This exam contains ten multiple-choice problems worth two points each, five true-false problems worth one point each, and four computational problems worth 25 points altogether, for an exam total of 50 points.

Part I. Multiple Choice. (2 points each)

For each of the following, mark your answer card with the letter corresponding to the only correct answer.

1. Consider the initial value problem $y'' + 5y' = 0$, $y(0) = -1$, $y'(0) = 6$. Find the solution, then describe its behavior as $t \to \infty$.

   (A) $y \to -\infty$
   (B) $y \to -5$
   (C) $y \to -\frac{6}{5}$
   (D) $y \to -\frac{1}{5}$
   (E) $y \to 0$
   (F) $y \to \frac{1}{5}$
   (G) $y \to \frac{6}{5}$
   (H) $y \to 5$
   (I) $y \to \infty$

   \[ r^2 + 5r = 0 \]
   \[ r = 0, -5 \]

   \[ y = c_1 + c_2 e^{-5t} \]

   \[ \lim_{t \to \infty} y = \frac{1}{5} \]

\[ y' = -5c_2 e^{-5t} \]

\[ 6 = -5c_2 \]

\[ c_2 = -\frac{6}{5}, c_1 = \frac{1}{5} \]

2. Consider the linear nonhomogeneous differential equation $y' + p(t)y' + q(t)y = g(t)$ as well as the corresponding homogeneous equation $y' + p(t)y' + q(t)y = 0$.

Suppose $y_1$ is a solution of the homogeneous differential equation and $y_2$ is a solution of the nonhomogeneous differential equation. Which of the following is a solution of the nonhomogeneous differential equation?

   (I) $y_1 + y_2$ ✓
   (II) $y_1 - y_2$ ×
   (III) $y_2 - y_1$ ✓

(A) none of these
(B) I only
(C) II only
(D) III only
(E) I and II only
(F) I and III only
(G) II and III only
(H) I, II, and III

Every solution of the nonhomogeneous DE is of this form: $y_2$ plus a solution of
the homogeneous DE.
3. Consider the linear nonhomogeneous differential equation \( y'' + y' - 2y = 7t^2e^t \). What is the correct form which must be used to find a particular solution \( Y \) when using the method of undetermined coefficients?

(A) \( Y = Ae^t \)
(B) \( Y = Ate^t \)
(C) \( Y = Ate^t + Be^t \)
(D) \( Y = At^2e^t \)
(E) \( Y = At^2e^t + Bte^t \)
(F) \( Y = At^2e^t + Bte^t + Be^t \)
(G) \( Y = At^2e^t \)
(H) \( Y = At^3e^t + Bte^t \)
(I) \( Y = At^3e^t + Bt^2e^t + Cte^t \)
(J) \( Y = At^3e^t + Bt^2e^t + Cte^t + De^t \)

4. Consider the linear nonhomogeneous differential equation \( y'' - y = 2\sin(e^t) \). Note that \( y_1 = e^{-t} \) and \( y_2 = e^t \) are linearly independent solutions of the corresponding homogeneous equation. Suppose you are using the method of variation of parameters to find a particular solution \( Y = u_1y_1 + u_2y_2 \) of the nonhomogeneous equation. What is \( u_1 \)? (Note that you do not have to find \( u_2 \) or \( Y \).)

(A) \( u_1 = \cos(e^t) \)
(B) \( u_1 = \sin(e^t) \)
(C) \( u_1 = 2\cos(e^t) \)
(D) \( u_1 = 2\sin(e^t) \)
(E) \( u_1 = e^{-t}\cos(e^t) \)
(F) \( u_1 = e^{-t}\sin(e^t) \)
(G) \( u_1 = e^t\cos(e^t) \)
(H) \( u_1 = e^t\sin(e^t) \)
(I) \( u_1 = \frac{\cos(e^t)}{e^t} \)
(J) \( u_1 = \frac{\sin(e^t)}{e^t} \)
5. Find the general solution of the linear homogeneous differential equation \( y^{iv} - 2y'' + y = 0 \).

