This exam should have 16 questions. Part I will have 12 multiple choice questions, 5 points each. Part II will have 4 hand graded questions, 10 points each. Please check to see that your exam is complete. If you do not have a PENCIL to mark your card, please ask to borrow one from your proctor.

Write your ID NUMBER (not your SS number) on the six blank lines at the top of your answer card, using one blank for each digit. Then shade in the corresponding boxes below. Also print your name at the top of your card.

As you work the exam, lightly shade in the correct answers on your answer card. At the end of your exam, when you are certain of all your choices, darken all your answer boxes. If your card becomes damaged please ask your proctor for a new one.

PART I: (60 points).

1) If \( f(x) = \ln(\cos(x)) \), then find \( f'(\frac{\pi}{3}) \).

\[
f'(x) = \frac{1}{\cos(x)} \cdot -\sin(x)
\]

\[
f'(x) = -\frac{\sin(x)}{\cos(x)}
\]

\[
f'(\frac{\pi}{3}) = -\frac{\sqrt{3}}{2} = -\sqrt{3}
\]
2) If we use \textbf{linear approximation} of the function \( f(x) = \sqrt{1 + x} \) at \( a = 0 \), to estimate \( \sqrt{1.01} \), we would get:

\[
\sqrt{1.01} = f(0.01) \approx f(0) + f'(0)(0.01)
\]

\[
f'(x) = \frac{1}{2\sqrt{1+x}} \quad f(0) = 1 \quad f'(0) = \frac{1}{2}
\]

\[
= 1 + \left(\frac{1}{2}\right) \left(\frac{1}{100}\right) = 1 + \frac{1}{200}
\]

3) For \( y = \sqrt{x} \), find \( dy \), for \( x = 1 \) and \( dx = 0.25 \):

\[
f'(x) = \frac{dy}{dx} = \frac{1}{3} x^{-2/3}
\]

\[
dy = f'(x) \ dx = \left(\frac{1}{3} x^{-2/3}\right) \ (dx)
\]

\[
\left( \text{at } x = 1 \right) \quad dy = \left(\frac{1}{3}\right) \left(\frac{1}{4}\right) = \frac{1}{12}
\]
4) Find \( \lim_{x \to 0} \frac{e^{2x} - 1}{2 \sin(3x)} \) by \( \frac{0}{0} \) rule

\[
= \lim_{x \to 0} \frac{4e^{2x}}{2 \cdot \cos(3x) \cdot 3} = \frac{4}{6} = \frac{2}{3}
\]

A) \( \infty \)
B) \( -\infty \)
C) 2
D) \( \frac{4}{3} \)
E) 3
F) 4
G) \( \frac{1}{2} \)
H) \( \frac{1}{4} \)
I) \( \frac{1}{3} \)
J) \( \frac{2}{3} \)

5) On what interval is the function \( f(t) = t \cdot \ln(t) \) increasing?

\[
f'(t) = \ln(t) + t \cdot \frac{1}{t} = \ln(t) + 1
\]

When \( \ln(t) = -1 \)

\[
t = e^{-1} = \frac{1}{e}
\]

\( f' \) dec \( - \frac{1}{e} \) inc

increasing for \( t > \frac{1}{e} \)
6) Find the **x-coordinates** of all the **critical numbers** of the function

\[
f(x) = \sqrt{x^2 - 6x + 8}
\]

\[
f'(x) = \frac{1}{2} \left( x^2 - 6x + 8 \right)^{-2/3} \cdot (2x-6)
\]

\[
= \frac{2x-6}{3 \left( x^2 - 6x + 8 \right)^{2/3}}
\]

i) \( f'(x) = 0 \) when \( x = 3 \)

ii) \( f'(x) \) DNE when \( x^2 - 6x + 8 = 0 \) or \( (x-2)(x-4) = 0 \)

\( x = 2, 4 \)
7) If Newton's Method is used to locate a root of \( x^3 - 7 = 0 \) with the initial approximation being \( x_1 = 2 \), then find the second approximation \( x_2 \).

\[
X_2 = X_1 - \frac{f(x_1)}{f'(x_1)}
\]

\[f'(x) = 3x^2\]

\[f(x_1) = 2^3 - 7 = 1\]

\[f'(x_1) = 3(2^2) = 12\]

So,

\[X_2 = 2 - \frac{1}{12} = \frac{23}{12}\]
8) If \( f(x) = x^x \) then find \( f'(e) \).

