T}) \]

\[ u = 0.03(T-t) \]
\[ du = -0.03 \, dt \]
\[ t=0 \rightarrow u = 0.03T \]
\[ t=T \rightarrow u = 0 \]

\[ 1000 = \frac{100}{0.03} (e^{0.03T} - 1) \]
\[ 30 = 100 (e^{0.03T} - 1) \]
\[ 0.3 = e^{0.03T} - 1 \]
\[ 1.3 = e^{0.03T} \]
\[ T = \frac{\ln(1.3)}{0.03} \]