(A) \( y = c_1 e^{-t} + c_2 e^{2t} + c_3 e^{t} + c_4 e^{-t} \)  
\( \lambda^4 - 2\lambda^2 + 1 = 0 \)

(B) \( y = c_1 e^t + c_2 e^{2t} + c_3 e^{t} + c_4 e^t \)  
\( (\lambda^2 - 1)(\lambda^2 - 1) = 0 \)

(C) \( y = c_1 e^{-t} + c_2 e^{t} + c_3 e^{2t} + c_4 e^{t} \)  
\( \lambda = \pm 1, \pm 1 \)

(D) \( y = c_1 e^{-t} + c_2 e^{-2t} + c_3 e^{t} + c_4 e^{2t} \)  

(E) \( y = c_1 e^{-t} + c_2 e^{-2t} + c_3 e^{t} + c_4 e^{2t} + c_5 e^{-t} \)  

(F) \( y = c_1 e^t + c_2 e^{2t} + c_3 e^{t} + c_4 e^{2t} \)  

(G) \( y = c_1 e^{-t} + c_2 e^{t} + c_3 \cos t + c_4 \sin t \)  

(H) \( y = c_1 e^{-t} + c_2 e^{t} + c_3 \cos t + c_4 \sin t \)  

(I) \( y = c_1 e^{t} + c_2 e^{-t} + c_3 \cos t + c_4 \sin t \)  

(J) \( y = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t \)  

6. Find the Taylor series for the function \( f(x) = \frac{1}{1+x} \) at \( x_0 = 0 \).

(A) \( \frac{1}{1+x} = 1 - x + \frac{1}{2} x^2 - \frac{1}{6} x^3 + \frac{1}{24} x^4 + \cdots \)

(B) \( \frac{1}{1+x} = 1 - (x+1) + \frac{1}{2} (x+1)^2 - \frac{1}{6} (x+1)^3 + \frac{1}{24} (x+1)^4 + \cdots \)

(C) \( \frac{1}{1+x} = 1 - \frac{1}{2} x + \frac{1}{3} x^2 - \frac{1}{4} x^3 + \frac{1}{5} x^4 + \cdots \)

(D) \( \frac{1}{1+x} = 1 - \frac{1}{2} (x+1) + \frac{1}{3} (x+1)^2 - \frac{1}{4} (x+1)^3 + \frac{1}{5} (x+1)^4 + \cdots \)

(E) \( \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + \cdots \)

(F) \( \frac{1}{1+x} = 1 - (x+1) + (x+1)^2 - (x+1)^3 + (x+1)^4 + \cdots \)

(G) \( \frac{1}{1+x} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 + \cdots \)

(H) \( \frac{1}{1+x} = 1 - 2(x+1) + 3(x+1)^2 - 4(x+1)^3 + 5(x+1)^4 + \cdots \)

(I) \( \frac{1}{1+x} = 1 - x + 2x^2 - 6x^3 + 24x^4 + \cdots \)

(J) \( \frac{1}{1+x} = 1 - (x+1) + 2(x+1)^2 - 6(x+1)^3 + 24(x+1)^4 + \cdots \)

\( f(x) = (1+x)^{-1} \)

\( f'(x) = -(1+x)^{-2} \)

\( f''(x) = 2(1+x)^{-3} \)

\( f'''(x) = -6(1+x)^{-4} \)

\( f^{iv}(x) = 24(1+x)^{-5} \)

\( f(0) = 1 \)

\( f'(0) = -1 \)

\( f''(0) = 2 \)

\( f'''(0) = -6 \)

\( f^{iv}(0) = 24 \)

\( a_0 = f(0)/0! = 1 \)

\( a_1 = f'(0)/1! = -1 \)

\( a_2 = f''(0)/2! = 1 \)

\( a_3 = f'''(0)/3! = -1 \)

\( a_4 = f^{iv}(0)/4! = 1 \)

\( \ldots \)
7. Consider the differential equation \((x - 2)y'' + (2x + 1)y' - y = 0\). The general solution (in the form of a power series solution about \(x_0 = 0\)) is \(y = a_0y_1 + a_1y_2\), where \(y_1\) and \(y_2\) are as shown below.

\[
y_1 = 1 - \frac{1}{4}x^2 - \frac{1}{12}x^3 + \cdots
\]
\[
y_2 = x + \frac{1}{4}x^2 + \frac{1}{6}x^3 + \cdots
\]

(You do not need to verify this.) Find the particular solution which satisfies the initial conditions \(y(0) = 3\), \(y'(0) = 1\).