\[
\text{If } y = x^x \\
\text{Then } \ln(y) = \ln(x^x) = x \cdot \ln(x) \Rightarrow \\
\frac{1}{y} \cdot \frac{dy}{dx} = \ln(x) + 1 \\
f'(x) = \frac{dy}{dx} = y \left( \ln(x) + 1 \right) \\
f'(x) = \frac{dy}{dx} = x^x \left( \ln(x) + 1 \right) \\
\frac{dy}{dx} = e^x \left( \ln(e) + 1 \right) = 2e^e.
\]
9) Find the **absolute minimum** (y-coordinate) of the function 
\( f(x) = x^4 - 2x^2 + 4 \) on the interval \([-2, 2]\).

\[ f'(x) = 4x^3 - 4x = 0 \]
\[ 4x(x^2 - 1) = 0 \]
\[ x = 0, 1, -1 \text{ are the critical numbers} \]

\[ f(-2) = 12 \quad f(-1) = 3 \quad f(0) = 4 \quad f(1) = 3 \quad f(2) = 12 \]

So, **absolute minimum** (y-coord.) is 3.
10) Calculate \( \lim_{x \to 0^+} (1 + 3x)^{\cot(x)} \)

A) \( y = e^4 \)
B) \( y = e^{-4} \)
C) \( y = e^{1/4} \)
D) \( y = e^{-1/4} \)
E) \( y = e^5 \)
F) \( y = e^{-3} \)
G) \( y = 4e \)
H) \( y = -4e \)
I) \( y = 3e^4 \)
J) \( y = 3e^{-4} \)

\[
\lim_{x \to 0^+} \ln \left( \frac{(1 + 3x)^{\cot(x)}}{e^x} \right)
\]

\[
= \lim_{x \to 0^+} \cot(x) \cdot \ln(1 + 3x)
\]

\[
= \lim_{x \to 0^+} \frac{\ln(1 + 3x)}{\cot(x)}
\]

By L'Hopital's Rule

\[
= \lim_{x \to 0^+} \frac{3}{1 + 3x} \cdot \csc^2(x)
\]

Hence \( \lim_{x \to 0^+} (1 + 3x)^{\cot(x)} = e^3 \)
11) Find the x-coordinates of the points of inflection of \( f(x) \), given that the derivative \( f'(x) = x^4 - \frac{3}{4} x^3 - 4x^2 \).

A) 1, 2, 3
B) 0, 1, 2
C) -1, 0, 1
D) -2, -1, 2
E) -3, 0, 2
F) 0, 1, 3
G) -1, 0, 2
H) -2, 1, 4
I) -1, 3, 4
J) -4, 0, 3

\[
f''(x) = 4x^3 - 4x^2 - 8x = 0
\]

\[
4x(x^2 - x - 2) = 0
\]

\[
4x(x-2)(x+1) = 0
\]

\[
f'': \quad \text{down} \quad -1 \quad \text{up} \quad 0 \quad \text{down} \quad 2 \quad \text{up}
\]

Therefore \( x = -1, 0, 2 \) are points of inflection.
12) Find \( \lim_{x \to \infty} \frac{\ln(1 + 2e^x)}{x} \). (\( \frac{\infty}{\infty} \))

A) 0

B) 1  \text{ By L'Hospital's Rule }

C) 2

D) \( \frac{2}{3} \)

E) \( \frac{3}{2} \)

F) 3

G) \( \frac{3}{4} \)

H) \( \frac{4}{3} \)

I) \( \infty \)

J) \(-\infty\)

\[
\lim_{x \to \infty} \frac{2e^x}{1 + 2e^x} = \lim_{x \to \infty} \frac{\frac{2e^x}{2e^x}}{\frac{1 + 2e^x}{2e^x}} = \frac{1}{1} = 1
\]
PART II : (40% of the Test points) In each problem show the work you did to find the answer. Generally, a correct answer without any supporting work will not receive credit. PUT YOUR NAME & ID # ON PAGES 11-14

13) Consider the curve \( y = \frac{x+1}{x-1} \). \( \text{Domain: } x \neq 1 \)

   a) Find the intervals where the curve is increasing, decreasing, and find all local max. and local min. points, if they exist. (3 pts)

   \[
   \frac{dy}{dx} = \frac{1 \cdot (x-1) - 1 \cdot (x+1)}{(x-1)^2} = \frac{-2}{(x-1)^2}
   \]

   always negative

   Hence curve is always decreasing

   with no local max. and no local min.

   b) Find the intervals where the curve is concave up, concave down, and all the inflection points, if they exist. (2 pts)

   \[
   f'(x) = -2 \cdot (x-1)^{-2}
   \]

   \[
   f''(x) = 4 \cdot (x-1)^{-3} = \frac{4}{(x-1)^3} \neq 0
   \]

   concave down on \((-\infty, 1)\)

   concave up on \((1, \infty)\), but

   no points of inflection
Part II (cont.)

