Name:__________________________________

Instructions: You have 2 hours to take this exam. You may use one 4 × 6 notecard. Calculators are not allowed.

Show your work.

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1. Differentiate the following functions.

(a) \( y = x^5 - 2x^4 + 3x^3 - 4x + 1 \)
\[
y' = 5x^4 - 8x^3 + 9x^2 - 4
\]

(b) \( y = x^2 + x + \sqrt{x} = x^2 + x + x^{\frac{1}{2}} \)
\[
y' = 2x + 1 + \frac{1}{2}x^{-\frac{1}{2}} = 2x + 1 + \frac{1}{2\sqrt{x}}
\]

(c) \( y = \frac{3}{x^2} + \frac{4}{x^3} = 3x^{-2} + 4x^{-3} \)
\[
y' = -6x^{-3} - 12x^{-4} = -\frac{6}{x^3} - \frac{12}{x^4}
\]

(d) \( y = \frac{x^2 + 4}{x} = x + 4x^{-1} \)
\[
y' = 1 - 4x^{-2} = 1 - \frac{4}{x^2}
\]

(e) \( y = x\sqrt{x} = x^{3/2} \)
\[
y' = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}
\]
2. Differentiate the following functions.

(a) \( y = x^2 \sin x \)

\[
y' = \sqrt{2x \sin x + x^2 \cos x}
\]

(b) \( y = \frac{x^2 + 1}{x^2 - 1} \)

\[
y' = \frac{(2x)(x^2 - 1) - (x^2 + 1)(2x)}{(x^2 - 1)^2}
\]

\[
= \frac{2x^3 - 2x - 2x^3 - 2x}{(x^2 - 1)^2}
\]

\[
= \frac{-4x}{(x^2 - 1)^2}
\]
3. Differentiate the following functions.

(a) \( y = (x^3 - 2x^2 + x)^{20} \)

\[
y' = \left[ 20(x^3 - 2x^2 + x)^{19} \right] (3x^2 - 4x + 1)
\]

(b) \( y = \cos \sqrt{x} \)

\[
y' = (-\sin \sqrt{x}) \left( \frac{1}{2\sqrt{x}} \right) = \frac{-1}{2\sqrt{x}} \sin \sqrt{x}
\]
4. Differentiate the following functions.

(a) \( y = x\sqrt{2x^5 + x} \)

\[
y' = \sqrt{2x^5 + x} + x \cdot \frac{1}{2} \left( 2x^5 + x \right)^{-\frac{1}{2}} (10x^4 + 1)
\]

\[
y' = \sqrt{2x^5 + x} + \frac{10x^4 + x}{2\sqrt{2x^5 + x}}
\]

(b) \( y = \frac{\sin(2x-1)}{x^3} \)

\[
y' = \frac{(\cos(2x-1)) \cdot x \cdot (x^3) - (\sin(2x-1)) \cdot (3x^2)}{x^6}
\]

\[
y' = \frac{2x^3 \cos(2x-1) - 3x^2 \sin(2x-1)}{x^6}
\]

\[
y' = \frac{2x^2 \cos(2x-1) - 3\sin(2x-1)}{x^4}
\]
5. Differentiate the following functions.

(a) \[ y = \frac{\sqrt{2x^3 - 3x^2}}{x+1} \]

\[ y' = \frac{\frac{1}{2}(2x^3 - 3x^2) - \frac{6x^2 - 6x}{(x+1)^2} - \sqrt{2x^3 - 3x^2}}{(x+1)^2} \]

\[ y' = \frac{3(x^3 - x)(x+1)}{\sqrt{2x^3 - 3x^2}} - \frac{\sqrt{2x^3 - 3x^2}}{(x+1)^2} = \frac{3x^3 - 3x}{(x+1)^2 \sqrt{2x^3 - 3x^2}} \]

\[ = \frac{x^3 + 3x^2 - 3x}{(x+1)^2 \sqrt{2x^3 - 3x^2}} \]

(b) \[ y = \sin(\cos x) \]

\[ y' = (\cos(\cos x))(-\sin x) = -\sin x \cos(\cos x) \]
6. Find the first, second, and third derivatives of the function $y = \frac{1}{2x+3}$.

\[ y' = (2x+3)^{-1} \]
\[ y' = - (2x+3)^{-2}(2) = -\frac{2}{(2x+3)^2} \]
\[ y'' = - (-2) (2x+3)^{-3}(2)(2) = \frac{8}{(2x+3)^3} \]
\[ y''' = 8 (-3)(2x+3)^{-4}(2) = -\frac{48}{(2x+3)^4} \]
7. Use implicit differentiation to find $dy/dx$ for the following equations:

(a) $x^3 + xy = 2$

$$3x^2 + y + xy' = 0$$

$$y' = \frac{-3x^2 - y}{x}$$

(b) $x^2y^3 = x + y$

$$2x^2y^3 + 2x^2y^2y' = 1 + y'$$

$$3x^2y^2 y' - y' = 1 - 2xy^2$$

$$y' = \frac{1 - 2xy^2}{3x^2y^2 - 1}$$
8. A ball is thrown off the top of a building on the surface of Mars. Its height above the ground, as a function of time (in seconds), is $H(t) = -3t^2 + 3t + 90$ (in feet).

(a) Find its velocity and acceleration as a function of time.

$$v(t) = -6t + 3 \quad (\text{ft/sec})$$
$$a(t) = -6 \quad (\text{ft/sec}^2)$$

(c) If the ball travels faster than 30 feet per second, it will disintegrate. Will this happen before it hits the ground?

$$0 = -3t^2 + 3t + 90$$

$$0 = t^2 - t - 30 = (t-6)(t+5)$$

roots: $t = 6, -5 \rightarrow$ discard negative root

hits ground at $t = 6$

$$v(6) = -6(6) + 3 = -33 \quad \text{ft/sec} \quad \text{(downward motion)}$$

yes, it surpasses 30 ft/sec
9. Find the intervals on which the following functions are increasing and decreasing.

(a) \( f(x) = 2x^3 - 3x^2 - 12x + 4 \)

\[ f'(x) = 6x^2 - 6x - 12 > 0 \]
\[ x^2 - x - 2 > 0 \]
\[ (x - 2)(x + 1) > 0 \quad \text{roots } x = 2, -1 \]
increasing when \( x < -1 \) and \( x > 2 \)

decreasing when \( -1 < x < 2 \)

(b) \( g(x) = \frac{x}{x-1} \)

\[ g'(x) = \frac{(1)(x-1) - x(1)}{(x-1)^2} = \frac{-1}{(x-1)^2} \]

\( g'(x) < 0 \) always

always decreasing \( (\text{except at } x=1) \)
10. Find the intervals on which the following functions are concave up and concave down.

(a) \( f(x) = 2x^3 - 3x^2 - 12x + 4 \)

\[ f'(x) = 6x^2 - 6x - 12 \]

\[ f''(x) = 12x - 6 \geq 0 \]

\[ 12x - 6 > 0 \]

\[ x > \frac{1}{2} \]

concave up when \( x > \frac{1}{2} \)

concave down when \( x < \frac{1}{2} \)

(b) \( g(x) = \frac{x}{x-1} \)

\[ g'(x) = \frac{1}{(x-1)^2} \quad \text{(see #9(b))} \]

\[ g''(x) = \frac{2}{(x-1)^3} \]

\[ g''(x) \geq 0 \quad \text{when} \quad (x-1)^3 > 0 \]

\[ \text{when} \quad x-1 > 0 \]

\[ \text{when} \quad x > 1 \]

concave up when \( x > 1 \)

concave down when \( x < 1 \)