(A) \(y = 1 - x - \frac{1}{2}x^2 - \frac{1}{4}x^3 + \cdots\)
(B) \(y = 1 - x - \frac{1}{4}x^2 + \frac{1}{6}x^3 + \cdots\)
(C) \(y = 1 + x - \frac{3}{4}x^2 - \frac{1}{4}x^3 + \cdots\)
(D) \(y = 1 + x + \frac{1}{12}x^3 + \cdots\)
(E) \(y = 1 + 3x + \frac{1}{4}x^2 + \frac{1}{2}x^3 + \cdots\)
(F) \(y = 1 + 3x + \frac{1}{2}x^2 + \frac{5}{12}x^3 + \cdots\)
(G) \(y = 3 - x - x^2 - \frac{5}{12}x^3 + \cdots\)
(H) \(y = 3 - x - \frac{1}{6}x^2 - \frac{5}{6}x^3 + \cdots\)
(I) \(y = 3 + x - x^2 - \frac{7}{12}x^3 + \cdots\)
(J) \(y = 3 + x - \frac{1}{2}x^2 - \frac{1}{12}x^3 + \cdots\)

\[
a_0 = 3, \quad a_1 = 1
\]
\[
y = 3\left[1 - \frac{1}{4}x^2 - \frac{1}{12}x^3 + \cdots\right] + \left[x + \frac{1}{4}x^2 + \frac{1}{12}x^3 + \cdots\right]
\]
\[
= 3 + x - \frac{1}{2}x^2 - \frac{1}{12}x^3 + \cdots
\]

8. Consider the differential equation \((x^2 + 1)y'' + 3xy' + (x - 4)y = 0\). Determine a lower bound for the radius of convergence of series solutions about the point \(x_0 = 3\) for this differential equation.

(A) 0
(B) 1
(C) 2
(D) 3
(E) 4
(F) \(\sqrt{5}\)
(G) \(\sqrt{10}\)
(H) \(\sqrt{13}\)
(I) \(\sqrt{17}\)
(J) \(\infty\)

\[
P(x) = x^2 + 1
\]

Roots of \(P(x)\): \(\pm i\)

\[
d(3, i) = \sqrt{10}
\]

\[
d(3, -i) = \sqrt{10}
\]
9. Consider the differential equation \((x + 2)y'' + (x - 2)y' + xy = 0\). Classify the point \(x_0 = -2\).

(A) \(x_0 = -2\) is an ordinary point of the differential equation.

(B) \(x_0 = -2\) is a regular singular point of the differential equation.

(C) \(x_0 = -2\) is an irregular singular point of the differential equation.

\[
P(-2) = 0, \text{ so } x_0 = -2 \text{ is a singular point.}
\]

\[
\lim_{x \to -2} (x+2) \cdot \frac{x-2}{x+2} = -4
\]

\[
\lim_{x \to -2} (x+2)^2 \cdot \frac{x}{x+2} = 0
\]

Since both limits are finite, \(x_0 = -2\) is a regular singular point.

10. Find the general solution of the differential equation \(x^2y'' + 13xy' + 36y = 0\).

(A) \(y = c_1e^{-6x} + c_2e^{-6x}\)

(B) \(y = c_1e^{-6x} + c_2xe^{-6x}\)

(C) \(y = c_1e^{-6x} + c_2e^{-6x}\ln x\)

(D) \(y = c_1e^{-4x} + c_2e^{-9x}\)

(E) \(y = c_1xe^{-4x} + c_2xe^{-9x}\)

(F) \(y = c_1x^{-6} + c_2x^{-6}\)

(G) \(y = c_1x^{-6} + c_2x^{-5}\)

(H) \(y = c_1x^{-6} + c_2x^{-6}\ln x\)

(I) \(y = c_1x^{-4} + c_2x^{-9}\)

(J) \(y = c_1x^{-4}\ln x + c_2x^{-9}\ln x\)

This is an Euler equation.