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13. c) Find all the **vertical** and **horizontal asymptotes**, if they exist.

Find **where** the curve crosses the **x** and **y** axes. (3 pts)

i) \( x = 1 \) is **vertical asymptote** since

\[
\lim_{x \to 1} \frac{x+1}{x-1} = \pm \infty
\]

ii) \( \lim_{x \to \pm \infty} \frac{x+1}{x-1} = 1 \) hence \( y = 1 \)

\((0, -1)\) is **y-intercept**

\((-1, 0)\) is **x-intercept**

d) Using the information in parts (a)-(c) sketch the curve. (2 pts).

\[ y = 1 \]

\((-1, 0)\) and \((1, 0)\)

\(x = 1\)
14) A box with a square base and an open top has a volume of 32 ft\(^3\). Find the exact dimensions of the box (length, width and height) if we want the box to have the smallest possible surface area.

a) Draw diagram, use symbols to name parts and write down given information. (3 points)

\[ \text{Given } x^2y = 32 \text{ ft}^3 \]

b) Express the problem, in terms of the symbols in part (a). (3 points)

Find \( x \) and \( y \) which gives the surface area \( A = x^2 + 4xy \) and its absolute minimum.

c) Solve the problem and state the dimensions of the box. (4 pts.)

Since \( y = \frac{32}{x^2} \), \( A = x^2 + 4x\left(\frac{32}{x^2}\right) \)

\[ A = x^2 + \frac{128}{x} \]

\[ \frac{dA}{dx} = 2x - \frac{128}{x^2} = \frac{2x^3 - 128}{x^2} = 0 \]

\[ x^3 = 64 \]

\[ x = 4 \text{ (given abs min)} \]

\[ y = \frac{32}{16} = 2 \]

So, dimensions are

\[ 4 \times 4 \times 2 \text{ feet} \]
15) At noon rowboat B is 1 km east of rowboat A. Then rowboat A rows north at 3 km/hr and rowboat B rows east at 3 km/hr. How fast is the distance between them changing at 1:00 PM?

a) Draw diagram, use symbols to name parts and write down given information.

\[
\text{Given: } \frac{dy}{dt} = 3 \text{ km/hr} \quad \text{and} \quad \frac{dx}{dt} = 3 \text{ km/hr}.
\]

b) Express the problem in terms of the symbols in part (a).

Find \( \frac{dz}{dt} \) when \( t = 1 \) (\( t = 0 \) is noon).

\[ z^2 = (x+1)^2 + y^2 \]

\[ \frac{d}{dt}z^2 = 2(x+1) \frac{dx}{dt} + 2y \frac{dy}{dt} \]

When \( t = 1 \): \( x = 3 \), \( y = 3 \) and \( z = 5 \) (3 x 4 x 5 right triangle)

So \( 5 \frac{dz}{dt} = (4)(3) + (3)(3) = 21 \)

\[ \frac{dz}{dt} = \frac{21}{5} \text{ km/hr} \]

C) Solve the problem. (5 pts.)

Part II (cont.)

Name__________________________________________

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16) **Use logarithmic differentiation** to find the derivative, \( \frac{dy}{dx} \), of the function \( y = \sqrt[10]{x} \cdot e^{x^2} \cdot (x^2 + 1)^{10} \), as an explicit function of \( x \).

(i.e. no \( y \) terms in your answer) (10 pts.)

\[
\ln(y) = \ln \left( x^{\frac{1}{10}} \cdot e^{x^2} \cdot (x^2 + 1)^{10} \right)
\]

\[
\ln(y) = \ln(x^{\frac{1}{10}}) + \ln(e^{x^2}) + \ln((x^2 + 1)^{10})
\]

\[
\ln(y) = \frac{1}{10} \ln(x) + x^2 + 10 \ln(x^2 + 1)
\]

\[
\frac{1}{y} \frac{dy}{dx} = \frac{1}{2x} + 2x + 10 \cdot \frac{2x}{x^4 + 1}
\]

\[
\frac{dy}{dx} = y \left( \frac{1}{2x} + 2x + \frac{20x}{x^4 + 1} \right)
\]

\[
\frac{dy}{dx} = \sqrt{x} e^{x^2} (x^2 + 1)^{10} \left[ \frac{1}{2x} + 2x + \frac{20x}{x^4 + 1} \right]
\]