Indicial equation:

\[
r(r-1) + 13r + 36 = 0
\]

\[
r^2 + 12r + 36 = 0
\]

\[
(r + 6)(r + 6) = 0
\]

\[
r = -6, -6
\]
Part II. True-False (1 point each)

Mark your answer card “A” if the statement is true and “B” if the statement is false.

11. The functions $y_1 = 3e^t$ and $y_2 = 4e^{2t}$ form a fundamental set of solutions for the differential equation $y'' - 3y' + 2y = 0$.

   A

   Both functions are solutions.

   The functions are linearly independent.

   B

   There are two functions for a second-order DE.

12. Consider spring systems with no damping and no external force. All other things being equal, a spring system with a greater mass vibrates with a greater frequency than one with a lesser mass.

   $mu'' + ku = 0 \quad mr^2 + k = 0 \quad r^2 = -\frac{k}{m}$

   $r = \pm \sqrt{\frac{k}{m}} \quad u = c_1 \cos \sqrt{\frac{k}{m}} t + c_2 \sin \sqrt{\frac{k}{m}} t$

   frequency: $\sqrt{\frac{k}{m}} \quad \text{if } m \text{ increases, } \sqrt{\frac{k}{m}} \text{ decreases}$

13. Consider spring systems with no damping and no external force. All other things being equal, a spring system with a greater spring constant, (in other words, a stiffer spring), vibrates with a greater frequency than one with a smaller spring constant.

   A

   If $k$ increases, $\sqrt{\frac{k}{m}}$ increases

14. Consider a spring system which involves both damping and an external force, and which has the following equation of motion.

   $u = 6\cos 3t + 2e^{-2t}\sin 4t$

   The steady-state solution is the term $2e^{-2t}\sin 4t$.

   B

   The steady-state solution is the part which is not “damped out”: the term $6\cos 3t$

15. The following functions are linearly independent.

   $y_1 = x + x^2 \quad y_2 = x^2 + x^3 \quad y_3 = x^3 + x \quad y_4 = x + x^2 + x^3$

   B

   $y_4 = \frac{1}{2} (y_1 + y_2 + y_3)$
Part III. Computational

16. (8 points)

Consider the linear homogeneous differential equation \( ty'' - 6y' + \frac{10}{t} y = 0, \ t > 0 \). One solution is \( y_1 = t^2 \). (You do not need to verify this.) Use the method of reduction of order to find a second solution \( y_2 \) so that \( y_1 \) and \( y_2 \) are linearly independent. Show all your work.

\[
\begin{align*}
y_2 &= ut^2 \\
y_2' &= 2vt + vt^2 \\
y_2'' &= 2v + 2v't + 2v't + vt^2 \\
t \left[ 2v + 4v't + vt^2 \right] - 6 \left[ 2vt + vt^2 \right] + \frac{10}{t} \left[ vt^2 \right] &= 0 \\
4v't^2 + vt^3 - 6vt^2 &= 0 \\
v't^3 - 2vt^2 &= 0 \\
w &= v' \\
w't^3 - 2wt^2 &= 0 \\
w' - \frac{2}{t}w &= 0 \\
\mu(t) &= e^{\int -\frac{2}{t} dt} = e^{-2\ln t} = t^{-2} \\
t^{-2}w' - 2t^{-3}w &= 0 \\
t^{-2}w &= C_1 \\
w &= C_1t^2 \\
v' &= C_1t^2 \\
v &= \frac{1}{3}C_1t^3 + C_2 \\
y_2 &= \left[ \frac{1}{3}C_1t^3 + C_2 \right]t^2 \\
Choose \ y_2 &= t^5. 
\end{align*}
\]
17. (5 points)

An 8 lb. object stretches a spring 2 feet.

(a) Find the general solution of the differential equation which describes the motion of this spring system. (Assume there is no damping and no external force.)

\[ m = \frac{8}{32} = \frac{1}{4} \quad 8 = k \cdot 2 \quad k = 4 \]

\[ \frac{1}{4} u'' + 4u = 0 \]

\[ \frac{1}{4} \lambda^2 + 4 = 0 \]

\[ \lambda^2 = -16 \quad \lambda = \pm 4i \]

\[ u = c_1 \cos 4t + c_2 \sin 4t \]

(b) Suppose the motion is damped with damping constant \( \gamma \). Find the value of \( \gamma \) for which the system is critically damped.

\[ \frac{1}{4} u'' + \gamma u' + 4u = 0 \]

\[ \frac{1}{4} \lambda^2 + \gamma \lambda + 4 = 0 \]

\[ \gamma^2 - 4 \left( \frac{1}{4} \right) (4) = 0 \]

\[ \gamma^2 = 4 \quad \gamma = 2 \quad \frac{1}{lb \cdot sec} \]

(c) Return to the original scenario in which there is no damping. Give an example of an external forcing function \( F(t) \) which would lead to resonance in the spring motion.

\[ F(t) = 3 \cos 4t \]

(any linear combination of \( \cos 4t \) and \( \sin 4t \) will do.)

(Note that all of the above questions can be answered without initial conditions being specified.)
18. (3 points)

Find the general solution of the following linear homogeneous differential equation.

\[ y''' + 8y = 0 \]

\[ r^3 + 8 = 0 \]

\[ (r + 2)(r^2 - 2r + 4) = 0 \]

\[ r = -2, \quad r = \frac{2 \pm \sqrt{4 - 16}}{2} = 1 \pm \sqrt{3} i \]

\[ y = c_1 e^{-2t} + c_2 e^t \cos \sqrt{3} t + c_3 e^t \sin \sqrt{3} t \]
19. (9 points)

Consider the differential equation \((1 - x^2)y'' - 3xy' - y = 0\). Find two linearly independent power series solutions \(y_1\) and \(y_2\) about \(x_0 = 0\) for this differential equation, following the steps below. You may give just the first three terms of each.

(a) Find the recurrence relation for the coefficients \(a_n\).

\[
\begin{align*}
\sum_{n=0}^{\infty} a_n x^n = y, \\
\sum_{n=0}^{\infty} n a_n x^{n-1} = y', \\
\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = y''.
\end{align*}
\]

\[
\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} n(n-1) a_n x^{n-1} - 3 \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0
\]

\[
\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} n(n-1) a_n x^n - 3 \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0
\]

\[
\sum_{n=0}^{\infty} \left[ (n+2)(n+1) a_{n+2} - n(n-1) a_n - 3na_n - a_n \right] x^n = 0
\]

For all \(n \geq 0\),

\[
(n+2)(n+1) a_{n+2} - n(n-1) a_n - 3na_n - a_n = 0
\]

\[
\begin{align*}
(n+2)(n+1) a_{n+2} &= (n^2 + 2n + 1) a_n \\
a_{n+2} &= \frac{n+1}{n+2} a_n
\end{align*}
\]

(b) Work out \(a_2\), \(a_3\), \(a_4\), and \(a_5\).

\(n=0\): \(a_2 = \frac{1}{2} a_0\)

\(n=1\): \(a_3 = \frac{2}{3} a_1\)

\(n=2\): \(a_4 = \frac{3}{4} a_2 = \frac{3}{8} a_0\)

\(n=3\): \(a_5 = \frac{4}{5} a_3 = \frac{8}{15} a_1\)

(c) Find \(y_1\) and \(y_2\) as specified above.

\[
\begin{align*}
y &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots \\
&= a_0 + a_1 x + \frac{1}{2} a_0 x^2 + \frac{2}{3} a_1 x^3 + \frac{3}{8} a_0 x^4 + \frac{8}{15} a_1 x^5 + \ldots
\end{align*}
\]

Set \(a_0 = 1\), \(a_1 = 0\) : \(y_1 = 1 + \frac{1}{2} x^2 + \frac{3}{8} x^4 + \ldots\)

Set \(a_0 = 0\), \(a_1 = 1\) : \(y_2 = x + \frac{2}{3} x^3 + \frac{8}{15} x^5 + \ldots\)